Φ-K-SUBGRADIENTS OF VECTOR-VALUED FUNCTIONS

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Dedicated to the memory of Professor Kiyoshi Iséki

Abstract. Let X be a metric spaces. Let Y be a Banach space partially ordered by a pointed closed convex cone K. Let Φ be a separable linear family (a class) of Lipschitz functions defined on X and with values in Y. Let Ω be an open subset of X. We say that a multifunction Γ mapping Ω into Y is *monotone* if for all $\phi_x \in \Gamma(x)$,$\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \geq_K 0.$$

In the paper it is proved that under certain conditions on Φ each monotone multifunction is single-valued and continuous on a dense G_δ-set.

1 Φ-K-subgradients and Φ-K-supergradients of vector valued functions Let a set X, called later the space, be given. Let $f(x)$ and $\phi(x)$ be two functions defined on X with values in a linear space Y ordered by a convex pointed cone $K \subset Y$. We assume that the intersections of K with finite dimensional spaces are closed. The cone K induced an order $\geq_K$ in the space Y in the following way

$$y \geq_K y_0 \text{ if } y \in y_0 + K, \tag{1.1}$$

$$y >_K y_0 \text{ if } y \in y_0 + \text{Int}_r K, \tag{1.1'}$$

where $\text{Int}_r K$ denote the relative interior of the cone K (see Jahn [2], [3]).

The function $\phi(x)$ will be called a *K-subgradient* (*K-supergradient*) of the function $f(x)$ at a point x_0 if

$$f(x) - f(x_0) \geq_K \phi(x) - \phi(x_0) \tag{1.2}$$

$$(\text{resp., } \; f(x) - f(x_0) \leq_K \phi(x) - \phi(x_0)\;) \tag{1.2'}$$

for all $x \in X$.

Of course, a function $\phi(x)$ is a K-subgradient of a function $f(x)$ at a point x_0 if and only if the function $f(x)$ is a K-supergradient of the function $\phi(x)$ at a point x_0.

Proposition 1.1. *Let $\phi(x)$ and $\psi(x)$ be two K-subgradients (K-supergradients) of a function $f(x)$ at a point x_0. Then for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, the function $\alpha\phi + \beta\psi$ is a K-subgradient (resp., K-supergradient) of the function $f(x)$ at a point x_0.*

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Proof. By the definition of subgradient

$$f(x) - f(x_0) \geq_K \phi(x) - \phi(x_0), \tag{1.2$_\phi$}$$

$$f(x) - f(x_0) \geq_K \psi(x) - \psi(x_0). \tag{1.2$_\psi$}$$

Multiplying $(1.2)_\phi$ by α and $(1.2)_\psi$ by β and adding them by the convexity of K we obtain that

$$f(x) - f(x_0) \geq_K [\alpha\phi(x) + \beta\psi(x)] - [\alpha\phi(x_0) + \beta\psi(x_0)], \tag{1.3}$$

which shows that $\alpha\phi(x) + \beta\psi(x)$ is a K-subgradient of a function $f(x)$ at a point x_0.

The proof for K-supergradients is similar.

Let Φ be a family (a class) of functions defined on X and with values in Y. Usually the class Φ consists of simpler functions that the function f. A function $\phi(x)$ will be called a *Φ-K-subgradient* (*Φ-K-supergradient*) of the function $f : X \to Y$ at a point x_0 if $\phi \in \Phi$ and $\phi(x)$ is a Φ-K-subgradient (resp., Φ-K-supergradient) of the function $f(x)$ at the point x_0.

As an immediate consequence of the definition is the following condition: if there are two classes Φ and Ψ such that $\Phi \subset \Psi$, then each Φ-K-subgradient (Φ-K-supergradient) of the function $f(x)$ at a point x_0 is automatically the Ψ-K-subgradient (Ψ-K-supergradient) of the function $f(x)$ at a point x_0.

The set of all Φ-K-subgradients (resp., Φ-K-supergradients) of the function f at a point x_0 we shall call *Φ-K-subdifferential* (resp., *Φ-K-superdifferential*) of the function f at a point x_0 and we shall denote it by $\partial_\Phi f\big|_{x_0}$ (resp., $\partial^\Phi f\big|_{x_0}$).

Proposition 1.2. *If the class Φ is additive (i.e., $\phi, \psi \in \Phi$ implies $\phi + \psi \in \Phi$), then*

$$\partial_\Phi(f + g)\big|_{x_0} \supset \partial_\Phi f\big|_{x_0} + \partial_\Phi g\big|_{x_0} \tag{1.4}$$

$$(\textit{resp.,}\quad \partial^\Phi(f + g)\big|_{x_0} \supset \partial^\Phi f\big|_{x_0} + \partial^\Phi g\big|_{x_0}\ ). \tag{1.4$'$}$$

If the class Φ is positive homogeneous (i.e., $\phi \in \Phi$ implies $t\phi \in \Phi$ for all positive t), then for $t > 0$

$$\partial_\Phi tf\big|_{x_0} = t\partial_\Phi f\big|_{x_0} \tag{1.5}$$

$$(\textit{resp.,}\quad \partial^\Phi tf\big|_{x_0} = t\partial^\Phi f\big|_{x_0}\ ). \tag{1.5$'$}$$

Proof. If $\phi \in \partial_\Phi f\big|_{x_0}$ and $\psi \in \partial_\Phi g\big|_{x_0}$, then by the definition we have

$$f(x) - f(x_0) \geq_K \phi(x) - \phi(x_0)$$

and

$$g(x) - g(x_0) \geq_K \psi(x) - \psi(x_0).$$

Adding both inequalities, we obtain

$$[f(x) + g(x)] - [f(x_0) + g(x_0)] \geq_K [\phi(x) + \psi(x)] - [\phi(x_0) + \psi(x_0)],$$

i.e., $\phi + \psi \in \partial_\Phi(f + g)\big|_{x_0}$ and (1.4) holds.

The proofs of the remaining formulae are similar.

Corollary 1.3. *If the class* Φ *is additive and positive homogeneous, then*

$$\partial_\Phi(af+bg)\big|_{x_0} \supset a\partial_\Phi f\big|_{x_0} + b\partial_\Phi g\big|_{x_0} \tag{1.4}$$

$$(\textit{resp.},\quad \partial^\Phi(af+bg)\big|_{x_0} \supset a\partial^\Phi f\big|_{x_0} + b\partial^\Phi g\big|_{x_0}\ ). \tag{1.4'}$$

We shall write $g \leq_K f$ without the argument if $g(x) \leq_K f(x)$ for all $x \in X$.

Let X, Y be two spaces. Let $\Gamma : X \to 2^Y$ be a multifunction, i.e., the mapping of the set X into subsets of Y. By the *domain* of Γ, $\mathrm{dom}(\Gamma)$, we shall call the set of those x, that $\Gamma(x) \neq \emptyset$,

$$\mathrm{dom}(\Gamma) = \{x \in X : \Gamma(x) \neq \emptyset\}.$$

By the *graph* of Γ, $G(\Gamma)$, we shall call the set $G(\Gamma) = \{(x,y) \in X \times Y : y \in \Gamma(x)\}$.

Let as before Y be a linear space. We assume that in Y the order is given by a convex cone K, such that the intersection of K by finite dimensional spaces are closed. We say that a multifunction Γ mapping X into Φ is *monotone* if for $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \geq_K 0. \tag{1.6}$$

In particular, when X is a linear space, and Φ is a linear space consisting of linear operators $\phi(x) = \langle \phi, x\rangle$, we can then rewrite (1.6) in the classical form

$$\langle \phi_x - \phi_y, x - y\rangle \geq_K 0. \tag*{$(1.6)_\ell$}$$

A multifunction Γ mapping X into Φ is called *n-cyclic monotone* if for arbitrary $x_0, x_1, ...,$ $x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $(i = 0, 1, 2, ..., n)$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \geq_K 0. \tag*{$(1.6)_n$}$$

A multifunction Γ mapping X into Φ is called *cyclic monotone* if it is n-cyclic monotone for $n = 2, 3, ...$. Of course, just from the definition a multifunction Γ is monotone if and only if it is 2-cyclic monotone.

Proposition 1.9. *For a given function* f *the subdifferential* $\partial^\Phi f\big|_x$*, considered as a multifunction of* x*, is cyclic monotone.*

Proof. Take arbitrary $x_0, x_1, ..., x_n = x_0 \in X$ and $\phi_{x_i} \in \partial^\Phi f\big|_{x_i}$, $i = 0, 1, 2, ..., n$. Since $\phi_{x_i} \in \partial^\Phi f\big|_{x_i}$ we have that for $i = 1, 2, \ldots, n$

$$f(x_i) - f(x_{i-1}) \geq_K \phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}). \tag*{$(1.2)^i$}$$

Adding all equations $(1.2)^i$ for $i = 1, 2, ..., n$ and changing the sign we obtain

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_i}(x_{i-1})] \geq_K 0. \tag*{$(1.6)_n$}$$

2 Monotone vector-valued multi-functions defined on metric spaces Let (X, d_X) be a metric space. Let $(Y, \|\cdot\|)$ be a Banach space. Till the end of the paper we assume that Y is an ordered Banach space by a closed pointed convex cone K such that the norm is *increasing* [1], i.e. for $a, b \in K$ $b \geq_K a$ we have

$$\|b\| \geq \|a\|. \tag{2.1}$$

Let $\mathcal{L}$ be the space of all Lipschitzian functions defined on X with values in Y. We define on $\mathcal{L}$ a quasi-norm

$$\|\phi\|_L = \sup_{\substack{x_1,x_2 \in X,\\ x_1 \neq x_2}} \frac{\|\phi(x_1) - \phi(x_2)\|}{d_X(x_1, x_2)}. \tag{2.2}$$

Observe that, if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function, i.e., there is $c \in Y$ such that $\phi_1(x) = \phi_2(x) + c$. Thus we consider the quotient space $\tilde{\mathcal{L}} = {}^{\mathcal{L}}\!/_{\mathbb{R}}$. The quasinorm $\|\phi\|_L$ induces the norm in the space $\tilde{\mathcal{L}}$. Since this will not lead to any misunderstanding, this norm we shall also denote by $\|\phi\|_L$.

Let Φ be a linear family of Lipschitz functions. If there is an element k, $\|k\| < 1$, belonging to the relative interior of K, $k \in \mathrm{Int}_r K$, such that for all $x \in X$ and all $\phi \in \Phi$ and all $t > 0$, there is a $y \in X$ such that $0 < d_X(x, y) < t$ and

$$\phi(y) - \phi(x) \geq_K \|\phi\|_L d_X(y, x) k, \tag{2.3}$$

we say that the family Φ has the *monotonicity property* with respect to the element k (briefly the family Φ has the *k-monotonicity property*). It is easy to see, that if $a \in K$, $0 \leq_K a \leq_K k$, then each family Φ having the k-monotonicity property also have a-monotonicity property.

Write for any $\phi \in \Phi$, $a \in Int_r K$, $x \in X$ (cf. Preiss and Zajiček [8], Rolewicz [9], Pallaschke and Rolewicz [6]).

$$K(\phi, a, x) = \{y \in X : \phi(y) - \phi(x) \geq_K \|\phi\|_L d_X(y, x) a\}. \tag{2.4}$$

The set $K(\phi, a, x)$ will be called an *a-cone with vertex at x and direction ϕ*. Of course, it may happen that $K(\phi, a, x) = \{x\}$. Since the norm is increasing it is so if $\|a\| > 1$. However, if $\|a\| < 1$ and the cone K has nonempty interior and $k \in a + \mathrm{Int}K$, it is obvious that the set $K(\phi, a, x)$ has a nonempty interior and, even more,

$$x \in \overline{\mathrm{Int}K(\phi, a, x)}. \tag{2.5}$$

Observe that just from the definition it follows that if $a_1 <_K a_2$, then $K(\phi, a_1, x) \supset K(\phi, a_2, x)$. A set $M \subset X$ is said to be *a-cone meagre* if for every $x \in M$ and arbitrary $\varepsilon > 0$ there are $z \in X, d_X(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \mathrm{Int}\ K(\phi, a, z) = \emptyset. \tag{2.6}$$

The arbitrariness of ε and (2.5) implies that an a-cone meagre set M is nowhere dense.

A set $M \subset X$ is called *a-angle-small* if it can be represented as a union of a countable number of a-cone meagre sets M_n,

[1] this notion is strictly connected with the notion of normal cone (compare Peressini [7] p.64, Jahn [3], p.28)

$$M = \bigcup_{n=1}^{\infty} M_n. \tag{2.7}$$

Of course, every angle-small set M is of the first category. Adapting the method of Preiss and Zajiček [8] to metric spaces we obtain

Theorem 2.1 (compare Rolewicz [9] , Pallaschke and Rolewicz[6]). *Let (X, d_X) be a metric space. Let $(Y, \|\cdot\|)$ be an Banach space, ordered by a closed pointed convex cone K, such that the norm be increasing. Let Φ be a linear family of Lipschitz functions mapping X into Y having the monotonicity property with respect to the element $k \in K$, $\|k\| < 1$. Assume that Φ is separable in the metric d_L. Let a multifunction Γ mapping X into 2^Φ be monotone and such that* dom $\Gamma = X$ *(i.e., $\Gamma(x) \neq \emptyset$ for all $x \in X$). Then for each a, $0 <_K a <_K k$, there is an a-angle-small set A such that Γ is single-valued and continuous on the set $X \setminus A$.*

Proof. It is sufficient to show that the set

$$A = \{x \in X : \lim_{\delta \to 0} \operatorname{diam} \Gamma(B(x, \delta)) > 0\}, \tag{2.8}$$

where by diam is denoted the diameter of the set measured in the Lipschitz metric d_L, is angle-small. Of course we can represent A as a union of sets

$$A_n = \{x \in X : \lim_{\delta \to 0} \operatorname{diam} \Gamma(B(x, \delta)) > \frac{1}{n}\}. \tag{2.9}$$

Let $\{\phi_m\}$ be a dense sequence in the space Φ in the metric d_L. Suppose that $0 <_K a <_K k$ and $\|a\| < 1$. Let

$$A_{n,m} = \{x \in A_n : \operatorname{dist}(\phi_m, \Gamma(x)) < \frac{\|a\|}{4n}\}, \tag{2.10}$$

where, as usual we denote $\operatorname{dist}(\phi_m, \Gamma(x)) = \inf\{\|\phi_m - \phi\|_L : \phi \in \Gamma(x)\}$. By the density of the sequence $\{\phi_m\}$ in Φ,

$$\bigcup_{m=1}^{\infty} A_{n,m} = A_n.$$

We will show that the sets $A_{n,m}$ are a-cone meagre. Suppose that $x \in A_{n,m}$. Let ε be an arbitrary positive number. Since $x \in A_n$, the are $0 < \delta < \varepsilon$ and $z_1, z_2 \in X$, $\phi_1 \in \Gamma(z_1)$, $\phi_2 \in \Gamma(z_2)$ such that $d_X(z_1, x) < \delta$, $d_X(z_2, x) < \delta$ and

$$\|\phi_1 - \phi_2\|_L > \frac{1}{n}. \tag{2.11}$$

Thus by the triangle inequality, for every $\phi \in \Gamma(x)$ either $\|\phi_1 - \phi\| > \frac{1}{2n}$ or $\|\phi_2 - \phi\| > \frac{1}{2n}$. By the definition of $A_{n,m}$, we can find $\phi_x \in \Gamma(x)$ such that $\|\phi_x - \phi_m\| < \frac{\|a\|}{4n}$. Therefore choosing as z either z_1 or z_2, we can say that there are a point $z \in X$ and $\phi_z \in \Gamma(z)$ such that $d_X(z, x) < \delta$ and

$$\|\phi_z - \phi_m\|_L \geq \|\phi_z - \phi_x\|_L - \|\phi_x - \phi_m\|_L > \frac{1}{2n} - \frac{\|a\|}{4n} > \frac{\|a\|}{4n}. \tag{2.12}$$

We shall show that

$$\begin{aligned} &A_{n,m} \cap K(\phi_z - \phi_m, a, z) \\ &= \{y \in A_{n,m} : \phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) \geq_K \|\phi_z - \phi_m\|_L d_X(y,z)a\} = \emptyset. \end{aligned} \tag{2.13}$$

Indeed, suppose that $y \in K(\phi_z - \phi_m, a, z)$. This means that

$$\phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) \geq_K \|\phi_z - \phi_m\|_L d_X(y,z)a.$$

Suppose that $\phi_y \in \Gamma(y)$. Then by the monotonicity of Γ,

$$\phi_y(y) - \phi_y(z) \geq_K \phi_z(y) - \phi_z(z)$$

and

$$\begin{aligned} \phi_y(y) + \phi_m(z) - \phi_m(y) - \phi_y(z) &\geq_K \phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) \\ &\geq_K \|\phi_z - \phi_m\|_L d_X(y,z)a \geq \tfrac{a}{4n} d_X(y,z). \end{aligned}$$

Using the fact that the norm is increasing we get

$$\|\phi_y(y) + \phi_m(z) - \phi_m(y) - \phi_y(z)\| \geq \frac{\|a\|}{4n} d_X(y,z).$$

This implies that

$$\|\phi_y - \phi_m\|_L \geq \frac{\|a\|}{4n}$$

and by the definition of $A_{n,m}$, $y \notin A_{n,m}$.

We recall that a set B of the second category is called *residual* if its complement is of the first category. Since the small-angle sets are always of the first category we immediately obtain the following extension of Kenderov [4] result on metric spaces and vector valued functions

Theorem 2.2. *Let (X, d_X) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y, \|\cdot\|)$ be a Banach space. We assume that $(Y, \|\cdot\|)$ is an ordered Banach space and that the order is given by a closed convex cone K such that the norm is increasing. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has the monotonicity property with respect to an element $k \in \mathrm{Int}_r K$, $\|k\| < 1$. Assume that Φ is separable in the metric d_L. Let Γ be a monotone multifunction mapping X into 2^Φ such that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a residual set B such that the multifunction Γ is single-valued and continuous on B.*

Corollary 2.3. *Let (X, d_X) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K and the norm is increasing. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to element $k \in Int_r K, \|k\| < 1$. Assume that Φ is separable in the metric d_L. Let $f(x)$ be a function having at each point a Φ-subgradient. Then there is a residual set B such that on B the subdifferential $\partial_\Phi f\big|_x$ is single-valued and it is continuous in the metric d_L.*

We shall say that a function $f(x)$ mapping a metric space (X, d_X) into a normed space $(Y, \|\cdot\|Y)$ is *Fréchet Φ-differentiable* at a point x_0 if there are a function $\gamma(t)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim \frac{\gamma(t)}{t} = 0$$

and a function $\phi_{x_0} \in \Phi$ such that

$$\|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]\|_Y \leq \gamma(d_X(x, x_0)).$$

The function ϕ will be called a *Fréchet Φ-gradient* of the function $f(x)$ at the point x_0. The function $\gamma(t)$ will be called the *modulus of smoothness*.

In the case of normed spaces the continuity of Gateaux differentials in the norm operator topology implies that these differentials are the Fréchet differential. Similarly, for metric spaces we obtain a following generalization of Asplund Theorem (Asplund [1]) (see also Mazur [5]).

Proposition 2.4 (compare Rolewicz [10], [11]). *Let (X, d_X) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K and the norm is increasing. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to an element $k \in \mathrm{Int}_r K$, $\|k\| < 1$. Assume that Φ is separable in the metric d_L. Let ϕ_{x_0} be a Φ-subgradient of the function $f(x)$ at a point x_0. Suppose that there is a neighbourhood U of x_0 such that for all $x \in U$ the subdifferential $\partial f\big|_x$ is not empty and it is lower semi-continuous at x_0 in the Lipschitz norm, i.e., for every $\varepsilon > 0$ there is a neighbourhood $V_\varepsilon \subset U$ such that for $x \in V_\varepsilon$ there is $\phi_x \in \partial_\Phi f\big|_X$ such that*

$$\|\phi_x - \phi_{x_0}\|_L \leq \varepsilon d_X(x, x_0). \tag{2.14}$$

Then ϕ_{x_0} is the Fréchet Φ-gradient of the function $f(x)$ at the point x_0.

Proof. Let

$$F(x) = [f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)].$$

It is easy to see that $F(x_0) = 0$. Since ϕ_{x_0} is a Φ-subgradient of the function $f(x)$ at a point x_0, then $F(x) \geq_K 0$. Let ε be an arbitrary positive number and let V_ε be a neighbourhood of x_0 such that for $x \in V_\varepsilon$ (2.14) holds. Since ϕ_x is a Φ-subgradient of the function $f(x)$ at a point x, $\psi_x = \phi_x - \phi_{x_0}$ is a Φ-subgradient of the function $F(x)$ at the point x. Thus

$$F(y) - F(x) \geq_K \psi_x(y) - \psi_x(x).$$

In particular, if $y = x_0$, then

$$F(x_0) - F(x) \geq_K \psi_x(x_0) - \psi_x(x). \tag{2.15}$$

Taking into account (2.14), we obtain that for $x \in V_\varepsilon$

$$0 \leq F(x) \leq \psi_x(x) - \psi_x(x_0) \leq_K \phi_x(x) - \phi_x(x_0). \tag{2.16}$$

Since the norm is increasing $0 \leq_K a \leq_K b$ implies that

$$\|a\| \leq \|b\|. \tag{2.17}$$

Thus from (2.14),(2.16) and (2.17) we obtain that

$$\|[f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)]\| \leq \varepsilon d_X(x, x_0). \tag{2.18}$$

Thus the arbitrarity of ε implies that ϕ_{x_0} is the Fréchet gradient of the function $f(x)$ at a point x_0.

If we assume that the function $f(x)$ is continuous, then we do not need to assume that there is a neighbourhood U of x_0 such that for all $x \in U$, the subdifferential $\partial f\big|_x$ is not empty. It is sufficient to assume that the subdifferential $\partial f\big|_x$ is not empty on a dense set.

Proposition 2.5 (compare Rolewicz [10],[11]). *Let (X, d_X) be a metric space. Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K and that the norm is increasing. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to element $k \in \mathrm{Int}_r K$. Assume that Φ is separable in the metric d_L. Let ϕ_{x_0} be a Φ-subgradient of the function $f(x)$ at a point x_0. Suppose that there is a dense set A in a neighbourhood U of x_0 such that for all $x \in A$ the Φ-subdifferential $\partial f_\Phi\big|_x$ is not empty and lower semi-continuous at x_0 in the Lipschitz norm. Then ϕ_{x_0} is the Fréchet Φ-gradient of the function $f(x)$ at the point x_0.*

Proof. The proof is going in the same line as proof of Proposition 2.4. We obtain that for $x \in A \cap V_\varepsilon$

$$\|[f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)]\| \leq \varepsilon d_X(x, x_0). \tag{2.18}$$

Thus, by the continuity of $f(x)$ and the density of A, we obtain that (2.18) holds for all $x \in U$. The remained part of the proof is the same.

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