PRESERVATION OF CONTINUITY

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Dedicated to the memory of Professor Kiyoshi Iséki

Abstract. This survey paper contains a brief account of the literature on preservation of continuity under various convergences from the beginning to current research. We include stronger forms of continuity such as uniform continuity, proximal continuity, strong uniform continuity and weaker forms of continuity such as quasi-continuity, near continuity and almost continuity. A few proofs of the results are given and references are given for others. A standard reference for this research area is [28]

This article is dedicated to the memory of Professor Kiyoshi Iséki, who was a pioneer in a number of research areas such as compact spaces, continuity, continuous convergence, measure theory, lattice theory, Hilbert spaces, number theory and semirings (see, *e.g.*,[19,22,23,20,21]).

1 Continuity. It is well-known that pointwise convergence does not preserve continuity. Here is a familiar example. Let (f_n) be a sequence of continuous functions on $[0,1]$ to itself given by $f_n(x) = x^n$. Then (f_n) converges pointwise to a discontinuous function f given by $f(0) = 0$ on $[0,1)$ and $f(1) = 1$.

The need to preserve continuity led to the discovery of *uniform convergence* [17] in 1847-1848 by Stokes and Seidel, independently. It was discovered in a paper dated 1841 but not published until 1894 by Weierstrass [37]. As is well-known, uniform convergence preserves continuity as well as uniform continuity. Uniform convergence is too strong. It is sufficient but not necessary for the preservation of continuity. For example, let f_n, f be functions on $\mathbb{R}$ to $\mathbb{R}$ given by

$$f_n(x) = \left(x + \frac{1}{n}\right)^2, f(x) = x^2.$$

Here, the sequence (f_n) of continuous functions converges pointwise to the continuous function f. The convergence is not uniform, since $f_n(n) - f(n) > 2$.

A search began to find necessary and sufficient conditions for the preservation of continuity. Modification of uniform convergence to achieve this objective was studied by Arzela, Dini, Young and others. Much of this interesting classical material is not found in modern texts on analysis and we have *Hobson's Choice* for this material [18]. Recently, there has been some literature on the preservation of continuity in topological spaces (see, *e.g.*,[8,14,5,9]).

There are two results from Dini in the monumental book by Hobson [18, §83, p. 124]. We state Dini's results in the current language. Let $(X,d),(Y,e)$ be metric spaces, $\mathcal{F}$ the set of functions on X to Y, and $C(X,Y)$ the set of continuous functions on X to Y. Let a sequence $(f_n : n \in \mathbb{N})$ in $C(X,Y)$ converge pointwise to $f \in \mathcal{F}$.

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Theorem 1.1. (Dini 1) *A necessary and sufficient condition that f is continuous is that, for each $x \in X$ and each $\varepsilon > 0$, eventually, for $n \in \mathbb{N}$, there is a neighbourhood U_x of x such that, for each $z \in U_x, e(f_n(z), f(z)) < \varepsilon$.*

Theorem 1.2. (Dini 2) *A necessary and sufficient condition that f is continuous is that, for each $x \in X$ and each $\varepsilon > 0$, frequently there exists $n \in \mathbb{N}$ and neighbourhood U_x of x such that, for each $z \in U_x, e(f_n(z), f(z)) < \varepsilon$.*

Bouleau [8] generalized Theorem 1.1 to uniform spaces. In doing so, Bouleau introduced what he called a *sticking* topology [1]. Bouleau defines *sticky convergence* to be proximity on a neighbourhood after possible humps. Let X be a Hausdorff space, $(Y, \mathcal{E})$ a uniform space ($\mathcal{E}$ is a family of pseudometrics), $\mathcal{F}$ the set of functions on X to Y, and $C(X, Y)$ the set of continuous functions. Let $(D, \succ)$ denote a directed set. A net $(f_n : n \in D)$ in $\mathcal{F}$ **sticky** converges to $f \in \mathcal{F}$ if, and only if, for each $x \in X, \varepsilon > 0$ and $e \in \mathcal{E}$, there is an $n_0 \in D$ such that, for each $n \succ n_0$, there is a neighbourhood U_x of x such that, for each $z \in U_x, e(f_n(z), f(z)) < \varepsilon$.

Theorem 1.3. *Let X be a Hausdorff space, $(Y, \mathcal{E})$ a uniform space, $\mathcal{F}$ the set of functions on X to Y and $C(X, Y)$ the set of continuous functions. Then*
(a) *$C(X, Y)$ is closed in $\mathcal{F}$ with sticky topology.*
(b)[8,Prop. 2] *Sticky topology and pointwise topology coincide on $C(X, Y)$.*

In other words, the pointwise limit of a net of continuous functions is continuous if, and only if, the convergence is sticky.

If X is compact, in sticky convergence of $(f_n : n \in D)$ to f, for each $n_0 \in D$, X has a finite open cover $(U_{n_k} : 1 \leq k \leq m)$ with $n_k \succ n_0$ and, for each $z \in X$, there is an n_k such that $z \in U_{n_k}$ and so $e(f_{n_k}(z), f(z)) < \varepsilon$. Thus we have the theorem of Arzelà[2,3], extended by Bartle [4]. This convergence is called **quasi-uniform**. Arzelà's theorem is very valuable. Bartle has given extensive coverage of its applications in functional analysis.

Theorem 1.4. *Let X be a compact Hausdorff space, $(Y, \mathcal{E})$ a uniform space. Then a net $(f_n : n \in D)$ of continuous functions on X to Y converges to a continuous function f if, and only if, (f_n) converges pointwise to f and, for each $\varepsilon > 0$ and $e \in \mathcal{E}$, eventually there exists a finite set $\{n_k : 1 \leq k \leq m\}$ such that, for each $x \in X$, there exists a k with $e(f_{n_k}(x), f(x)) < \varepsilon$.*

Next, we consider Dini's well-known result on monotone nets found in many texts. A net of functions $(f_n : n \in D)$ on a Hausdorff space X to $\mathbb{R}$ is **monotone increasing** if, and only if, for each $n \geq m$ and for all $x \in X$, $f_n(x) \geq f_m(x)$. Monotone decreasing is defined similarly. Suppose $(f_n : n \in D)$ is a monotone increasing net of real-valued continuous functions on a compact Hausdorff space and converges pointwise to a continuous function f. Since f is continuous, the convergence is quasi-uniform by Theorem 1.4. It follows that, for each $\varepsilon > 0$, eventually there exists a finite set $\{f_{n_k} : 1 \leq k \leq m\}$ such that, for each $x \in X$, there exists k with $f(x) < f_{n_k}(x) + \varepsilon$. Hence, eventually $f(x) < f_n(x) + \varepsilon$. This proves Dini's monotone result.

Theorem 1.5. *Let (f_n) be a net of real-valued continuous monotone functions on a compact Hausdorff space converging pointwise to a function f. Then f is continuous if, and only if, the convergence is uniform.*

Alexandroff [1] generalized Arzelà's result (Theorem 1.4) to the case where the domain is any topological space and the range is a metric space.

[1]The sticking topology τ on $C(X, Y)$ is the coarsest topology preserving continuity in the sense that the restriction of τ to $C(X, Y)$ coincides with pointwise convergence [8, §1]. See, also,[10]

Theorem 1.6. *Let (f_n) be a sequence of functions from a topological space X to a metric space (Y, e). Then (f_n) is* Alexandroff convergent *to a function $f : X \to Y$ if, and only if, (f_n) converges pointwise to f and, for every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a countable open cover $\{G_k\}$ of X and a sequence of positive integers $(n_k) : n_k \succ n_0$ such that, for each $x \in G_k, e(f_{n_k}(x), f(x)) < \varepsilon$. If each f_n is continuous, then f is continuous if, and only if, (f_n) Alexandroff converges to f.*

Ewert [14] generalized Arzelà's result (Theorem 1.4) to an arbitrary domain X and a regular range Y. She used open covers of Y. Her results are given in Theorem 1.7.

Theorem 1.7. *Let $(f_n : n \in D)$ be a net of functions on a space X to a regular space Y. Then (f_n) converges strongly to f if, and only if,*
(*a*) *(f_n) converges pointwise to f, and*
(*b*) *for each open cover $\mathcal{G}$ of Y and $n \in D$, there is a finite set $F \subset D$ with each member of F greater than n such that, for each $x \in X$, there exists a $k \in F$ and $G \in \mathcal{G}$ with $\{f_k(x), f(x)\} \subset \mathcal{G}$.*
(*c*) *Further, if each f_n is continuous and converges strongly to f, then f is continuous.*
(*d*) *If Y is a uniform space, then strong convergence implies quasi-uniform convergence. If Y is compact, then the two convergences are equal.*
(*e*) *X is compact if, and only if, strong convergence equals pointwise convergence.*

2 Proximal Convergence S. Leader discovered proximal convergence in 1959 [26,27]. For an overview of recent work on proximity space theory, see, *e.g.*,[32,34]. First, we give a motivation by looking at pointwise convergence from a proximity point of view. Let $f(x)\ \underline{\eta}\ B$ denote that $f(x)$ is *far* (*remote from*) B and let $f(A) \ll_\eta U$ denote $f(x)\ \underline{\eta}\ U^c$, *i.e.*, U is a *proximal neighbourhood* of $f(x)$. Further, let $(X, \delta), (Y, \eta)$ be EF-proximity spaces, $\mathcal{F}$ a set of functions on X to Y and (f_n) a net in $\mathcal{F}$ that converges pointwise to $f \in \mathcal{F}$. This is equivalent to the following proximity result.

$$\text{For each } x \in X, B \subset Y, f(x)\ \underline{\eta}\ B \Rightarrow \text{ eventually } f_n(x)\ \underline{\eta}\ B.$$

We get proximal convergence by replacing x by any subset A of X.

Precisely, (**prxc**). A net $(f_n) \in \mathcal{F}$ **converges proximally** or **Leader converges** to f if, and only if,

$$\text{for each } A \subset X, B \subset Y, f(A)\ \underline{\eta}\ B \Rightarrow \text{ eventually } f_n(A)\ \underline{\eta}\ B.$$

Notice that proximal convergence (prxc) is equivalent to

$$\text{for each } A \subset X, U \subset Y, f(A) \ll_\eta U \Rightarrow \text{ eventually } f_n(A) \ll_\eta U.$$

Leader showed that uniform convergence implies proximal convergence but the converse is not true. Uniform convergence and proximal convergence are equal in the following three cases [27,33,12,13].

(**ucpc**.a) Uniform space Y is totally bounded.
(**ucpc**.b) A net (f_n) in $\mathcal{F}$ is a sequence or the directed set of the net is linearly ordered.
(**ucpc**.c) X is compact.

Proximal convergence preserves continuity as well as proximal continuity. The proofs are analogous. Suppose that in the above convergence, each f_n is proximally continuous. We show that f is proximally continuous. For sets $A, B \subset X$, $f(A)\ \underline{\eta}\ f(B)$ implies that there is an $E \subset Y$ such that $f(A)\ \underline{\eta}\ E$ and $(Y - E)\ \underline{\eta}\ f(B)$. Proximal convergence implies

$$\text{eventually } f_n(A)\ \underline{\eta}\ E \text{ and } (Y - E)\ \underline{\eta}\ f_n(B),$$

which shows that $f_n(A)\ \underline{\eta}\ f_n(B)$. Since f_n is proximally continuous, $A\ \underline{\delta}\ B$, showing thereby that f is proximally continuous.

Theorem 2.1. *Let $(X, \delta), (Y, \eta)$ be EF-proximity spaces, $\mathcal{F}$ a set of functions on X to Y. If a net (f_n) of continuous (proximallycontinuous) functions in $\mathcal{F}$ converges proximally to $f \in \mathcal{F}$, then f is continuous (proximallycontinuous).*

A generalization of (prxc) substitutes *each* $A \subset X$ with *each set* A *from a network* α *of nonempty subsets of* X[13]. The resulting *proximal set open topology* contains, as special cases, all set open topologies. A. Di Concilio [11] has used them in the study of homeomorphism groups.

It is obvious from the proof of Theorem 2.1 that, for the continuity of f, it is sufficient that $f_n(A)\ \underline{\eta}\ f_n(B)$ for some n. However, since we would like f_n to be near f (to get convergence), we follow Dini and add the condition that pointwise convergence to the condition *frequently* $f_n(A)\ \underline{\eta}\ f_n(B)$. Thus, we get an analogue of Dini's *simple uniform convergence* (in which *eventually* is substituted by *pointwise convergence plus frequently*). Its precise formulation is given next. A net (f_n) of functions in $\mathcal{F}$ **simply converges proximally** to $f \in \mathcal{F}$ if, and only if,the convergence is pointwise and frequently for sets $A, B \subset X$, $f(A)\ \underline{\eta}\ f(B)$ implies $f_n(A)\ \underline{\eta}\ f_n(B)$. It is easy to see that simple proximal convergence preserves continuity as well as proximal continuity.

There is an analogue of Arzelà's theorem in the proximal setting. A net (f_n) of functions in $\mathcal{F}$ **proximal Arzelà converges** to $f \in \mathcal{F}$ if, and only if, the convergence is pointwise and frequently, for each $p \in X, B \subset X$, whenever $f(p)\ \underline{\eta}\ f(B)$, there is a finite subset

$$\{n_k : 1 \leq k \leq q\} \subset D \text{ and } B = \bigcup \{B_k : 1 \leq k \leq q\} \text{ such that } f_{n_k}(p)\ \underline{\eta}\ f_{n_k}(B_k).$$

Theorem 2.2. *Let $(f_n : n \in D)$ be a net of continuous functions on a compact space X to an EF-proximity space (Y, η) converge pointwise to f. Then f is continuous if, and only if, the convergence is proximal Arzelà.*

3 Strong Uniform Continuity and Convergence In topology and analysis, the concepts of uniform continuity and uniform convergence on compacta are important and widely used. It is well-known that a continuous function restricted to a compact set is uniformly continuous on that set. In the setting of metric spaces, Beer and Levi[6] observed that a continuous function is actually uniformly continuous on a ε neighbourhood of that compact set. This led to the concept of *strong uniform continuity* on a set that is not necessarily compact. The concepts of uniform continuity and strong uniform continuity agree on the whole space but they may differ on a subset. This phenomenon is interesting because it is rare. Generally, one comes across concepts that differ globally but agree locally.

Uniform convergence preserves uniform continuity but not necessarily strong uniform continuity. In [6,5,9], one finds *strong uniform convergence* that preserves strong uniform continuity. Let $(X, \mathcal{U}), (Y, \mathcal{V})$ denote non-trivial Hausdorff uniform spaces. A function $f : X \to Y$ is **strongly uniformly continuous** on a set $B \subset X$ if, and only if, for $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that, for each $E \subset B$, $f(U(E)) \subset V(f(E))$ (the corresponding $\varepsilon - \delta$ definition is given in [6,Def. 1.1]). This statement is stronger than the usual definition of uniform continuity in which, for all $\{p, q\} \in B$, $(p, q) \in U$ implies $(f(p), f(q)) \in V$.

Uniform convergence does not preserve strong uniform continuity. This observation led to strong uniform convergence. A typical entourage in the **strong uniformity** for $B \subset X$ and $V \in \mathcal{V}$ is

$$\{(f, g) : \text{ there is a } U \in \mathcal{U} \text{ such that } \forall x \in U(B), (f(x), g(x)) \in V\}.$$

Strong uniform convergence preserves strong uniform continuity. For a fascinating account of this topic, with history and results comparing it with other related topics, see [5,6,9,10].

4 Non-Continuous Functions Several interesting cases of non-continuous functions occur naturally in analysis and topology. Here we study a few important cases.

Let X, Y be two topological spaces and let $f : X \to Y$ be a function on X to Y. Then $G(f) = \{(x, f(x)) : x \in X\}$ denotes the graph of f. If $K \subset X$, then the graph of f restricted to K is denoted by $G(f|K)$ and defined to be

$$G(f|K) = \{(x, f(x)) : x \in K\}.$$

A function f is **almost continuous** if, and only if, for each open set W in $X \times Y$ containing $G(f)$ also contains $G(g)$, where g is a continuous function. A space X has the **fixed point property** if, and only if, every continuous self map on X has a fixed point, *i.e.*, there is a point $p \in X$ such that $f(p) = p$.

In a Hausdorff space with the fixed point property, every almost continuous self map also has a fixed point [36]. The definition of almost continuous functions leads to the concept of **graph topology** Γ on the space of all functions on X to Y [31]. A typical basis element of Γ is the set of all functions on X to Y, graphs of which lie in an open set in $X \times Y$. Another motivation suggested by Poppe [35] is that graph topology is a subspace topology on graphs of functions induced by the upper Vietoris topology on the hyperspace $X \times Y$. An interesting fact is that if X is T_1, then the Vietoris topology on graphs equals the graph topology Γ.

Theorem 4.1. *The set of almost continuous functions on X to Y is closed in the space of all functions on X to Y with graph topology* Γ.
A function f is said to be **connected**, if f preserves connected sets [24] (see, also, [29]). A function f is called a **semi-connectivity** if, and only if, for each component $K \subset X, G(f|K)$ is connected [30,§2].

Theorem 4.2. Naimpally [30] *If $X \times Y$ is completely normal, then the set of semi-connectivity functions on $X \times Y$ is closed in the space of all functions on X to Y with graph topology.*

A function f is called a **connectivity** if, and only if, for each connected set $K \subset X, G(f|K)$ is connected. To prove the Brouwer fixed point theorem in the interval $I = [0, 1]$, it is sufficient that the self-map is a connectivity. Nash asked and Stallings proved that every connectivity self map on the n-cube I^n has a fixed point [34]. There is a vast literature on properties of connectivity functions that are stronger than connected (Darboux) functions which preserve connectedness.

Theorem 4.3. Naimpally[30] If Y is a uniform space, then the set of connectivity functions on X to Y is closed in the space of all functions on X to Y with uniform convergence topology.

If X and Y are Banach spaces, every linear function f on X to Y is **nearly continuous**, *i.e.*, for each $x \in X$ and neighbourhood V of $f(x)$, closure $f^{-1}(V)$ is a neighbourhood of x. Nearly continuous functions play an important role in the closed graph and open mappings theorems in functional analysis. H. Blumberg [7] introduced the notion of near continuity on Euclidean spaces using the term *densely approaching* and proved that every function $f : \mathbb{R} \to \mathbb{R}$ is nearly continuous on a dense set of $\mathbb{R}$. The implications of the closed graph theorem have a long history in functional analysis (see, *e.g.*,[38,39,15]).

A complementary concept is quasi-continuous. A function f is **quasi-continuous** if, and only if, for each $x \in X$, an open neighbourhood U of x and a neighbourhood V of

$f(x)$, there is an open set $W \subset U$ such that $f(W) \subset V$[25]. Quasi-continuous functions arise naturally in solving problems involving separate versus joint continuity of functions of two variables. Interestingly, near continuity and quasi-continuity give a decomposition of continuity when Y is regular[16].

Theorem 4.4. Garg and Naimpally [16] *If Y is a uniform space, then the set of nearly continuous (respectively, quasi-continuous) functions on X to Y is closed in the space of all functions on X to Y with uniform convergence topology.*

Open Problems Let X, Y be two topological spaces and let a net (f_n) of connectivity (respectively, nearly continuous, quasi-continuous) functions on X to Y converge pointwise to f. Find necessary and sufficient conditions for f to be connectivity (respectively, nearly continuous, quasi-continuous).

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