# ON NERI'S MEAN FIELD EQUATION WITH HYPERBOLIC SINE VORTICITY DISTRIBUTION

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Received February 8, 2012; revised December 5, 2012

Abstract. Motivated by equations derived by Neri, Pointin-Lundgren and Joyce-Montgomery in the context of Onsager's statistical theory of vortices, we study the blow-up properties of some sinh-Gordon type extensions of the standard mean field equation with exponential nonlinearity on two-dimensional compact surfaces.

**1 Introduction and main results** Our study is motivated by the equation:

$$(1.1)\qquad \begin{cases} -\Delta_g v = \lambda\left[\dfrac{te^v-(1-t)e^{-v}}{\int_\Omega[te^v+(1-t)e^{-v}]\,dv_g} - \dfrac{1}{|\Omega|}\dfrac{\int_\Omega[te^v-(1-t)e^{-v}]\,dv_g}{\int_\Omega[te^v+(1-t)e^{-v}]\,dv_g}\right] & \text{in } \Omega,\\ \displaystyle\int_\Omega v\,dv_g = 0, \end{cases}$$

where $(\Omega, g)$ is a compact orientable two-dimensional Riemannian manifold without boundary, $dv_g$ denotes the volume element on $\Omega$, $|\Omega|$ denotes the volume of $\Omega$, $t \in [0,1]$ and $\lambda > 0$ is a constant. Equation (1.1) is a special case of the mean field equation derived by Neri [10] in the context of the statistical mechanics description of two-dimensional turbulence, as initiated by Onsager [15] and further developed by Joyce and Montgomery [7], Pointin and Lundgren [16]. This special case captures the main features of the interaction between the positive part and the negative part of the exponential nonlinearity.

More precisely, Neri's equation is given by

$$(1.2)\qquad \begin{cases} -\Delta_g v = \lambda\,\dfrac{\int_I \alpha(e^{\alpha v} - \frac{1}{|\Omega|}\int_\Omega e^{\alpha v}dv_g)\,\mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha v}\,\mathcal{P}(d\alpha)dv_g} & \text{in } \Omega\\ \displaystyle\int_\Omega v = 0, \end{cases}$$

where $v$ corresponds to the stream function and $\mathcal{P} = \mathcal{P}(d\alpha)$, $\alpha \in I = [-1,1]$, is the probability distribution of the relative circulations of the vortices, which are assumed to be independent identically distributed random variables. Equation (1.2) reduces to (1.1) when $\mathcal{P}$ is of the "hyperbolic sine type", namely

$$(1.3)\qquad \mathcal{P}(d\alpha) = t\delta_1(d\alpha) + (1-t)\delta_{-1}(d\alpha).$$

The mathematical analysis for equation (1.2) is quite recent. An existence result for solutions to equation (1.2) under Dirichlet boundary conditions was obtained by Neri himself in [10]. Results on the blow-up properties of solution sequences to (1.2), as well as the

2010 *Mathematics Subject Classification.* Primary: 76B03, Secondary: 35B44, 76B47.
*Key words and phrases.* Mean field equation, sinh-Gordon equation, blow-up analysis.

corresponding optimal Moser-Trudinger inequality are contained in our previous work [18], where the following more general problem is considered:

$$(1.4)\qquad \begin{cases} -\Delta_g v = \lambda \displaystyle\int_I V(\alpha,x,v)e^{\alpha v}\,\mathcal{P}(d\alpha) - \frac{\lambda}{|\Omega|}\int_{I\times\Omega} V(\alpha,x,v)e^{\alpha v}\,\mathcal{P}(d\alpha)dv_g & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

On the other hand, a mean field equation similar to (1.2) may be derived by the method of [7, 16], assuming that $\mathcal{P}$ is the probability measure which determines the distribution of the relative circulations, see [4, 19]. Under such assumptions, the stream function $v$ satisfies:

$$(1.5)\qquad \begin{cases} -\Delta v = \lambda \displaystyle\int_I \alpha \left( \frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}\,dv_g} - \frac{1}{|\Omega|}\right)\mathcal{P}(d\alpha) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

A blow-up analysis for (1.5) was obtained in [11] together with a related Moser-Trudinger type inequality, which is seen to be dual to the logarithmic Hardy-Littlewood-Sobolev inequality of Shafrir-Wolansky [21] when $\mathcal{P}$ is atomic. The optimal constant in the above mentioned Moser-Trudinger inequality was recently obtained in [17]. We note that the general equation (1.4) contains equation (1.2) and equation (1.5) as special cases.

The above mentioned results were motivated by the mathematical analysis carried out by Ohtsuka and Suzuki in [13, 14] for equation (1.5) in the special hyperbolic sine assumption (1.3). In such a case, (1.5) takes the form:

$$(1.6)\qquad \begin{cases} -\Delta_g v = \lambda t \left( \dfrac{e^{v}}{\int_\Omega e^{v}\,dv_g} - \dfrac{1}{|\Omega|}\right) - \lambda(1-t)\left( \dfrac{e^{-v}}{\int_\Omega e^{-v}\,dv_g} - \dfrac{1}{|\Omega|}\right) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

In particular, the articles [13, 14] contain some refined blow-up results for (1.6) whose extension to the general equation (1.4) appears to be difficult. Therefore, as a first step it is natural to try to extend such refined blow-up results to the special "sinh case" of equation (1.2), namely to equation (1.1). Indeed, this is the main objective in this note.

Before stating our main results, we note that under the assumption $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, the Dirac measure concentrated at $\alpha = 1$, both equation (1.2) and equation (1.5) reduce to the well known mean field equation

$$(1.7)\qquad \begin{cases} -\Delta_g v = \lambda \left( \dfrac{e^{v}}{\int_\Omega e^{v}\,dv_g} - \dfrac{1}{|\Omega|}\right) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0, \end{cases}$$

which has been extensively studied in recent years in connection with the Nirenberg problem, chemotaxis, Chern-Simons vortex theory as well as statistical hydrodynamics. See, e.g., [8, 20] and the references therein. Therefore, equation (1.1) and equation (1.6) may be viewed as physically relevant sinh-Gordon type extensions of the standard mean field equation (1.7).

Towards our objectives, we shall take the point of view of studying the slightly more general problem

$$
\begin{cases} -\Delta_g v = \lambda\left[t\dfrac{e^v}{\mathcal{I}_1(v)} - (1-t)\dfrac{e^{-v}}{\mathcal{I}_2(v)}\right] - \lambda\kappa(v) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0, \end{cases} \tag{1.8}
$$

where $\mathcal{I}_1$, $\mathcal{I}_2$, are real functionals defined on $\mathcal{E}$ and where the constant $\kappa(v)$ is defined by $\kappa(v) = \kappa_1(v) - \kappa_2(v)$,

$$
\kappa_1(v) = \frac{t}{|\Omega|}\int_\Omega \frac{e^v}{\mathcal{I}_1(v)}\,dv_g, \qquad \kappa_2(v) = \frac{1-t}{|\Omega|}\int_\Omega \frac{e^{-v}}{\mathcal{I}_2(v)}\,dv_g \tag{1.9}
$$

so that the integral of the r.h.s. in (1.8) is zero. We denote

$$
\mathcal{E} = \left\{v \in H^1(\Omega) : \int_\Omega v\,dv_g = 0\right\}.
$$

We make the following assumption on $\mathcal{I}_1, \mathcal{I}_2$:

($\mathcal{I}$) There exists $C_1 > 0$ such that $t\int_\Omega e^v\,dv_g \le C_1\mathcal{I}_1(v)$ and $(1-t)\int_\Omega e^{-v}\,dv_g \le C_1\mathcal{I}_2(v)$ for all $v \in \mathcal{E}$.

It follows from assumption ($\mathcal{I}$) that

$$
|\kappa(v)| \leqslant \frac{C_1}{|\Omega|} \tag{1.10}
$$

for all $v \in \mathcal{E}$. For later convenience, we also observe that in view of Jensen's inequality we have $\mathcal{I}_1(v) \ge tC_1^{-1}|\Omega|$ and $\mathcal{I}_2(v) \ge (1-t)C_1^{-1}|\Omega|$ for all $v \in \mathcal{E}$. In particular, in the "strictly hyperbolic" case $t \in (0,1)$ there exists $c_0 > 0$ such that

$$
\mathcal{I}_1(v) \ge c_0, \qquad \mathcal{I}_2(v) \ge c_0 \tag{1.11}
$$

for all $v \in \mathcal{E}$. Clearly, (1.8) reduces to Neri's "sinh" case (1.1) when

$$
\mathcal{I}_1(v) = \mathcal{I}_2(v) = \int_\Omega [te^v + (1-t)e^{-v}]\,dv_g
$$

and it reduces to (1.6) when

$$
\mathcal{I}_1(v) = \int_\Omega e^v\,dv_g, \qquad \mathcal{I}_2(v) = \int_\Omega e^{-v}\,dv_g.
$$

Thus, by extending the results by Ohtsuka and Suzuki [13, 14] to equation (1.8), we conclude that equation (1.1) and equation (1.6) share analogous blow-up properties.

In order to state our main results, we denote by $G = G(x,y)$ the Green's function associated to $-\Delta_g$ on $\Omega$. Namely, $G$ is defined by

$$
\begin{cases} -\Delta_g G(\cdot,y) = \delta_y - \dfrac{1}{|\Omega|} & \text{in } \Omega \\ \displaystyle\int_\Omega G(\cdot,y)\,dv_g(y) = 0. \end{cases} \tag{1.12}
$$

We shall be mainly concerned with blow-up sequences to (1.8). More precisely, we consider solution sequences $v_n$ to equation (1.8) with $\lambda = \lambda_n$ and $\lambda_n \to \lambda_0$. We define the measures $\mu_{1,n}, \mu_{2,n} \in \mathcal{M}(\Omega)$ by

$$\mu_{1,n} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}, \qquad \mu_{2,n} = \lambda_n (1-t) \frac{e^{-v_n}}{\mathcal{I}_2(v_n)}. \tag{1.13}$$

By assumption $(\mathcal{I})$ we may assume that $\mu_{i,n} \stackrel{*}{\rightharpoonup} \mu_i \in \mathcal{M}(\Omega)$ weakly in the sense of measures, $i = 1, 2$. We set

$$u_{1,n} = G * \mu_{1,n}, \qquad u_{2,n} = G * \mu_{2,n}. \tag{1.14}$$

Then, $v_n = u_{1,n} - u_{2,n}$. We note that $(u_{1,n}, u_{2,n})$ satisfies a Liouville-type system as extensively analyzed in [2, 3, 21]:

$$\begin{cases} -\Delta u_{1,n} = \lambda \left( t \dfrac{e^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)} - \kappa_1(v_n) \right) & \text{in } \Omega \\ -\Delta u_{2,n} = \lambda \left( (1-t) \dfrac{e^{u_{2,n}-u_{1,n}}}{\mathcal{I}_2(v_n)} - \kappa_2(v_n) \right) & \text{in } \Omega \\ \displaystyle\int_\Omega u_{1,n}\, dv_g = 0 = \int_\Omega u_{2,n}\, dv_g. \end{cases}$$

We define as usual the blow-up sets:

$$\mathcal{S}_\pm = \{p \in \Omega : \exists\, p_{\pm,n} \to p \text{ s.t. } v_n(p_{\pm,n}) \to \pm\infty)\}$$

and we denote $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. By adapting to our case some results for the general equation (1.4) as contained in Theorem 2.1, Theorem 2.2 and Theorem 4.1 in [18], we derive the following results for (1.8), which extend the alternatives as discovered in [1, 9].

**Theorem 1.1** ([18], Brezis-Merle type alternative)**.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. Then, the following alternative holds:*

1) *Compactness:* $\limsup_{n\to\infty} \|v_n\| < +\infty$. *We have $\mathcal{S}_+ \cup \mathcal{S}_- = \emptyset$ and there exist a solution $v \in \mathcal{E}$ to* (1.8) *with $\lambda = \lambda_0$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$.*

2) *Concentration:* $\limsup_{n\to\infty} \|v_n\| = +\infty$. *We have $\mathcal{S} \neq \emptyset$ and*

$$\mu_1 = \sum_{p \in \mathcal{S}_+} m_+(p)\delta_p + r_1\, dv_g$$
$$\mu_2 = \sum_{p \in \mathcal{S}_-} m_-(p)\delta_p + r_2\, dv_g$$

*where $\delta_p$ denotes the Dirac delta centered at $p \in \mathcal{S}$, the constants $m_\pm(p)$ satisfy the "minimum mass property"*

$$m_\pm(p) \geqslant 4\pi, \tag{1.15}$$

*and $r_i \in L^1(\Omega) \cap L^\infty_{loc}(\Omega \setminus \mathcal{S})$. Moreover the following facts hold:*

*2-i) If there exists $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$ then $m_\pm(p_0) = 8\pi$ and $m_\mp(p_0) = 0$.*

*2-ii) For every $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, we have the quadratic identity*

$$8\pi[m_+(p_0) + m_-(p_0)] = [m_+(p_0) - m_-(p_0)]^2 \tag{1.16}$$

*and*

$$m_+(p_0) + m_-(p_0) \geqslant 4(3 + \sqrt{5})\pi. \tag{1.17}$$

Our first aim is to improve the minimum mass (1.15). We obtain the following.

**Theorem 1.2.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. In the conclusion of Theorem 1.1, Alternative 2), the lower bound* (1.15) *is improved as follows:*

$$m_\pm(p) \geqslant 8\pi \tag{1.18}$$

*for any $p \in \mathcal{S}_\pm$. Consequently,* (1.17) *is also improved:*

$$m_+(p_0) + m_-(p_0) \geqslant 24\pi.$$

Clearly, when $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, Theorem 1.2 follows by Theorem 1.1, 2-*i*). Therefore, in the proof of Theorem 1.2 we need only consider the two-sided blow-up case $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$. This result is obtained by performing the rescaling argument as in [13] using a Brezis-Merle type lemma and the classification of solutions to Liouville type equations in $\mathbb{R}^2$ obtained by Chen and Lin in [5, 6].

Our second result concerns residual vanishing in the one-sided blow-up case.

**Theorem 1.3** (One-sided concentration, residual vanishing)**.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. Suppose $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$. Then,*

$$r_1 \equiv 0, \qquad \mu_1 = 8\pi \sum_{p \in \mathcal{S}_+} \delta_p,$$

*and $\mu_{1,n} \to 0$ in $L^\infty(\omega)$ for every $\omega \Subset \Omega \setminus \mathcal{S}_+$. Consequently, $u_{1,n} \to u_1$ in $H^1_{loc}(\Omega \setminus \mathcal{S}_+)$, where $u_1$ is given by*

$$u_1 = 8\pi \sum_{p \in \mathcal{S}_+} G(\cdot, p).$$

*Moreover, there exists $u_2 \in \mathcal{E}$ and a subsequence $\{u_{2,n}\}$ such that $u_{2,n} \to u_2$ in $\mathcal{E}$ and*

$$\begin{cases} -\Delta u_2 = \beta\lambda_0 \left[K(x)e^{u_2} - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega K(x)e^{u_2}\right] & \text{in } \Omega, \\ \displaystyle\int_\Omega u_2 = 0, \end{cases} \tag{1.19}$$

*where $\beta \in [0, \frac{1}{|\Omega|}]$ and $K(x) = e^{-\sum_{p \in \mathcal{S}_+} 8\pi G(x,p)}$. Analogous results hold in the converse case $p_0 \in \mathcal{S}_- \setminus \mathcal{S}_+$.*

Our third result concerns the location of the blow-up points in the one-sided blow-up case. In order to state it, we define a suitable local chart centered at the blow-up point $p_0$. More precisely, given $p_0 \in \mathcal{S}$ we denote by $(\Psi, \mathcal{U})$ an iso-thermal chart satisfying $\bar{\mathcal{U}} \cap \mathcal{S} = \{p_0\}$, $\Psi(\mathcal{U}) = \mathcal{O} \subset \mathbb{R}^2$ and

$$\Psi(p_0) = 0, \qquad g(X) = e^{\xi(X)}(dX_1^2 + dX_2^2), \qquad \xi(0) = 0, \tag{1.20}$$

where $X = (X_1, X_2)$ are Euclidean coordinates on $\mathcal{O}$. Then, the Laplace-Beltrami operator $\Delta_g$ is mapped to the operator $e^{-\xi(X)}\Delta_X$ on $\mathcal{O}$, where $\Delta_X = \partial^2_{X_1^2} + \partial^2_{X_2^2}$. We denote by $H_\Psi(x, y)$ the regular part of the Green's function $G(x, y)$ relative to the chart $(\mathcal{U}, \Psi)$, i.e.,

$$H_\Psi(x, y) = G(x, y) + \frac{1}{2\pi}\log(|\Psi(x) - \Psi(y)|) \tag{1.21}$$

for $x, y \in \mathcal{U}$. By $G_\mathcal{O}(X, Y)$ we denote the Green's function of $-\Delta_X$ on $\mathcal{O}$ with Dirichlet boundary conditions and by $H_\mathcal{O}(X, Y)$ we indicate its regular part. Namely,

$$\begin{cases} -\Delta_X G_\mathcal{O}(X, Y) = \delta_Y & \text{in } \mathcal{O} \\ G_\mathcal{O}(X, Y) = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

and

$$H_\mathcal{O}(X, Y) = G_\mathcal{O}(X, Y) + \frac{1}{2\pi}\log(|X - Y|). \tag{1.22}$$

With this notation, we have the following.

**Theorem 1.4.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. Suppose that there exists $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$. Then, the following relation holds in the iso-thermal chart* (1.20) *centered at $p_0$:*

(1.23)
$$\nabla_X \Big(8\pi H_\Psi(\Psi^{-1}(X), p_0) + 8\pi \sum_{p \in \mathcal{S}_+ \setminus \{p_0\}} G(\Psi^{-1}(X), p) - u_2(\Psi^{-1}(X)) + \xi(X)\Big)\Big|_{X=0} = 0,$$

*where $u_2$ is given by* (1.19). *Analogous results hold in the converse case $p_0 \in \mathcal{S}_- \setminus \mathcal{S}_+$.*

We note that (1.23) coincides with the relation obtained in [14], Theorem 1.2, thus confirming that the hyperbolic sine mean field equations (1.1) and (1.6) exhibit analogous blow-up properties.

**Notation** For the sake of simplicity, in the local coordinate patch $\mathcal{O} \subset \mathbb{R}^2$ defined in (1.20) we denote $\nabla = \nabla_X$ and $\Delta = \Delta_X$.

**2 Proof of Theorem 1.2** Since Theorem 1.2 is already known in the one-sided blow-up case $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, without loss of generality we assume $t \in (0, 1)$. In particular, we assume that the lower bound (1.11) holds. Similarly to [13], the main observation towards obtaining our improved blow-up results is the local reduction of equation (1.8) to the following Liouville system:

$$\begin{cases} -\Delta w_1 = V_1 e^{w_1} - V_2 e^{w_2} & \text{in } \mathcal{O} \\ -\Delta w_2 = -V_1 e^{w_1} + V_2 e^{w_2} & \text{in } \mathcal{O}, \end{cases} \tag{2.1}$$

where $0 \leqslant V_i \leqslant C$ and $\int_\mathcal{O} e^{w_i} \leqslant C$, with $C$ a constant independent of $w_i$, $i = 1, 2$, to which the blow-up analysis developed in [13] may be applied, see Lemma 2.1 below. Indeed, we take a coordinate patch $(\Psi, \mathcal{U})$ as defined in (1.20). In particular, identifying $v(X) = v(\Psi^{-1}(X))$ for any function $v$ defined on $\Omega$, we have that a solution $v$ to equation (1.8) satisfies

$$-\Delta v = \lambda\left[\frac{te^v}{\mathcal{I}_1(v)} - \frac{(1-t)e^{-v}}{\mathcal{I}_2(v)}\right]e^\xi - \lambda\kappa(v)e^\xi \qquad \text{in } \mathcal{O}.$$

We define $h_\xi$ by

$$\begin{cases} -\Delta h_\xi = e^\xi & \text{in } \mathcal{O}, \\ h_\xi = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{2.2}$$

Correspondingly, we define $w_i : \mathcal{O} \to \mathbb{R}$ by setting

$$\begin{aligned} w_1 &= v - \log \mathcal{I}_1(v) + \lambda\kappa(v)h_\xi, \\ w_2 &= -v - \log \mathcal{I}_2(v) - \lambda\kappa(v)h_\xi. \end{aligned}$$

Then,

$$\frac{e^v}{\mathcal{I}_1(v)} = e^{w_1 - \lambda\kappa(v)h_\xi}, \qquad \frac{e^{-v}}{\mathcal{I}_2(v)} = e^{w_2 + \lambda\kappa(v)h_\xi}.$$

It follows that $w_1$ satisfies the equation

$$\begin{aligned} -\Delta w_1 =& -\Delta v + \kappa(v)(-\Delta)h_\xi = \lambda \left[ \frac{te^v}{\mathcal{I}_1(v)} - \frac{(1-t)e^{-v}}{\mathcal{I}_2(v)} \right] e^\xi \\ =& \lambda t\, e^{\xi - \lambda\kappa(v)h_\xi} e^{w_1} - \lambda(1-t)\, e^{\xi + \lambda\kappa(v)h_\xi} e^{w_2} \end{aligned}$$

in $\mathcal{O}$. Setting

$$V_1 = \lambda t\, e^{\xi - \lambda\kappa(v)h_\xi}, \qquad V_2 = \lambda(1-t)\, e^{\xi + \lambda\kappa(v)h_\xi} \tag{2.3}$$

we conclude that $w_1$ satisfies the first equation in (2.1). Similarly, $w_2$ satisfies the second equation in (2.1) with $V_i$ defined by (2.3), $i = 1, 2$.

Now, we consider a solution sequence $v_n$ to (1.8) with $\lambda = \lambda_n \to \lambda_0$. Similarly, we set

$$\begin{aligned} w_{1,n} &= v_n - \log \mathcal{I}_1(v_n) + \lambda_n\kappa(v_n)h_\xi, \\ w_{2,n} &= -v_n - \log \mathcal{I}_2(v_n) - \lambda_n\kappa(v_n)h_\xi, \end{aligned}$$

where $h_\xi$ is the function defined in (2.2) and, as before, we identify $v_n(X) = v_n(\Psi^{-1}(X))$. For later use, we note that in view of (1.11) we have

$$w_{1,n} + w_{2,n} = -\log \mathcal{I}_1(v_n) - \log \mathcal{I}_2(v_n) \leqslant -2\log c_0 \tag{2.4}$$

for some $c_0 > 0$. In view of the arguments above, we conclude that $(w_{1,n}, w_{2,n})$ satisfies the Liouville system

$$\begin{cases} -\Delta w_{1,n} = V_{1,n}e^{w_{1,n}} - V_{2,n}e^{w_{2,n}} & \text{in } \mathcal{O} \\ -\Delta w_{2,n} = -V_{1,n}e^{w_{1,n}} + V_{2,n}e^{w_{2,n}} & \text{in } \mathcal{O}, \end{cases} \tag{2.5}$$

where

$$V_{1,n} = \lambda_n t\, e^{\xi - \lambda_n\kappa(v_n)h_\xi}, \qquad V_{2,n} = \lambda_n(1-t)\, e^{\xi + \lambda_n\kappa(v_n)h_\xi} \tag{2.6}$$

and

$$0 \leqslant V_{i,n} \leqslant C, \qquad \int_{\mathcal{O}} e^{w_{i,n}} \leqslant C \tag{2.7}$$

$i = 1, 2$, for some $C > 0$ independent of $n$. In view of estimate (1.10), we may assume that $\kappa(v_n) \to \kappa_0$ and consequently

$$V_{1,n} \to V_1 = \lambda_0 t\, e^{\xi - \lambda_0 \kappa_0 h_\xi},$$
$$V_{2,n} \to V_2 = \lambda_0 (1-t)\, e^{\xi + \lambda_0 \kappa_0 h_\xi},$$

uniformly on $\overline{\mathcal{O}}$. We define

$$\mathcal{S}_i^0 = \{X \in \mathcal{O} : \exists X_n \to X \text{ s.t. } w_{i,n}(X_n) \to +\infty\}.$$

We recall the following Brezis-Merle type Lemma for (2.5) from [13], Lemma 2.1.

**Lemma 2.1** ([13])**.** *Suppose $\{w_{1,n}, w_{2,n}\}_n$ is a solution sequence to the Liouville system* (2.5)*, satisfying* (2.7)*. Then, up to subsequences, exactly one of the following alternatives holds true.*

1. *Both $\{w_{1,n}\}_n$ and $\{w_{2,n}\}_n$ are locally uniformly bounded in $\mathcal{O}$.*
2. *There is $i \in \{1, 2\}$ such that $\{w_{i,n}\}_n$ is uniformly bounded in $\mathcal{O}$ and $\{w_{j,n}\}_n \to -\infty$ locally uniformly in $\mathcal{O}$ for $j \neq i$.*
3. *Both $w_{1,n} \to -\infty$ and $w_{2,n} \to -\infty$ locally uniformly in $\mathcal{O}$.*
4. *For the blow-up sets $\mathcal{S}_1^0$, $\mathcal{S}_2^0$ defined for this subsequence, we have $\mathcal{S}_1^0 \cup \mathcal{S}_2^0 \neq \emptyset$ and $\sharp(\mathcal{S}_1^0 \cup \mathcal{S}_2^0) < +\infty$. Furthermore, for each $i \in \{1, 2\}$, either $\{w_{i,n}\}_n$ is locally uniformly bounded in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$ or $w_{i,n} \to -\infty$ locally uniformly in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$. Here, if $\mathcal{S}_i^0 \setminus (S_1^0 \cap \mathcal{S}_2^0) \neq \emptyset$ then $w_{i,n} \to -\infty$ locally uniformly in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$, and for each $x_0 \in \mathcal{S}_i^0$ there exists $m_i(x_0) \geqslant 4\pi$ such that*

$$V_{i,n}(x) e^{w_{i,n}} \rightharpoonup \sum_{x_0 \in \mathcal{S}_i^0} m_i(x_0) \delta_{x_0} \qquad * \text{-weakly in } \mathcal{M}(\mathcal{O}).$$

We will start by giving the proof of the following preliminary lemma, which relies on arguments from [14].

**Lemma 2.2.** *Let $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$. There exists a sequence $x_{1,n} \to p_0$ and a sequence $x_{2,n} \to p_0$ such that:*

*i)* $v_n(x_{1,n}) \to +\infty, \qquad v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) \to +\infty,$

*ii)* $-v_n(x_{2,n}) \to +\infty, \qquad -v_n(x_{2,n}) - \log \mathcal{I}_2(v_n) \to +\infty.$

*Proof.* We prove only relation *i*). The proof of *ii*) is similar. Recall that

$$\mu_{1,n} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}, \tag{2.8}$$

and that by Theorem 1.1 we have

$$\mu_{1,n} \overset{*}{\rightharpoonup} \mu_1 = \sum_{p \in \mathcal{S}_+} m_+(p) \delta_p + r_1\, dv_g. \tag{2.9}$$

Since $\mu_1$ is singular in $p_0$, $\mu_{1,n}$ is $L^\infty$-unbounded around $p_0 \in \mathcal{S}_+$. Hence, we can assume

$$\lim_{n \to \infty} \sup_{B(p_0, r_0)} (v_n - \log \mathcal{I}_1(v_n)) = +\infty, \qquad \forall r_0 > 0.$$

Hence, we find $r_0 > 0$ such that $\overline{B(p_0, r_0)} \cap \mathcal{S}_+ = \{p_0\}$ and a sequence of points $x_{1,n} \in \overline{B(p_0, r_0)}$, such that

$$v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) = \max\left\{v_n(x) - \log \mathcal{I}_1(v_n) : x \in \overline{B(p_0, r_0)}\right\} \to +\infty.$$

Moreover, in view of (1.11), we have

$$v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) \leqslant v_n(x_{1,n}) - \log c_0,$$

and therefore we also have $v_n(x_{1,n}) \to +\infty$. It remains to prove that $x_{1,n} \to p_0$. Suppose the contrary. Then up to subsequence, we may assume $x_{1,n} \to \bar{p} \neq p_0$, $\bar{p} \in B(p_0, r_0)$ and hence $\bar{p}$ is not in $\mathcal{S}_+$. This means $\limsup_{n\to\infty} v_n(x_{1,n}) < +\infty$, a contradiction. □

In view of Lemma 2.2 there exist $\{x_{1,n}\}$ and $\{x_{2,n}\}$ such that $x_{1,n} \to p_0$, $v_n(x_{1,n}) \to +\infty$, $x_{2,n} \to p_0$ and $-v_n(x_{2,n}) \to +\infty$, and furthermore

$$\begin{aligned} X_{1,n} = \Psi(x_{1,n}) \to 0 \quad &\text{and} \quad w_{1,n}(X_{1,n}) \to +\infty \\ X_{2,n} = \Psi(x_{2,n}) \to 0 \quad &\text{and} \quad w_{2,n}(X_{2,n}) \to +\infty. \end{aligned}$$

In particular, $0 \in \mathcal{S}_i^0$, $i = 1, 2$, and

$$\mathcal{S}_1^0 = \Psi(\mathcal{U} \cap \mathcal{S}_+) = \{0\} = \Psi(\mathcal{U} \cap \mathcal{S}_-) = \mathcal{S}_2^0. \tag{2.10}$$

On the other hand, since

$$V_{1,n} e^{w_{1,n}} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}\, e^{\xi}$$

and

$$V_{2,n} e^{w_{2,n}} = \lambda_n (1-t) \frac{e^{-v_n}}{\mathcal{I}_2(v_n)}\, e^{\xi},$$

recalling that $\xi(0) = 0$, from (2.9) we derive that

$$\begin{aligned} V_{1,n} e^{w_{1,n}} &\overset{*}{\rightharpoonup} m_+(p_0)\delta_0(dX) + s_1(X)\,dX, \\ V_{2,n} e^{w_{2,n}} &\overset{*}{\rightharpoonup} m_-(p_0)\delta_0(dX) + s_2(X)\,dX, \end{aligned} \tag{2.11}$$

$*$-weakly in $\mathcal{M}(\bar{\mathcal{O}})$, where $s_i(X) = r_i(\Psi^{-1}(X))e^{\xi(X)}$, $s_i \in L^1(\mathcal{O}) \cap L^\infty_{loc}(\bar{\mathcal{O}} \setminus \{0\})$, $i = 1, 2$, and $\min(m_+(p_0), m_-(p_0)) \geqslant 4\pi$. In view of (2.10) there exist $Y_{1,n}, Y_{2,n} \in \mathcal{O}$, $Y_{1,n}, Y_{2,n} \to 0$ such that

$$\begin{aligned} w_{1,n}(Y_{1,n}) &= \sup_{\mathcal{O}} w_{1,n} \to +\infty \\ w_{2,n}(Y_{2,n}) &= \sup_{\mathcal{O}} w_{2,n} \to +\infty. \end{aligned}$$

*Proof of Theorem 1.2.* We define the rescaling parameters

$$\begin{aligned} \varepsilon_{1,n} &= e^{-w_{1,n}(Y_{1,n})/2} \\ \varepsilon_{2,n} &= e^{-w_{2,n}(Y_{2,n})/2}. \end{aligned}$$

Correspondingly, we rescale the Liouville system (2.5) two times.

Namely, we first rescale (2.5) around $Y_{1,n}$ with respect to $\varepsilon_{1,n}$. We define the expanding domain

$$\mathcal{O}_n^1 = \{X \in \mathbb{R}^2 : \; Y_{1,n} + \varepsilon_{1,n} X \in \mathcal{O}\}$$

and we define $\widetilde{w}^1_{1,n}, \widetilde{w}^1_{2,n} : \mathcal{O}^1_n \to \mathbb{R}$ by setting

$$\begin{aligned}\widetilde{w}^1_{1,n}(X) =& w_{1,n}(Y_{1,n} + \varepsilon_{1,n}X) - w_{1,n}(Y_{1,n})\\ \widetilde{w}^1_{2,n}(X) =& w_{2,n}(Y_{1,n} + \varepsilon_{1,n}X) - w_{1,n}(Y_{1,n}).\end{aligned}$$

We note that $\widetilde{w}^1_{1,n}$ is a standard rescaling. Then, $\widetilde{w}^1_{1,n}, \widetilde{w}^1_{2,n}$ is a solution for the Liouville system

$$\begin{cases} -\Delta\widetilde{w}^1_{1,n} =& \widetilde{V}^1_{1,n} e^{\widetilde{w}^1_{1,n}} - \widetilde{V}^1_{2,n} e^{\widetilde{w}^1_{2,n}} \\ -\Delta\widetilde{w}^1_{2,n} =& -\widetilde{V}^1_{1,n} e^{\widetilde{w}^1_{1,n}} + \widetilde{V}^1_{2,n} e^{\widetilde{w}^1_{2,n}} \end{cases} \tag{2.12}$$

in $\mathcal{O}^1_n$, where $\widetilde{V}^1_{1,n}(X) = V_{1,n}(Y_{1,n} + \varepsilon_{1,n}X)$ and $\widetilde{V}^1_{2,n}(X) = V_{2,n}(Y_{1,n} + \varepsilon_{1,n}X)$. We note that (2.4) implies that

$$\widetilde{w}^1_{1,n} + \widetilde{w}^1_{2,n} = -\log \mathcal{I}_1(v_n) - \log \mathcal{I}_2(v_n) - 2w_{1,n}(Y_{1,n}) \to -\infty. \tag{2.13}$$

Similarly, we rescale (2.5) around $Y_{2,n}$ with respect to $\varepsilon_{2,n}$. We define the expanding domain

$$\mathcal{O}^2_n = \{X \in \mathbb{R}^2 :\ Y_{2,n} + \varepsilon_{2,n}X \in \mathcal{O}\}$$

and we define $\widetilde{w}^2_{1,n}, \widetilde{w}^2_{2,n} : \mathcal{O}^2_n \to \mathbb{R}$ by setting

$$\begin{aligned}\widetilde{w}^2_{1,n}(X) =& w_{1,n}(Y_{2,n} + \varepsilon_{2,n}X) - w_{2,n}(Y_{2,n})\\ \widetilde{w}^2_{2,n}(X) =& w_{2,n}(Y_{2,n} + \varepsilon_{2,n}X) - w_{2,n}(Y_{2,n}).\end{aligned}$$

We note that $\widetilde{w}^2_{2,n}$ is a standard rescaling. Then, $\widetilde{w}^2_{1,n}, \widetilde{w}^2_{2,n}$ is a solution for the Liouville system

$$\begin{cases} -\Delta\widetilde{w}^2_{1,n} =& \widetilde{V}^2_{1,n} e^{\widetilde{w}^2_{1,n}} - \widetilde{V}^2_{2,n} e^{\widetilde{w}^2_{2,n}} \\ -\Delta\widetilde{w}^2_{2,n} =& -\widetilde{V}^2_{1,n} e^{\widetilde{w}^2_{1,n}} + \widetilde{V}^2_{2,n} e^{\widetilde{w}^2_{2,n}} \end{cases} \tag{2.14}$$

in $\mathcal{O}^2_n$, where $\widetilde{V}^2_{1,n}(X) = V_{1,n}(Y_{2,n} + \varepsilon_{2,n}X)$ and $\widetilde{V}^2_{2,n}(X) = V_{2,n}(Y_{2,n} + \varepsilon_{2,n}X)$. Furthermore, as above,

$$\widetilde{w}^2_{1,n} + \widetilde{w}^2_{2,n} = -\log \mathcal{I}_1(v_n) - \log \mathcal{I}_2(v_n) - 2w_{2,n}(Y_{2,n}) \to -\infty. \tag{2.15}$$

We observe that

$$0 \leqslant \widetilde{V}^k_{i,n}(X) \leqslant C, \qquad \int_{\mathcal{O}^k_n} e^{\widetilde{w}^k_{i,n}}\, dX \leqslant C$$

for $i, k = 1, 2$, for some $C > 0$ independent of $X$ and $n$. Therefore, the Brezis-Merle alternative Lemma 2.1 may be applied locally to the Liouville systems (2.12)–(2.14).

At this point, the remaining part of the proof is completely analogous to [13], and therefore we just outline it briefly. In view of the identities (2.13)–(2.15), we rule out Alternative 1. On the other hand, since $\widetilde{w}^1_{1,n}(0) = 0 = \widetilde{w}^2_{2,n}(0)$, we rule out Alternative 3. Let $\widetilde{w}_k : \mathbb{R}^2 \to \mathbb{R}$ be such that $\widetilde{w}^k_{k,n} \to \widetilde{w}_k$ uniformly on compact subsets of $\mathbb{R}^2$. In view of (2.15), the rescaled functions $\widetilde{w}^2_{1,n}$ and $\widetilde{w}^2_{2,n}$ cannot converge locally uniformly to a locally bounded function, and therefore the residual term in the rescaled equations is necessarily either zero, or a finite sum of negative Dirac masses. It follows that the limit equations for $\widetilde{w}^k_{k,n}$ are one of the following.

Either the standard Liouville equation

$$-\Delta\tilde{w}_k = V_k(0)e^{\tilde{w}_k} \qquad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{\tilde{w}_k} < +\infty,$$

$k = 1, 2$, in which case, in view of [5], we derive

$$m_k \geqslant \int_{\mathbb{R}^2} V_k(0)e^{\tilde{w}_k} = 8\pi,$$

where for convenience we denote $m_1 = m_+(p_0)$, $m_2 = m_-(p_0)$.

Or, the singular Liouville equation

$$-\Delta\tilde{w}_k = V_k(0)e^{\tilde{w}_k} - \sum_{X_0\in\mathcal{S}} \alpha(X_0)\delta_{X_0} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{\tilde{w}_k} < +\infty, \tag{2.16}$$

where $\mathcal{S} \subset \mathbb{R}^2$ is a finite set and $\alpha(X_0) \geqslant 4\pi$ for any $X_0 \in \mathcal{S}$, in which case we derive

$$m_k \geqslant \int_{\mathbb{R}^2} V_k(0)e^{\tilde{w}_k} > 4\pi + \sum_{x_0\in\mathcal{S}} \alpha(x_0) > 8\pi$$

in view of [6]. In particular, we conclude that $m_k \geqslant 8\pi$ and (1.18) is established. □

**3 Proof of Theorem 1.3** We derive the proof of Theorem 1.3 by adapting arguments from [12], Theorem 2.1–(III). See also [18], Theorem 4.1. Under the assumptions of Theorem 1.3, since $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$, we necessarily have $t > 0$. therefore, assumption $(\mathcal{I})$ implies that

$$\mathcal{I}_1(v) \geq c_1 \int_\Omega e^v \tag{3.1}$$

for some $c_1 > 0$, independent of $v \in \mathcal{E}$. In order to prove Theorem 1.3 it clearly suffices to show the following.

**Lemma 3.1.** *Under the assumptions of Theorem 1.3, we have*

$$\lim_{n\to\infty} \mathcal{I}_1(v_n) = +\infty.$$

*Proof.* We denote $G^T(\cdot, p_0) = \min\{T, G(\cdot, p_0)\}$. We estimate:

$$u_{1,n} = G * \lambda_n t \,\frac{e^{v_n}}{\mathcal{I}_1(v_n)} \geq G^T * \lambda_n t \,\frac{e^{v_n}}{\mathcal{I}_1(v_n)} \to G^T * \sum_{p\in\mathcal{S}_+} (n_{+,p}\delta_p + r_1).$$

In particular, recalling that in view of Theorem 1.1, case 2–i) we have $n_{+,p_0} = 8\pi$, we have

$$u_{1,n}(x) \geq 8\pi G^T(x, p_0) - C.$$

In a local coordinate patch $(\psi, \mathcal{U})$, identifying $u_{1,n}(X) = u_{1,n}(\psi^{-1}(X))$, we derive that

$$e^{u_{1,n}(X)} \geq c \left[\frac{1}{|X|^4}\right]^T,$$

where for a general function $f$ defined on $\Psi(\mathcal{U})$ we denote $f^T = \min\{T, f\}$. In particular, for any $T > 0$ we have

$$\liminf_{n\to\infty} e^{u_{1,n}(X)} \geq c \left[\frac{1}{|X|^4}\right]^T.$$

By Fatou's lemma, we derive

$$\liminf_{n\to\infty}\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX \geq c\int_{\Psi(\mathcal{U})}\left[\frac{1}{|X|^4}\right]^T\,dX.$$

Since $T$ is arbitrary, we conclude that $\lim_{n\to\infty}\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX = +\infty$. Finally, since $u_{2,n}$ is bounded, using (3.1) we conclude that

$$\lim_{n\to\infty}\mathcal{I}_1(v_n) \geq c_1\int_\Omega e^v dv_g \geq c_2\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX = +\infty.$$

□

**4 Proof of Theorem 1.4** We recall a particular case of Lemma 4.1 of [12] that will be useful in the sequel. Here $\mathcal{O}$ denotes a bounded domain in $\mathbb{R}^2$ with smooth boundary, and $0 \in \mathcal{O}$ is assumed without loss of generality.

**Lemma 4.1** ([12]). *Let $f_n$ be a sequence in $W^{1,\infty}(\mathcal{O})$ satisfying*

$$\nabla f_n \to \mathbf{F} \qquad \text{in } L^\infty(\mathcal{O})^2$$

*for some $\mathbf{F} \in C(\mathcal{O})^2$. Moreover, suppose that $\{v_n\} \subset W_0^{1,2}(\mathcal{O})$ is a solution sequence to*

$$\begin{cases} -\Delta v_n = e^{v_n+f_n} & \text{in } \mathcal{O} \\ v_n = 0 & \text{on } \partial\mathcal{O} \end{cases} \tag{4.1}$$

*and that*

$$e^{v_n+f_n} \to 8\pi\delta_0 \qquad * - \text{ weakly in } \mathcal{M}(\bar{\mathcal{O}}).$$

*Then,*

$$\mathbf{F}(0) = -8\pi\nabla_X H_{\mathcal{O}}(X,0)\big|_{X=0}.$$

Let $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$ and let $(\mathcal{U},\Psi)$ satisfy (1.20) around $p_0$. We may assume that $\partial\mathcal{O}$ is smooth. We identify $u_{1,n}(X) = u_{1,n}(\Psi^{-1}(X))$, where $u_{1,n}$ is defined in (1.14). Then,

$$-\Delta u_{1,n} = \lambda_n\left(\frac{te^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)} - \kappa_1(v_n)\right)e^{\xi(X)} \qquad \text{in } \mathcal{O}. \tag{4.2}$$

Setting

$$\tilde{\mu}_{1,n} = \lambda_n t\frac{e^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)}e^{\xi} \tag{4.3}$$

we have $\tilde{\mu}_{1,n} \stackrel{*}{\rightharpoonup} 8\pi\delta_0$ in $\mathcal{M}(\bar{\mathcal{O}})$. Moreover, by the assumption on $\mathcal{U}$ we also have

$$\limsup_{n\to\infty}\|u_{1,n}\|_{W^{2-\frac{1}{r},r}(\partial\mathcal{O})} < +\infty$$

for any $r \in (2,+\infty)$. Hence, the function $\tilde{h}_n(X)$ defined by

$$\begin{cases} \Delta\tilde{h}_n = 0 & \text{in } \mathcal{O} \\ \tilde{h}_n = u_{1,n} & \text{on } \partial\mathcal{O}, \end{cases} \tag{4.4}$$

satisfies

$$\limsup_{n\to\infty} \|\tilde{h}_n\|_{W^{2,r}(\mathcal{O})} < +\infty.$$

It follows that we may assume

$$\tilde{h}_n \to \tilde{h}_\infty \qquad \text{in } C^1(\bar{\mathcal{O}}).$$

Now, we set

$$\tilde{u}_{1,n} = u_{1,n} + \lambda_n \kappa_1(v_n) h_\xi - \tilde{h}_n$$

where $h_\xi$ is defined by (2.2). Then, $\tilde{u}_{1,n}$ satisfies

$$\begin{cases} -\Delta \tilde{u}_{1,n} = \lambda_n t \frac{e^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)} e^{\xi} = \lambda_n t \frac{\exp\{-\lambda_n \kappa_1(v_n) h_\xi + \tilde{h}_n - u_{2,n} + \xi\}}{\mathcal{I}_1(v_n)} e^{\tilde{u}_{1,n}} & \text{in } \mathcal{O} \\ \tilde{u}_{1,n} = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{4.5}$$

That is, $\tilde{u}_{1,n}$ satisfies (4.1) with $f = f_n$ given by

$$f_n = -\lambda_n \kappa_1(v_n) h_\xi + \tilde{h}_n + \log \lambda_n - u_{2,n} - \log(\mathcal{I}_1(v_n)) + \xi.$$

Now we are ready to complete the proof of Theorem 1.4.

*Proof of Theorem 1.4.* By taking subsequences, we may assume that $\kappa_1(v_n) \to \kappa_1$. Then, we also have $f_n \in W^{1,\infty}(\mathcal{O})$,

$$\nabla f_n \to \nabla(-\lambda_0 \kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi) \qquad \text{in } L^\infty(\mathcal{O})^2$$

and

$$\nabla(-\lambda_0 \kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi) \qquad \text{belongs to } C(\mathcal{O})^2.$$

Hence, applying Lemma 4.1 to the sequence $\{\tilde{u}_{1,n}\}$ we conclude that

$$\nabla(-\lambda_0 \kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi)\Big|_{X=0} = -8\pi \nabla_X H_{\mathcal{O}}(X, 0)|_{X=0}\,. \tag{4.6}$$

We claim that

$$-\lambda_0 \kappa_1 h_\xi + \tilde{h}_\infty = 8\pi \sum_{p \in \mathcal{S}_+} G(\Psi^{-1}(\cdot), p) - 8\pi G_{\mathcal{O}}(\cdot, 0). \tag{4.7}$$

To see this, we note that (1.13) implies

$$\mu_{1,n}(\Omega) \to 8\pi\, \sharp \mathcal{S}_+.$$

On the other hand, by definition of $\kappa_1$ we also have

$$\mu_{1,n}(\Omega) \to \lambda_0 \kappa_1 |\Omega|.$$

Hence, we conclude that

$$\lambda_0 \kappa_1 |\Omega| = 8\pi\, \sharp \mathcal{S}_+. \tag{4.8}$$

It follows that $w = -\lambda_0 \kappa_1 h_\xi + \tilde{h}_\infty$ satisfies the Dirichlet problem

$$\begin{cases} -\Delta w = -\dfrac{8\pi\, \sharp \mathcal{S}_+}{|\Omega|} e^{\xi} & \text{in } \mathcal{O} \\ w = 8\pi \displaystyle\sum_{p \in \mathcal{S}_+} G(\Psi^{-1}(\cdot), p) & \text{on } \partial\mathcal{O}. \end{cases} \tag{4.9}$$

By uniqueness, we conclude that $w = 8\pi \sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot), p) - 8\pi G_{\mathcal{O}}(\cdot, 0)$ so that (4.7) is established. Finally, we note that by (1.21) with $y = p_0$ we obtain

$$H_\Psi(\Psi^{-1}(X), p_0) = G(\Psi^{-1}(X), p_0) + \frac{1}{2\pi}\log|X|.$$

Therefore, using (1.22), we have

$$G(\Psi^{-1}(X), p_0) - G_{\mathcal{O}}(X, 0) = H_\Psi(\Psi^{-1}(X), p_0) - H_{\mathcal{O}}(X, 0)$$

so that

$$\sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot), p) - G_{\mathcal{O}}(\cdot, 0) = \sum_{p\in\mathcal{S}_+\setminus\{p_0\}} G(\Psi^{-1}(\cdot), p) + H_\Psi(\Psi^{-1}(\cdot), p_0) - H_{\mathcal{O}}(\cdot, 0). \tag{4.10}$$

Combining (4.6), (4.7) and (4.10), we derive the asserted necessary condition (1.23). □

**Acknowledgments** T.R. was partially supported by FP7-MC-IRSES-2009-247486 *Mathematical studies on critical non-equilibrium phenomena via mean field theories and theories of nonlinear partial differential equations* and by PRIN 2009 *Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations.* G.Z. was partially supported by PRIN 2008 *Equazioni ellittiche e omeomorfismi di Sobolev.*

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