HOPF HYPERSURFACES WITH $\eta$-PARALLEL RICCI TENSORS IN A NONFLAT COMPLEX SPACE FORM

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Abstract. We give a classification theorem of Hopf hypersurfaces $M^{2n-1}$ with $\eta$-parallel Ricci tensors in a nonflat complex space form $\tilde{M}_n(c), n \geq 2$. There exist non-homogeneous Hopf hypersurfaces $M^3$ with $\eta$-parallel Ricci tensors in $\tilde{M}_2(c), c \neq 0$. Note that these real hypersurfaces do not have $\eta$-parallel shape operators in this ambient space.

1 Introduction We denote by $\tilde{M}_n(c)$ a complex $n$-dimensional nonflat complex space form of constant holomorphic sectional curvature $c \neq 0$. That is, $\tilde{M}_n(c)$ is holomorphically congruent to either a complex projective space of constant holomorphic sectional curvature $c > 0$ or a complex hyperbolic space of constant holomorphic sectional curvature $c < 0$.

The study of real hypersurfaces isometrically immersed into $\tilde{M}_n(c)$ is one of the most interesting objects in differential geometry. There are many nice results in this field (cf [6]). For $n \geq 3$ some results can be proved affirmatively, but for $n = 2$ it is difficult to get those same results or there exist counter examples to them.

The classification theorem of Hopf hypersurfaces $M$ (namely real hypersurfaces $M$ such that the characteristic vector $\xi$ of $M$ is principal at its each point) with $\eta$-parallel Ricci tensors in $\tilde{M}_n(c)$ is one of such results. We here review the definition of the $\eta$-parallelism for a tensor field $T$ of type $(1, 1)$ on a real hypersurface $M$ in $\tilde{M}_n(c)$. $T$ is called $\eta$-parallel if $g((\nabla_X T)Y, Z) = 0$ for all vectors $X, Y$ and $Z$ which are orthogonal to $\xi$ on $M$.

The purpose of this paper is to prove the following two theorems.

Theorem 1. Let $M$ be a connected Hopf hypersurface in $\mathbb{C}P^n(c), n \geq 2$. Suppose that $M$ has $\eta$-parallel Ricci tensor. Then $M$ is either locally congruent to one of homogeneous real hypersurfaces of types $(A_1)$, $(A_2)$ and $(B)$ in $\mathbb{C}P^n(c), n \geq 2$, or a non-homogeneous real hypersurface with $\mathcal{A}\xi = 0$ in $\mathbb{C}P^2(c)$. This non-homogeneous real hypersurface $M$ is locally congruent a tube of radius $\pi/2\sqrt{-c}$ over a non-totally geodesic complex curve which does not have the principal curvatures $\pm \sqrt{-c}/2$ in $\mathbb{C}P^2(c)$.

Theorem 2. Let $M$ be a connected Hopf hypersurface in $\mathbb{C}H^n(c), n \geq 2$. Suppose that $M$ has $\eta$-parallel Ricci tensor. Then $M$ is either locally congruent to one of homogeneous real hypersurfaces of types $(A_0)$, $(A_{1,0})$, $(A_{1,1})$, $(A_2)$ and $(B)$ in $\mathbb{C}H^n(c), n \geq 2$, or a non-homogeneous real hypersurface with $\mathcal{A}\xi = 0$ in $\mathbb{C}H^2(c)$.

In [7], the classification problem of Hopf hypersurfaces with $\eta$-parallel Ricci tensors in $\tilde{M}_n(c), n \geq 2$ was discussed. However, there are some serious gaps in that paper. Non-homogeneous real hypersurfaces $M^3$ with $\mathcal{A}\xi = 0$ in $\tilde{M}_2(c) (= \mathbb{C}P^2(c) \text{ or } \mathbb{C}H^2(c))$ are counter examples to results of [7].
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2 Preliminaries Let $M^{2n-1}$ be a real hypersurface immersed into a nonflat complex space form $\tilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $N$. The Riemannian connections $\tilde{\nabla}$ of $\tilde{M}_n(c)$ and $\nabla$ of $M$ are related by the following formulas of Gauss and Weingarten:

\begin{equation}
\tilde{\nabla}_XY = \nabla_XY + g(AX, Y)N,
\end{equation}

\begin{equation}
\tilde{\nabla}_XN = -AX
\end{equation}

for arbitrary vector fields $X$ and $Y$ on $M$, where $g$ is the Riemannian metric of $M$ induced from the ambient space $\tilde{M}_n(c)$ and $A$ is the shape operator of $M$ in $\tilde{M}_n(c)$. An eigenvector of the shape operator $A$ is called a principal curvature vector of $M$ in $\tilde{M}_n(c)$ and an eigenvalue of $A$ is called a principal curvature of $M$ in $\tilde{M}_n(c)$. We set $\mathcal{V}_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal distribution associated to the principal curvature $\lambda$.

It is well-known that $M$ has an almost contact metric structure induced from the Kähler structure $(J, g)$ of the ambient space $\tilde{M}_n(c)$. That is, we have a quadruple $(\phi, \xi, \eta, g)$ defined by

\[g(\phi X, Y) = g(JX, Y), \quad \xi = -J\eta \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, N).\]

Then they satisfy

\[\phi^2 = -X + \eta(X)\xi, \quad \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\]

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi \xi = 0$ and $\eta(\phi X) = 0$. In the following, we call $\phi$, $\xi$ and $\eta$ the structure tensor, the characteristic vector and the contact form on $M$, respectively.

It follows from (2.1), (2.2), $\nabla J = 0$ and $JX = \phi X + \eta(X)N$ that

\begin{equation}
(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi,
\end{equation}

\begin{equation}
\nabla_X\xi = \phi AX.
\end{equation}

Denoting the curvature tensor of $M$ by $R$, we have the equation of Gauss given by

\begin{equation}
\begin{aligned}
g((R(X, Y)Z, W) &= (c/4)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\
&\quad + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W) \} \\
&\quad + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W).
\end{aligned}
\end{equation}

The following is called the equation of Codazzi.

\begin{equation}
(\nabla_X\phi)Y - (\nabla_Y\phi)X = (c/4)(\eta(\xi)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi).
\end{equation}

We usually call $M$ a Hopf hypersurface if the characteristic vector $\xi$ of $M$ is a principal curvature vector at each point of $M$. The following lemma clarifies fundamental properties of principal curvatures of a Hopf hypersurface $M$ in $\tilde{M}_n(c)$ (cf. [6]).

**Lemma 1.** Let $M$ be a Hopf hypersurface of a nonflat complex space form $\tilde{M}_n(c)$, $n \geq 2$. Then the following hold.
1. If a nonzero vector $v \in TM$ orthogonal to $\xi$ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta \lambda + (c/2))\phi v$, where $\delta$ is the principal curvature associated with $\xi$. In particular, when $c > 0$, we have $A\phi v = ((\delta \lambda + (c/2))/(2\lambda - \delta))\phi v$.

2. The principal curvature $\delta$ associated with $\xi$ is constant locally on $M$.

In $CP^n(c) \ (n \geq 2)$, a connected Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (see [4, 8]):

(A) A geodesic sphere of radius $r$, where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius $r$ around a totally geodesic $CP^\ell(c) \ (1 \leq \ell \leq n - 2)$, where $0 < r < \pi/\sqrt{c}$;
(C) A tube of radius $r$ around a complex hyperquadric $CP^{n-1}(c)$, where $0 < r < (2\sqrt{c})$ and $n \geq 5$ is odd;
(D) A tube of radius $r$ around the Plücker embedding of a complex Grassmannian $CG_{2,5}$, where $0 < r < (2\sqrt{c})$ and $n = 9$;
(E) A tube of radius $r$ around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A) and (B), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively.

In $CH^n(c) \ (n \geq 2)$, a connected Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (see [1]):

(A) A horosphere in $CH^n(c)$;
(B) A geodesic sphere of radius $r$, where $0 < r < \infty$;
(C) A tube of radius $r$ around a totally geodesic hypersurface $CH^{n-1}(c)$, where $0 < r < \infty$;
(D) A tube of radius $r$ around a totally geodesic $CH^\ell(c) \ (1 \leq \ell \leq n - 2)$, where $0 < r < \infty$;
(E) A tube of radius $r$ around a totally real totally geodesic $RH^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A), (B), (C), (D) and (E). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}) \log_4(2 + \sqrt{3})$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{|3|}/2$ and $\lambda_2 = \sqrt{|3|}/(2\sqrt{3})$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. Unifying real hypersurfaces of types (A), (B), (C), (D) and (E), we call them hypersurfaces of type (A).

3 Proof of Theorems 1 and 2 First of all we recall the following two classification theorems of Hopf hypersurfaces with $\eta$-parallel shape operators in a nonflat complex space form.

**Theorem A** ([5]). Let $M$ be a connected Hopf hypersurface of $CP^n(c), n \geq 2$. Then $M$ has $\eta$-parallel shape operator if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of types (A), (A) and (B) in $CP^n(c)$.
Theorem B ([7]). Let $M$ be a connected Hopf hypersurface of $\mathbb{C}H^n(c), n \geq 2$. Then $M$ has $\eta$-parallel shape operator if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of types $(A_0)$, $(A_{1,0})$, $(A_{1,1})$, $(A_2)$ and (B) in $\mathbb{C}H^n(c)$.

The Ricci tensor $S$ of an arbitrary real hypersurface $M$ in $\overline{M}_n(c)$ is expressed as (see (2.5)):

$$ SX = (c/4)(2n + 1)X - 3\eta(X)\xi) + (\text{trace } A)AX - A^2X. $$

By (3.1) we get the following equation on an arbitrary real hypersurface $M$ in the ambient space $\overline{M}_n(c)$:

$$ g((\nabla_X S)Y, Z) = (X(\text{trace } A))g(AY, Z) + (\text{trace } A)g((\nabla_X A)Y, Z) $$

$$ - g((\nabla_X A^2)Y, Z) = 0 \quad \text{for } X, Y, Z(\perp \xi) \in TM. $$

Remark 1. It follows from

$$ g((\nabla_X A^2)Y, Z) = g((\nabla_X A)Y, AZ) + g(AY, (\nabla_X A)Z) $$

and Equation (3.2) that if a Hopf hypersurface $M$ in a nonflat complex space form $\overline{M}_n(c)$ has $\eta$-parallel shape operator $A$ and trace $A$ is constant on $M$, then $M$ has $\eta$-parallel Ricci tensor $S$.

We shall prove Theorem 1. Without loss of generality, we may set $c = 4$.

We first study the case of $n \geq 3$. Our discussion here is essentially due to [7]. We suppose that the Ricci tensor $S$ of our Hopf hypersurface $M$ is $\eta$-parallel. Then, for a unit vector $Y(\perp \xi)$ with $AY = \lambda Y$, putting $h = \text{trace } A$ in Equation (3.2), we find

$$ \lambda(Xh) + h(X\lambda) - X(\lambda^2) = 0 \quad \text{for any } X(\perp \xi), $$

which means that $X(\lambda h - \lambda^2) = 0$ for any $X(\perp \xi)$. On the other hand, for any $Y(\perp \xi)$ such that $AY = \lambda Y$ we have $(\nabla_\xi A)Y = (\xi)Y + (\lambda I - A)\nabla_\xi Y$. Thus, from (2.6) we get $\xi \lambda = g((\nabla_\xi A)Y, Y) = g((\nabla_Y A)\xi, Y) = 0$. This, together with Lemma 1(2), implies that $\xi h = 0$. Hence, the function $\lambda h - \lambda^2$ is constant on $M$. Thus, for any principal curvatures $\lambda, \mu$ with $AX = \lambda X(\perp \xi)$, $AY = \mu Y(\perp \xi)$, we can put

$$ \lambda h - \lambda^2 = a $$

and

$$ \mu h - \mu^2 = b. $$

We here consider both cases of $a = b$ and $a \neq b$. It follows from Lemma 1(1), (3.4) and (3.5) that

$$ (2h\delta - \delta^2 - 4b)\lambda^2 - \{(\delta^2 - 4)h + 4\delta - 4b\delta\}\lambda - (2\delta h + b\delta^2 + 4) = 0. $$

This, combined with $h\lambda = \lambda^2 + a$ (see (3.4)), yields the following algebraic equation:

$$ 2\delta \lambda^4 - (2\delta^2 + 4b - 4)\lambda^3 + 2(a\delta + 2b\delta - 3\delta)\lambda^2 - (a\delta^2 - 4a + b\delta^2 + 4)\lambda - 2a\delta = 0. $$

Except the case of $\delta = 0$ and $a = b = 1$, $\lambda$ satisfies the algebraic equation (3.6) with constant coefficients, so that $\lambda$ is constant locally on $M$. 

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We next consider the case that $\delta = 0$ and $a = b = 1$. Note that in this case coefficients in Equation (3.6) are all vanishing. So, from Lemma 1 we must consider the case that $M$ has at most three distinct principal curvatures $\delta = 0$ with multiplicity $1, \lambda$ with multiplicity, say $k$, and $1/\lambda$ with multiplicity $2n - 2 - k$ having the equation $h = \lambda + (1/\lambda)$ (see (3.4)).

When $k$ satisfies either $k = 0$ or $2n - 2 - k = 0$, $\lambda$ must satisfy $\lambda = 1/\lambda$, so that $\lambda = 1$ or $-1$.

We shall show that the case where $k \geq 1$ and $2n - 2 - k \geq 1$ does not occur. It follows from $h = \lambda + (1/\lambda)$ that

$$(k - 1)\lambda^2 + 2n - 3 - k = 0.$$ 

Since $k - 1 \geq 0$ and $2n - 3 - k \geq 0$, this equation holds if and only if $k = 1$ and $n = 2$, which contradicts to the assumption $n \geq 3$. Therefore our Hopf hypersurface with $\eta$-parallel Ricci tensor in a nonflat complex space form must be homogeneous in $\mathbb{C}P^n(4)$.

In order to prove our Theorem, we shall show that our real hypersurface satisfying the assumption has $\eta$-parallel shape operator. Let $M$ be homogeneous in $\mathbb{C}P^n(4)$. We take three principal curvature vectors $X \in V^0_\lambda$, $Y \in V^0_\mu$, $Z \in V^0_\nu$. Here, for example $V^0_\lambda$ is defined by $V^0_\lambda = \{X \in TM | AX = \lambda X, X \perp \xi\}$. Note that Codazzi equation (2.6) shows that $g((\nabla_X A)Y, Z)$ is symmetric for all $X, Y, Z \in T^0M$. We have

$$g((\nabla_X A)Y, Z) = g(\nabla_X (AY) - A\nabla_X Y, Z) = g((\mu I - A)\nabla_X Y, Z) = (\mu - \nu)g(\nabla_X Y, Z).$$

On the other hand, it follows from (3.2) that

$$g((\nabla_X S)Y, Z) = h \cdot g((\nabla_X A)Y, Z) - g((\nabla_X A^2)Y, Z) = h \cdot g((\nabla_X A)Y, Z) - g((\nabla_X A)Y, AZ) - g(AY, (\nabla_X A)Z) = (h - \mu - \nu)g((\nabla_X A)Y, Z),$$

which, together with the assumption that the Ricci tensor $S$ is $\eta$-parallel, shows

$$g((\nabla_X A)Y, Z) = 0.$$

When $\mu = \nu$, Equation (3.7) implies $g((\nabla_X A)Y, Z) = 0$. So, in the following it suffices to consider the case that $\mu \neq \nu$.

When $h - \mu - \nu \neq 0$, Equation (3.8) yields $g((\nabla_X A)Y, Z) = 0$. Thus it remains to consider the case that $h - \mu - \nu = 0$. Changing $X$ and $Y$ in (3.8), we get

$$g((\nabla_X A)X, Z) = 0.$$ 

If $\lambda \neq \mu$, then $h - \mu - \nu = 0$ implies $h - \lambda - \nu \neq 0$. This, combined with (3.9), yields $g((\nabla_X A)Y, Z) = g((\nabla_X A)X, Z) = 0$. If $\lambda = \mu$, Equation (3.7) gives $g((\nabla_X A)Y, Z) = g((\nabla_Z A)X, Y) = 0$. Therefore we can see that our Hopf hypersurface $M$ has $\eta$-parallel shape operator. So $M$ is locally congruent to one of homogeneous real hypersurfaces of types $(A_1), (A_2)$ and $(B)$ (see Theorem A).

Conversely, let $M$ be of either type $(A_1)$, type $(A_2)$ or type $(B)$. Then, from Theorem A and Remark 1 we see easily that $M$ has $\eta$-parallel Ricci tensor.

We next study the case of $n = 2$. When $\delta \neq 0$, Equation (3.6) means that $\lambda$ is constant locally. Hence $M$ is a Hopf hypersurface with constant principal curvatures with $A\xi \neq 0$ in $\mathbb{C}P^2(4)$. In this case, $M$ is of either type $(A_1)$ of radius $r(\neq \pi/4)$ or type $(B)$ of each radius $r \in (0, \pi/4]$ in $\mathbb{C}P^2(4)$.

Next, let consider the case that $\delta = 0$ and $tr A$ is constant locally. Then $M$ is a Hopf hypersurface with constant principal curvatures with $A\xi = 0$ in $\mathbb{C}P^2(4)$. In this case, $M$ is of type $(A_1)$ of radius $\pi/4$ in $\mathbb{C}P^2(4)$.
In these two cases, $M$ has $\eta$-parallel shape operator. Then we can see that $M$ has $\eta$-parallel Ricci tensor by Theorem A and Remark 1.

We finally consider the case that $\delta = 0$ and $h$ is a non-constant function on $M$. We shall verify that every non-homogeneous real hypersurface $M$ with $A\xi = 0$ in $\mathbb{CP}^2(4)$ has $\eta$-parallel Ricci tensor. Such a real hypersurface $M$ has three distinct principal curvatures $\delta = 0, \lambda$ and $1/\lambda$, where $\lambda$ is a non-constant smooth function on $M$. We take two unit vectors $X$ and $Y$ with $AX = \lambda X$ and $AY = (1/\lambda)Y$. Using Equation (3.2) and $h = \lambda + (1/\lambda)$ repeatedly, we obtain the following:

$$g((\nabla_X S)X, X) = \left(X\lambda - \frac{X\lambda}{\lambda^2}\right)g(AX, X) + \left(\lambda + \frac{1}{\lambda}\right)g((\nabla_X A)X, X)$$

$$- g((\nabla_X A)X, AX) - g(AX, (\nabla_X A)X)$$

$$= \lambda \left(X\lambda - \frac{X\lambda}{\lambda^2}\right) + \left(\frac{1}{\lambda} - \lambda\right)g((\nabla_X A)X, X)$$

$$= \lambda \left(X\lambda - \frac{X\lambda}{\lambda^2}\right) + (X\lambda)(\frac{1}{\lambda} - \lambda) = 0,$$

and

$$g((\nabla_X S)Y, Y) = \left(X\lambda - \frac{X\lambda}{\lambda^2}\right)g(AY, Y) + \left(\lambda + \frac{1}{\lambda}\right)g((\nabla_X A)Y, Y)$$

$$- g((\nabla_X A)Y, AY) - g(AY, (\nabla_X A)Y)$$

$$= \frac{1}{\lambda} \left(X\lambda - \frac{X\lambda}{\lambda^2}\right) + \left(\frac{1}{\lambda} - \lambda\right)g((\nabla_X A)Y, Y)$$

$$= \frac{1}{\lambda} \left(X\lambda - \frac{X\lambda}{\lambda^2}\right) - \frac{X\lambda}{\lambda^2}(\frac{1}{\lambda} - \lambda) = 0,$$

and

$$g((\nabla_X S)X, Y) = (Xh)g(AX, Y) + h \cdot g((\nabla_X A)X, Y)$$

$$- g((\nabla_X A)X, AY) - g(AX, (\nabla_X A)Y)$$

$$= \left(h - \lambda - \frac{1}{\lambda}\right)g((\nabla_X A)X, Y) = 0.$$

We have similarly the following:

$$g((\nabla_Y S)Y, Y) = g((\nabla_Y S)X, Y) = g((\nabla_Y S)X, X) = 0.$$

These, combined with the symmetry of $S$, show that $M$ has $\eta$-parallel Ricci tensor.

The rest of the proof is to guarantee the existence of a non-homogeneous real hypersurface $M$ with $A\xi = 0$ in $\mathbb{CP}^2(4)$. To do this, we recall the following fact due to [2]:

**Fact.**

1. Every tube $M$ of sufficiently small constant radius around each Kähler submanifold of $\mathbb{CP}^n(c)$ is a Hopf hypersurface in this ambient space. However, in general $M$ has singular points, namely $M$ is not smooth at these points.

2. Let $M^{2n-1}$ be a Hopf hypersurface with $A\xi = \delta\xi$ of $\mathbb{CP}^n(c), n \geq 2$. Suppose that all principal curvatures of $M$ in the ambient space $\mathbb{CP}^n(c)$ have constant multiplicities on $M$. Then $M$ is locally congruent to a tube of constant radius $r(> 0)$ around a certain Kähler submanifold $N$ of $\mathbb{CP}^n(c)$. Moreover, $\delta = \sqrt{c} \cot(\sqrt{c}r)$ and all other principal curvatures $\lambda$ of $M$ are expressed as either $\lambda = (\sqrt{c}/2) \cot(\sqrt{c}r/2)$, $\lambda = -(\sqrt{c}/2) \tan(\sqrt{c}r/2)$ or $\lambda = (\sqrt{c}/2) \cot(\sqrt{c}r/2 \pm \theta)$, where $(\pm \sqrt{c}/2) \cot \theta$ are principal curvatures of the Kähler submanifold $N$. 
This fact means that every non-homogeneous real hypersurface $M$ with $Aξ = 0$ in $\mathbb{CP}^2(4)$ is locally congruent a tube of radius $\pi/4$ over a non-totally geodesic complex curve in the ambient space $\mathbb{CP}^2(4)$. Here, note that to delete singular points of $M$ we have only to consider the complex curve without having the principal curvatures $±1$ in $\mathbb{CP}^2(4)$.

Therefore we obtain the desired conclusion of Theorem 1. □

We next prove Theorem 2. Without loss of generality, we may set $c = -4$. We first investigate the case of $n \geq 3$. The “only if” part is obvious from Theorem B and Remark 1. So we shall prove the “if” part.

We suppose that the Ricci tensor $S$ of our Hopf hypersurface $M$ (with $Aξ = δξ$) is $η$-parallel. By the same discussion as that in the proof of Theorem 1 we also have Equations (3.4) and (3.5).

We first take a unit vector $X ∈ V_0^0$ with $2λ − δ \neq 0$. Then $AφX = µφX$ with $µ = (δλ − 2)/(2λ − δ)$ (see Lemma 1). This, together with (3.5), yields

\[(2δh − δ^2 − 4b)λ^2 + (4δ + 4bδ − (δ^2 + 4)h)λ + 2δh − 4 − bδ^2 = 0.\]

It follows from (3.4) and (3.10) that

\[2δλ^4 − 2(δ^2 + 2b + 2)λ^3 + 2δ(a + 2b + 3)λ^2 − (aδ^2 + bδ^2 + 4a + 4)λ + 2aδ = 0,
\]

which corresponds to Equation (3.6). Except the case of $δ = 0$ and $a = b = −1$, $λ$ satisfies the algebraic equation (3.11) with constant coefficients, so that $λ$ is constant locally on $M$.

We next consider the case of $δ = 0$ and $a = b = −1$. Note that in this case coefficients in Equation (3.11) are all vanishing. Here, from Lemma 1 we must consider the case that $M$ has at most three distinct principal curvatures $δ = 0$ with multiplicity 1, $λ$ with multiplicity, say $k$, and $-1/λ$ with multiplicity $2n − 2 − k$ having the equation $h = λ(1/λ)$ (see (3.4)). When $k$ satisfies either $k = 0$ or $2n − 2 − k = 0$, $λ$ must satisfy $λ = −1/λ$, which is a contradiction.

So we only to study the case that $k ≥ 1$ and $2n − 2 − k ≥ 1$. Since $h = λ(1/λ)$, we have

\[(k − 1)λ^2 − (2n − 3 − k) = 0.
\]

We note that $k \neq 1$, since $n ≥ 3$. Hence, $λ$ is also constant locally on $M$. However, there does not exist such a Hopf hypersurface with three constant principal curvatures $δ = 0, λ, −1/λ$ in $\mathbb{CH}^n(-4)$ (see [6]).

We finally consider the case of $2λ − δ = 0$ at some point of $M$. We shall verify that $2λ − δ$ vanishes identically on $M$. Assume that $2λ − δ \neq 0$ at some point $x_0 ∈ M$, and set $y_0 = (2λ − δ)(x_0)$. Let $N$ be the subset of these points $x ∈ M$ such that $(2λ − δ)(x) = y_0$. Clearly $N$ is a non-empty closed subset of $M$. It is also open, since the discussion in the case of $2λ − δ \neq 0$ means that the function $2λ − δ$ is constantly equal to $y_0 \neq 0$ on some neighborhood of each point $x ∈ N$. Since $M$ is connected, we find that $N = M$, which is a contradiction. So we find that $λ = δ/2$ on $M$.

Therefore we can see that every principal curvature $λ$ of $M$ is constant locally, so that $M$ is locally congruent to one of homogeneous real hypersurfaces of types $(A_0), (A_{1,0}), (A_{1,1}), (A_2)$ and (B).

We next study the case of $n = 2$. By the same discussion as that in the proof of Theorem 1, we can see that our real hypersurface $M$ satisfying the assumption is locally congruent to either a homogeneous Hopf hypersurface or a non-homogeneous Hopf hypersurface with $Aξ = 0$ in the ambient space $\mathbb{CH}^2(-4)$.

At the end of this paper we explain briefly the construction of a non-homogeneous real hypersurface $M^3$ with $Aξ = 0$ in $\mathbb{CH}^2(-4)$. In [3], T.A. Ivey and P.J. Ryan construct the
class of Hopf hypersurfaces in $CH^2(-4)$ with $A\xi = \delta \xi$ and $0 \leq \delta \leq 2$. Moreover, they show that every such Hopf hypersurface for $\delta < 2$ can be characterized in terms of Weierstrass-type data which take the form of a pair of embedded contact curves in a unit sphere $S^3$ (for details, see Theorem 2 in [3]).

Hence we obtain the desired conclusion.

References


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