THE WORK OF PROFESSOR KIYOSHI ISÉKI ON TOPOLOGY

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Abstract. We shall review the work of Professor Kiyoshi Iséki on topology.

Professor Kiyoshi Iséki was interested in several fields in mathematics, e.g., Functional Analysis, Topology, Algebra, Mathematical Foundation including Mathematical Logic and Set Theory. He published his first paper in 1949, and then he published more than three hundreds articles concerning the fields above. His first published paper was involved with general topology, since then he wrote more than fifty papers concerning general topology in the period 1950–1970. Professor Iséki contributed to the development of the early stage of general topology, especially, the theory of covering properties (countably compactness, feeble compactness, pseudocompactness, countable paracompactness etc.), the theory of uniform spaces and the extensions of continuous functions.

The first development of general topology was achieved in the first half of the 20th century by distinguished mathematicians such as P. S. Alexandroff, K. Borsuk, M. Fréchet, K. Kuratowski, M. H. Stone, etc. The decade 1950's was the opening stage of the second development of modern general topology. The fundamental theorems in "modern general topology" were discovered in 1950's; e.g., Nagata-Smirnov Metrization Theorem (1951), Michael's Selection Theorems (1956), Michael's Characterizations of Paracompact Spaces (1953, 1957 and 1959), Dowker's Characterization of Paracompact Spaces (1951), Morita-Katětov Coincidence Theorem for dim and Ind (1952 (Katětov), 1954 (Morita)), etc. Japanese topologists also made a big contribution to the development of general topology; Professors K. Kunugui, K. Morita, S. Hanai, H. Terasaka, J. Nagata, K. Nagami, Y. Kodama, A. Okuyama, etc. Professor Iséki was working in general topology among them and he found many theorems on covering properties, uniform spaces, the theory of the extensions of continuous functions, etc. We shall review the work of Professor Kiyoshi Iséki from the view of topology.

In the article, a space means a topological space with no separation axioms unless we mention any additional assumptions. We refer the reader to [3] and [22] for undefined notions and the results on general topology.

1 Characterizations of generalized compact spaces First, we recall some notions which are generalizations of compact spaces.

- A space X is called a *countably compact* if every countable open cover of X has a finite subcover.
- A space X is called a *feebly compact* if every locally finite family of open sets of X is finite.

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- A space X is called a *pseudocompact* if every real-valued continuous function on X is bounded.

Countably compact spaces were introduced by M. Fréchet in 1906, feebly compact spaces were introduced by S. Mardešić and P. Papić in 1955 and pseudocompact spaces were introduced by E. Hewitt in 1948 (see [3] for the historical remarks about these spaces). Today, we know the details about these spaces. Many fundamental results around these spaces were obtained by several authors including Professor Iséki in 1950's. Every countably compact space is feebly compact, and every feebly compact space is pseudocompact, but the converses do not hold in general. If spaces are assumed to be normal, then three notions above are equivalent. We refer to the reader to [25] and [23] for the brief surveys on countably compact spaces, feebly compact spaces and pseudocompact spaces. Now, we recall the results obtained by Iséki which characterize countably compact spaces, feebly compact spaces and pseudocompact spaces. Iséki called a family $\mathcal{A}$ of a space X has the *AU-property* if $\{\mathrm{Cl}A : A \in \mathcal{A}\}$ is a cover of X.

Theorem 1 ([12],[13],[16]) *Let X be a regular space. Then the following conditions are equivalent.*

(a) *X is feebly compact.*

(b) *Every point finite open covering of X has a finite subfamily which has the AU-property.*

(c) $\cap_{n=1}^{\infty}\mathrm{Cl}G_n \neq \emptyset$ *for each countable family $\{G_n\}_{n=1}^{\infty}$ of open sets with the finite intersection property.*

(d) *Every locally finite open covering has a finite subfamily which has the AU-property.*

(e) *Every countable open covering has a finite subfamily which has the AU-property.*

Theorem 2 ([15]) *Let X be a regular T_0-space. Then the following conditions are equivalent.*

(a)*X is feebly compact.*

(b) *Every point finite countable open covering of X has a proper subfamily which has the AU-property.*

(c)*Every point finite countable open covering of X has a proper subfamily $\mathcal{U}$ such that* $\mathrm{Cl}(\cup\mathcal{U}) = X$.

Iséki introduced a notion of the convergence of continuous functions, and characterized pseudocompact spaces by means of it. A sequence $\{f_n\}_{n=1}^{\infty}$ of (real-valued) functions on a space X is said to be *strictly continuously convergent to f* if for each sequence $\{x_n\}_{n=1}^{\infty}$ of X whenever $\{f(x_n)\}_{n=1}^{\infty}$ converges, then $\{f_n(x_n)\}_{n=1}^{\infty}$ converges to the same limit.

Theorem 3 ([17]) *Let X be a Tychonoff space. Then the following three conditions are equivalent:*

(a) *X is pseudocompact.*

(b) *Every sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued continuous functions on X convergent strictly continuously to a real-valued continuous function f converges uniformly to f.*

(c) *Every equicontinuous sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued continuous functions converging pointwise to a real-valued continuous function f converges uniformly to f, where $\{f_n\}_{n=1}^{\infty}$ is equicontinuous if for each $\varepsilon > 0$ and each $x \in X$ there is a neighborhood U of x such that* $|f_n(x) - f_n(y)| < \varepsilon$ *for each n and each $y \in U$.*

A sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on X is said to *converge uniformly to f at $x \in X$* if for every $\varepsilon > 0$, there are a natural number n_0 and a neighborhood U of x such that $|f_n(y) - f_(y)| < \varepsilon$ for all $n \geq n_0$ and $y \in U$.

Theorem 4 ([14]) *A Tychonoff space X is pseudocompact if and only if every sequence of real-valued continuous functions on X that converges to a continuous function uniformly at every point of X converges uniformly on all of X.*

Theorem 5 ([15]) *Let X be a regular T_0-space. Then the following are equivalent.*
(a) X is countably compact.
(b) Every point finite infinite open covering of X has a proper subcovering.
(c) Every point finite countably infinite open covering of X has a proper subcovering.

2 Uniform spaces The notion of uniform spaces was introduced and studied by A. Weil [26] in 1936. J. W. Tukey redefined the uniform spaces by use of open coverings [24]. Since then, many authors investigated on uniform spaces as a generalization of the notion of metrizable spaces. Today, the uniform spaces and their generalization, quasi uniform spaces, are focused again by the applications to the theory of computational topology and domain theory.

Now, we recall the (classical) results on uniform spaces which are obtained by Iséki.

Let X be a Hausdorff uniform space with a uniform structure $\mathcal{V}$. An open covering $\mathcal{U}$ of X has the *Lebesgue property* if there exists $V \in \mathcal{V}$ such that for every $x \in X$, $V(x) = \{y \in X : (x, y) \in V\} \subset U$ for some $U \in \mathcal{U}$.

Theorem 6 ([9]) *If X is a uniform space, where every finite open covering has the Lebesgue property, and if its completion $\tilde{X}$ is normal, then* $\dim X = \dim \tilde{X}$ (*where dim denotes the covering dimension*).

Theorem 7 ([10]) *Let $(X, \mathcal{V})$ be a uniform space, f an upper semi-continuous real-valued function on X and g a lower semi-continuous real-valued function on X. If every countable open covering of X has the Lebesgue property, and let $f(x) < g(x)$ for each $x \in X$, then for every $\varepsilon > 0$ there is $V \in \mathcal{V}$ such that $f(x') < g(x'') + \varepsilon$ for each $(x', x'') \in V$.*

3 Extensions of continuous functions *Let $\mathcal{P}$ be a class of normal spaces containing every closed subspace of each of its members. A space X is called an absolute retract* (*AR*) *for the class $\mathcal{P}$ if $X \in \mathcal{P}$ and X is a retract for every space $Y \in \mathcal{P}$ with $X \subset Y$. A space X is called an absolute neighborhood retract* (*ANR*) *for the class $\mathcal{P}$ if $X \in \mathcal{P}$ and for every space $Y \in \mathcal{P}$ with $X \subset Y$ there is an open set U of Y with $X \subset U$ and X is a retract of U. We refer to the reader to [2], [5] and [27] for the brief surveys on AR (ANR). A space X is called an absolutely G_δ if X is a G_δ-set of Y for every space Y with $X \subset Y$. O. Hanner [4] has shown that every metric ANR for the class of paracompact spaces is an absolute G_δ. By using the same technique as Hanner's proof, Iséki showed the following.*

Theorem 8 *If X is a metric ANR for the class of hereditarily normal spaces, then X is an absolutely G_δ.*

Iséki generalized the result due to N. Aronszajn and K. Borsuk [1] for the cases where all spaces involved are separable metric or compact metric spaces to hereditarily normal spaces.

Theorem 9 ([6]) *If $X = X_1 \cup X_2$, X_1 and X_2 are closed in X, and $X_1 \cap X_2$ is an AR* (*resp. ANR*) *for the class of hereditarily normal spaces, then X is an AR* (*resp. ANR*) *for the class of hereditarily normal spaces if and only if both X_1 and X_2 are AR* (*resp. ANR*) *for the class of hereditarily normal spaces.*

A space X is called an absolute extensor (*AE*) *for the class $\mathcal{P}$ if for every $Y \in \mathcal{P}$, each closed subset A of Y and each continuous mapping $f : A \to X$ there exists a continuous extension $g : Y \to X$ of f. A space X is called an absolute neighborhood extensor* (*ANE*) *for the class $\mathcal{P}$ if for each $Y \in \mathcal{P}$, each closed subset A of Y and each continuous mapping $f : A \to X$ there exist an open set U of Y with $A \subset U$ and a continuous extension $g : U \to X$ of f. We refer to the reader to [2] and [5] for the brief surveys on AE (ANE). The following is a generalization of the results by O. Hanner for the classes of normal, collectionwise normal, perfectly normal, or paracompact spaces.*

Theorem 10 ([7]) *Let $\mathcal{P}$ be either the class of hereditarily normal spaces or the class of normal and countably paracompact spaces. Then X is an AR (resp. ANR) for the class $\mathcal{P}$ if and only if X is an AE (resp. ANE) for the class $\mathcal{P}$.*

4 Other topics *Iséki also studied algebraic structures in sets of continuous mappings on topological spaces. In particular, he considered relations between ideals in the subring $C^+(X)$ of all non-negative functions in the ring $C^*(X)$ of all continuous bounded functions on a Hausdorff space X and filteres of closed sets of X in [11] and [19].*

Let f be a real-valued function on a space X and $x \in X$. Then the oscillation $\omega(x; f)$ of f at x is defined by $\omega(x; f) = \inf\{\operatorname{diam} f(U) : U$ is a neighborhood of $x\}$. Then Iséki and Miyanaga showed the following theorem by use of a partition of unity subordinate to a locally finite open covering.

Theorem 11 ([20]) *Let f be a real-valued function defined on a paracompact space X. If for each $x \in X$ the oscillation $\omega(x; f)$ of f at x is $\leq \alpha$, then for each $\varepsilon > 0$, there exists a real-valued continuous function g on X such that $|f(x) - g(x)| \leq \alpha + \varepsilon$ for each $x \in X$.*

A function f on a metric space (X, ρ) is said to be completely compact if for each two sequences $\{x_n\}$, $\{x'_n\}$ satisfying $\rho(x_n, x'_n) \to 0$ as $n \to \infty$, there are subsequences $\{x_{n_k}\}$, $\{x'_{n_k}\}$ of $\{x_n\}$, $\{x'_n\}$ respectively such that $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(x'_{n_k})$ has a finite value. Then Iséki characterized the compactness in terms of completely compact functions.

Theorem 12 ([18]) *Let X be a metric space. Then X is compact if and only if every continuous function is completely compact.*

Iséki also gave a simple example of a hereditarily paracompact Hausdorff space which is not perfectly normal in [8]. This answered the question asked by K. Nagami [21]. Today, we know that the Michael line is such example.

5 A memory of Professor Kiyoshi Iséki *I joined the seminar organized by Professor Iséki at Kobe University about 30 years ago when I was a graduate student at Kobe University and a research assistant at Osaka Kyoiku Universiy. The seminar was held at every Saturday, and the main topics discussed there were set-theoretic topology and functional analysis. Professor Iséki was the center of the discussions. We used to have a lunch with Professor Iséki after the morning session of the seminar, and he friendly talked with us about several topics not only mathematics. He was interested in everything, and had a big energy to study mathematics which enables him to write more than three hundreds published papers.*

People who were learned mathematics from Professor Iséki including me never forget him as a great mathematician and an excellent teacher.

References

[1] N. Aronszajn and K. Borsuk, Sur la somme et le produit combinatoire des ré tractes absolus, Fund. Math. 18 (1932), 193-197.

[2] A. Dranishnikov, Extensors; in K. P. Hart, J. Nagata and J. E. Vaughan eds., Encyclopedia of General Topology, Elsevier (2004), 122-125.

[3] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.

[4] O. Hanner, Retraction and extension of mappings of metric and non-metric spaces, Ark. Mat. 2 (1952), 315-360.

[5] R. W. Heath, Extensions of maps; in K. P. Hart, J. Nagata and J. E. Vaughan eds., Encyclopedia of General Topology, Elsevier (2004), 100-102.

[6] K. Iséki, A note on retraction in completely normal spaces, Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 3 (1954), 176-180.

[7] K. Iséki, On Hannerisation of two countably paracompact normal spaces, Proc. Japan Acad. 30 (1954), 443-444.

[8] K. Iséki, On a conjecture of K. Nagami, Proc. Japan Acad. 31 (1955), 430.

[9] K. Iséki, On the property of Lebesgue in uniform spaces, VI. Proc. Japan Acad. 32 (1956), 117-119.

[10] K. Iséki, Notes on topological spaces, II. Some properties of topological spaces with Lebesgue property, Proc. Japan Acad. 32 (1956), 171-173.

[11] K. Iséki, Notes on topological spaces V. On structure spaces of semiring, Proc. Japan Acad. 32 (1956), 426-429.

[12] K. Iséki, On weakly compact topological spaces, Proc. Japan Acad. 33 (1957), 182.

[13] K. Iséki, On weakly compact regular spaces, I, Proc. Japan Acad. 33 (1957), 252-254.

[14] K. Iséki, A characterisation of pseudo-compact spaces, Proc. Japan Acad. 33 (1957), 320-322.

[15] K. Iséki, On weakly compact and countably compact topological spaces, Proc. Japan Acad. 33 (1957), 323-324.

[16] K. Iséki, AU-covering theorem and compactness, Proc. Japan Acad. 33 (1957), 363-367.

[17] K. Iséki, Pseudo-compactness and strictly continuous convergence, Proc. Japan Acad. 33 (1957), 424-428.

[18] K. Iséki, A characterisation of compact metric spaces, Proc. Japan Acad. 34 (1958), 255-256.

[19] K. Iséki and Y. Miyanaga, Notes on topological spaces. IV, Function semiring on topological spaces, Proc. Japan Acad. 32 (1956), 392-395.

[20] K. Iséki and Y. Miyanaga, A theorem on paracompact spaces, Proc. Japan Acad. 32 (1956), 396-398.

[21] K. Nagami, On the dimension of paracompact Hausdorff spaces, Nagoya Math. J. 8 (1955), 69-70.

[22] J. Nagata, Modern general topology, second revised edition, North-Holland, Amsterdam, 1985.

[23] R. M. Stephenson,Jr., Pseudocompact spaces; in K. P. Hart, J. Nagata and J. E. Vaughan eds., Encyclopedia of General Topology, Elsevier (2004), 177-181.

[24] J. W. Tukey, Convergence and uniformity in topology, Ann. Math. Studies 2, Princeton, 1940.

[25] J. Vaughan, Countably compact spaces; in K. P. Hart, J. Nagata and J. E. Vaughan eds., Encyclopedia of General Topology, Elsevier (2004), 174-176.

[26] A. Weil, Les recouvrements des espaces topologique: espaces completes, espaces bicompacts, C. R. Acad. Paris 202 (1936), 1002-1005.

[27] J. E. West, Absolute retracts; in K. P. Hart, J. Nagata and J. E. Vaughan eds., Encyclopedia of General Topology, Elsevier (2004), 117-121.

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