A NONLINEAR MODEL OF A SEARCH ALLOCATION GAME WITH FALSE CONTACTS

Toshiyuki Kekka and Ryusuke Hohzaki

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Abstract. This paper deals with a two-person zero-sum search game called *search allocation game* (SAG) with a searcher and a target as players, taking account of false contacts. The searcher distributes his searching resource in a search space to detect the target and the target moves to evade the searcher. The searcher obtains a profit of target value on detection of target but expends cost for the search. The payoff of the game is the expected reward defined by obtained target value minus expended searching cost. The searcher's strategy is denoted by a distribution plan about where and when he distributes his searching resource and the target strategy is the selection of a path to follow from some options. In the search operation, any sensor cannot get rid of false contacts caused by signal processing noises and real objects similar to the true target under noisy environment. On their happening, they make the searcher waste some time for investigation and interrupt the search operation for a while. There have been few researches dealing with the SAG with the false contacts. In this paper, we model the game with false contacts by a stochastic process and discuss a general procedure to derive an equilibrium point through a nonlinear programming method for a searcher's best response to the target's behavior.

1 Introduction This paper deals with a search game called *search allocation game* (SAG) [5, 7], where a searcher distributes his searching resource in a search space expecting the detection of a target and the target moves to evade the searcher. This game is categorized in the so-called *search-and-evasion game* (SEG), where searchers and moving targets compete each other. In the search operation, any sensor cannot get rid of false contacts caused by signal processing noises and real objects similar to the true target under noisy environment. In this paper, we analyze optimal strategies of the searcher and the target, taking account of the occurrence of the false contacts.

We can see the original of the SEG in datum search game. An exposed position of targets is referred to as datum point and the datum is a generic of information about the target including the datum point. We call the search operation kicked off by the datum the datum search. In an early research of search theory, entitled "Search and Screening" [21], Koopman discussed an optimal datum search against a submarine moving in a randomized direction from a datum point on a plane.

Meinardi [22] dealt with a datum search game but focused on the diffusive motion of a target on a one-dimensional line and tried to obtain the target motion such that the distribution probability of the target is as uniformly as possible on the line. Most modelings of the datum search were taken in military operations such as anti-submarine warfare (ASW) of a submarine vs. ASW airplanes. Danskin [2] showed us optimal strategies of players in the datum search game, where a submarine chooses a fixed speed and course at first and keeps them through the game, and an AWS airplane selects a point to dip his active sound buoys each time. There are some other models of the datum search game applied to the

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ASW operation, such as Baston and Bostock [1] and Garnaev [4]. In their models, a submarine hides on a one-dimensional space and an ASW airplane tries to maximize the probability of destroying the submarine by dropping several depth charges. Those authors implicitly assumed that the searcher can distribute his searching resource, e.g. sonars or depth charges, wherever he likes because the mobility of ASW vehicles is superior to that of submarines. The model of the SAG requires the assumption on the searcher's superior mobility.

If there is not such superiority between the moving capabilities of searchers and targets, we'd better consider the moving strategy for the searcher as well as the target. Washburn [26] discussed the moving strategies of the searcher and the target in a multi-stage game, which had the payoff of the total traveling cost of the searcher until the coincidence of players' positions. Kikuta [18] adopted searching cost as the payoff of the game with a moving searcher and a hiding target. Assuming that the positions of two moving players determined a temporary payoff each time and its total gave a comprehensive payoff of the game, Eagle and Washburn [3] investigated a single-stage game.

The research on the SAG started with stationary target models, where the target strategy is to choose the position of hiding himself. For the SAG with stationary targets, Nakai [23] and Iida et al. [15] studied several types of payoffs such as the detection probability of target or the reward of searcher. Hohzaki and Iida [16, 11, 13] extended the stationary target model to the SAG with moving targets. Hohzaki and Iida [12] generalized the model further and proposed a general numerical algorithm to derive an equilibrium of the SAG. Most of the previous researches assume comparatively simple target motions such as the selection of a path from a limited number of options or the limited mobility to neighbor places from its current position. Washburn and Hohzaki [27], Hohzaki et al. [14] and Hohzaki [7] considered energy constraints on target motion and a large number of options for target paths to tackle more practical SAG models. Hohzaki [9] discussed a multi-stage SAG and Hohzaki [10] first invented a cooperative SAG game with a coalition of multiple searchers against the target.

The past researches surveyed so far handled only the detection of true targets. In the search operation, however, there inevitably occur false contacts caused by environmental noise, signal processing noise or real objects other than true targets. They often interrupt the search operation and make the searcher waste some time for their investigation, by which the searcher figures out if it is what he is looking for. The searcher usually takes a two-phase operation consisting of broad search and then investigation for contact signals. There have been several papers on the false contact search model but most of them discussed a one-sided optimization problem for the searcher. Stone [25] solved an optimal distribution problem of searching resource to minimize the expected total amount of searching time plus investigation time until detecting true targets, assuming that objects exist in the search space and they bring about false contacts. But he did not take account of the two-phase operation and therefore his problem is not different from the true-target model in essence. Kisi [19] considered the problem with noise-type false contacts. In his model, contacts occur according to Poisson distribution. In order to maximize the detection probability of the true target, the searcher has to make decision about how much time he should spend for investigation each time the contact occurs. Iida et al. [17, 20] refined the Kisi's study but their basic approach is the same as the Kisi's. In the studies, their assumption of Poisson distribution with the stationary occurrence rate of contacts makes the problem easy to be solved because they can regard the search as a renewal process. As seen above, the previous researches with false contacts discussed only the one-sided problem for the searcher and never dealt with the game from two-sided point of view.

Hohzaki [6, 8] are the first researches that tackled the SAG model with false contacts.

He assumed that the searcher's strategy, i.e. the distribution of searching resource, affects the occurrence probability of false contacts such that more sonars or sound hydrophones would much easily pick up signals caused from false targets or noises. But, as the payoff of the game, he did not deal with an exact function of the detection probability of true targets but a linear function of resources accumulated on the target path. This paper discusses the SAG with false contacts and adopt the precise expression of the expected reward as payoff, which is a general criterion of the search operation and includes the detection probability as a special case, in the practical two-phase operation model with broad search and investigation process. In this paper, we aim to derive an equilibrium for the game and clarify the characteristics of optimal strategies in a competitive search operation between the searcher and the target.

In the next section, we describe some assumptions about a SAG with a searcher and a target, and model the detection of a true target and false contacts in a stochastic process. In Section 3, we define the payoff of the game by the expected reward caused by the target detection and the expenditure of searching cost, which would be given by a nonlinear function of decision variables of the searcher's strategy. In Section 4, we devise a computational algorithm to obtain an optimal distribution of searching resource by the searcher given the target strategy. As our final result, we propose a numerical algorithm to derive an equilibrium point for our SAG in Section 5. We take some numerical examples to investigate some properties of optimal strategies of players and do some sensitivity analyses in Section 6.

2 Description of Model and Instance of Events We consider with a two-person zero-sum search game with false contacts, which a search and a target play.

A1. A search space consists of a discrete cell space $\boldsymbol{K} = \{1, \cdots, K\}$ and a discrete time space $\boldsymbol{T} = \{1, \cdots, T\}$ and it is denoted by $\boldsymbol{K} \times \boldsymbol{T}$.

A2. A target chooses a path running through the search space to evade a searcher. An entire set of paths is denoted by Ω. A path $\omega \in \Omega$ is a mapping from $\boldsymbol{T}$ to $\boldsymbol{K}$, namely $\omega : \boldsymbol{T} \to \boldsymbol{K}$, and is assumed to pass through cell $\omega(t)$ at time $t \in \boldsymbol{T}$.

A3. The searcher wants to detect the target by distributing searching resource after time point τ, that is, during a time duration $\widehat{\boldsymbol{T}} \equiv \{\tau, \cdots, T\} \subseteq \boldsymbol{T}$. The searcher can use $\Phi(t)$ resources at each time point $t \in \widehat{\boldsymbol{T}}$. Let us denote the amount of resource to be distributed in cell i at time t by $\varphi(i,t)$. But the distribution costs $c_0(i,t) > 0$ per unit resource.

If the target is in cell $i \in \boldsymbol{K}$ at time $t \in \widehat{\boldsymbol{T}}$, the distribution of $\varphi(i,t)$ resources there brings the searcher detection probability

$$1 - \exp\left(-\alpha_i \varphi(i,t)\right), \tag{1}$$

where parameter α_i indicates the efficiency of unit resource in cell i for detection.

A4. There could exist two types of contact events: the detection of true target and the contact of false target or noise after the search begins. Each event occurs once at most at each time independent of the other event and the occurrence probability of false contact is Q_t at time t. If there is no detection event, the search continues at the next time. If a contact happens, the searcher starts an investigation phase for the contact, which requires $t_f - 1$ time points. During the investigation phase, the searcher has to stop the search. The searcher gets the faultless diagnosis about the contact at the earlier time of the completion time of the inspection or the last time T. If the true-target detection is diagnosed, he gets a target value and terminates the search game. If only the false contact occurs, the search operation resumes after the investigation phase.

A5. The searcher is given target value $V(t)$ of the time t when the contact of the true target occurs, after getting its diagnosis. The target value is nonnegative and decreases as time elapses, namely, for any time $t = 1, \dots, T-1$,

$$V(t) \geq V(t+1) \geq 0. \tag{2}$$

A6. The game ends on detection of the true target or at the final time point T.

A7. The payoff of the game is the reward on the searcher's side. The reward is defined by the target value at the detection time minus the distribution cost expended until the detection or only the distribution cost in the case of no detection. The searcher wants to maximize the payoff and the target desires to minimize it.

In Assumption A3, we denote a searcher's strategy of resource distribution by $\varphi = \{\varphi(i,t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$. As assumed in A2, the target chooses a path ω, which is his pure strategy. We take a mixed strategy for him, that is, the target selects $\omega \in \Omega$ with probability $\pi(\omega)$. The respective feasible regions of the searcher and the target strategies are as follows:

$$\Psi \equiv \left\{ \varphi \,\middle|\, \sum_{i \in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}},\ \ \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}, \tag{3}$$

$$\Pi \equiv \left\{ \pi \,\middle|\, \sum_{\omega \in \Omega} \pi(\omega) = 1,\ \pi(\omega) \geq 0,\ \omega \in \Omega \right\}. \tag{4}$$

Corresponding to all combinations of players' strategies, there could occur four types of events or states every time after the beginning of the search: (1) detection event including true target detection, (2) false contact, (3) no-detection and (4) investigation state. We denote the four events by 'D', 'F', 'S' and 'I', respectively. The four types of events occurs in probabilistic manner during a period $\widehat{\boldsymbol{T}}$ with $T - \tau + 1$ time points. We are interested in the detection of true target and therefore we consider the enumeration of a sequence of events exclusive to the true detection, namely, events not including symbol 'D'. From Assumption A4, we can make every instance by the following rule.

(1) Put 'S' or 'F' at an initial time τ.

(2) As a general combination of false contact and investigation, we can think of a 'F' followed by $t_f - 1$ successive 'I's. As some special cases of the combination, the time reaches the last time T in the middle of investigation. To represent the situation, we put a 'F' followed by y 'I's for $0 \leq y \leq t_f - 2$.

(3) At time points not assigned to the above two types of sequences, We put 'S'.

Figure 1 shows an instance of the sequence of contacts without any 'D' in the case of $T = 9$ and $t_f = 4$. The last inspection phase is truncated.

During a time period $[\tau, L]$ with the final time L, a set of instances without true detection, A_L, has the following cases. The first is the case that time is up to the last time point L in the middle of investigation process after a false contact. Let the repetition number of investigation be y. y would be in an interval $0 \leq y \leq t_f - 2$. The number of false contacts, M, could be $\lfloor (L - y - \tau)/t_f \rfloor$ at most because the contacts occur from time τ until just

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t=  1  2  3  4  5  6  7  8  T
 --S--F--I--I--I--S--S--F--I-->
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Figure 1: An instance of contacts sequence

before the last occurrence time $L-y-1$. Therefore, M must be $0 \leq M \leq \lfloor (L-\tau-y)/t_f \rfloor$. There are M pairs of false contact and a sequent investigation process, and the last false contact, namely, the $M+1$-th false contact, occurs followed by y investigation time points until the last time. At other time points not assigned to $M+1$ groups of the false contact, which we count $L-\tau-Mt_f-y$, event 'S' would be assigned. We can specify an instance of those 'S's by x_j $(j=1,\dots,M+1)$, where x_j is the number of successive 'S's just before the j-th false contact. The numbers $\{x_j\}$ have to satisfy the following condition:

$$\sum_{j=1}^{M+1} x_j = L-\tau-Mt_f-y. \tag{5}$$

The total number of feasible sets for $\{x_j\}$ is ${}_{L-\tau-M(t_f-1)-y}C_M$.

The second is the case that the completion of the last investigation process is not intercepted by the last time L. Every investigation process with t_f time points is put correctly inside of a whole time period of $L-\tau+1$. The number of false contacts, M, must be $0 \leq M \leq \lfloor (L-\tau+1)/t_f \rfloor$. An instance of 'S' occurrence at $L-\tau-Mt_f+1$ time points is also specified by $\{x_j, j=1,\dots,M\}$ satisfying

$$\sum_{j=1}^{M+1} x_j = L-\tau-Mt_f+1.$$

The total number of such specification is ${}_{L-\tau-M(t_f-1)+1}C_M$. The second case is realized by applying $y=-1$ to the first case. Now we can construct an algorithm to enumerate all instances A_L without any true detection as follows:

(i) Change y among $-1,0,1,\dots,t_f-2$.

(ii) For each y, set M to each of $0,1,\dots,\lfloor (L-y-\tau)/t_f \rfloor$.

(iii) For a combination of y and M, enumerate all vectors $(x_j, j=1,\cdots,M+1)$ so as to satisfy Eq. (5).

A combination of y, M and $\{x_j\}$ gives us an instance. In the next section, we calculate the occurrence probability of each instance and derive the expected payoff of the game.

3 Derivation of Payoff Function In the end of Section 2, we explain an algorithm to enumerate an entire set of instances without any detection of true target. In an instance, we use symbols 'S', 'F' and 'I' to represent events. Here we replace $\{S,F,I\}$ with $\sigma(t) \in \{1,-1,0\}$, respectively, to formulate the payoff of the game. We can express any instance of a sequence of events by a vector $(\sigma(t), t \in \widehat{\boldsymbol{T}})$.

Let us assume that the target takes a path ω and the searcher distributes his searching resource by a plan φ. Because event S denoted $\sigma(t)=1$ indicates that there is no detection of true target and no false contact, its occurrence probability is $(1-Q_t)\exp(-\alpha_{\omega(t)}\varphi(\omega(t),t))$

from Assumption A3 and A4. Similarly, the event of $\sigma(t) = -1$ is a false contact without any true contact and has its occurrence probability $Q_t \exp(-\alpha_{\omega(t)}\varphi(\omega(t), t))$. In the investigation process denoted by $\sigma(t) = 0$, no event happens because of no execution of the search. From the discussion above, we have the following occurrence probability for each event $\sigma(t) \in \{1, -1, 0\}$:

$$\left(1 - \sigma(t)Q_t - \delta_{\sigma(t),-1}\right) \exp\left(-|\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right),$$

where $\delta_{i,j}$ is the Kronecker's delta which is 1 for $i = j$ and 0 for $i \neq j$. Using the expression, we obtain the detection probability of true target, $P(\varphi, \omega)$, during time period $[\tau, T]$ as the occurrence probability of instances exclusive to the enumerated ones by A_T.

$$P(\varphi, \omega) = 1 - \sum_{\sigma \in A_T} \left[\prod_{t=\tau}^{T} \left(1 - \sigma(t)Q_t - \delta_{\sigma(t),-1}\right)\right] \exp\left(-\sum_{t=\tau}^{T} |\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right) \quad (6)$$

Especially in the case of a constant probability of false contact with $Q_t = Q$, Eq. (6) ends up to a simple expression.

$$P(\varphi, \omega) = 1 - \sum_{\sigma \in A_T} Q^{m(\sigma)}(1 - Q)^{n(\sigma)} \exp\left(-\sum_{t=\tau}^{T} |\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right), \quad (7)$$

where, $m(\sigma)$ is the number of false contacts and $n(\sigma)$ is the number of no-contacts in the search.

Furthermore, we are going to derive the expected payoff of the game. ¿From the discussion above and Assumption A3, the occurrence probability of an instance σ, the detection probability of target and searching cost $C_t(\varphi, \sigma)$ during a period $[\tau, t]$ are given by the followings.

$$Q_t(\sigma) = \prod_{\zeta=\tau}^{t} \left(1 - \sigma(\zeta)Q_\zeta - \delta_{\sigma(\zeta),-1}\right) \quad (8)$$

$$P_t(\varphi, \omega, \sigma) = 1 - \exp\left(-\sum_{\zeta=\tau}^{t} |\sigma(\zeta)|\alpha_{\omega(\zeta)}\varphi(\omega(\zeta), \zeta)\right) \quad (9)$$

$$C_t(\varphi, \sigma) = \sum_{\zeta=\tau}^{t} |\sigma(\zeta)| \sum_{i \in K} c_0(i, \zeta)\varphi(i, \zeta) \quad (10)$$

The searcher gets a reward $V(t) - C_t(\varphi, \sigma)$ on detection of target at time $t \in \widehat{\boldsymbol{T}}$ but just loses searching cost $C_T(\varphi, \sigma)$ in the case of no detection. Taking account of both cases, we can evaluate the expected payoff by

$$\begin{aligned} R(\varphi, \omega) \;\; = \;\; & \sum_{t=\tau}^{T} \sum_{\sigma \in A_t} (V(t) - C_t(\varphi, \sigma))Q_t(\sigma) \left(P_t(\varphi, \omega, \sigma) - P_{t-1}(\varphi, \omega, \sigma)\right) \\ & - \sum_{\sigma \in A_T} C_T(\varphi, \sigma)Q_T(\sigma) \left(1 - P_T(\varphi, \omega, \sigma)\right). \end{aligned}$$

To calculate $P_{t-1}(\varphi, \omega, \sigma)$, we need just the part $(\sigma(\tau), \cdots, \sigma(t-1))$ of an entire vector $\sigma \in A_t$. However we use notation $P_{t-1}(\varphi, \omega, \sigma)$ instead of $P_{t-1}(\varphi, \omega, \sigma|_{A_{t-1}})$ for the sake of simplicity. Let us transform the above expression into a simpler form, as follows.

$$\begin{aligned}
R(\varphi,\omega) &= \sum_{t=\tau}^{T}\sum_{\sigma\in A_t}(V(t)-C_t(\varphi,\sigma))Q_t(\sigma)P_t(\varphi,\omega,\sigma) \\
&\quad -\sum_{t=\tau-1}^{T-1}\sum_{\sigma\in A_{t+1}}(V(t+1)-C_{t+1}(\varphi,\sigma))Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma) \\
&\quad -\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\left(1-P_T(\varphi,\omega,\sigma)\right) \\
&= \sum_{t=\tau}^{T-1}\left\{\sum_{\sigma\in A_t}(V(t)-C_t(\varphi,\sigma))Q_t(\sigma)P_t(\varphi,\omega,\sigma)\right. \\
&\quad \left.-\sum_{\sigma\in A_{t+1}}(V(t+1)-C_{t+1}(\varphi,\sigma))Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)\right\} \\
&\quad +\sum_{\sigma\in A_T}(V(T)-C_T(\varphi,\sigma))Q_T(\sigma)P_T(\varphi,\omega,\sigma) \\
&\quad -\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\left(1-P_T(\varphi,\omega,\sigma)\right) \qquad (11)
\end{aligned}$$

Any instance of enumeration A_{t+1} is made by adding an event $\sigma(t+1)$ to an enumeration A_t. For the additional event, there are two cases. In the case of $\sigma(t+1)\in\{1,-1\}$, we can make an instance until time $t+1$ using a common sequence of instances $\{\sigma(\tau),\cdots,\sigma(t)\}$. In the case of $\sigma(t+1)=0$ of investigation, however, $\{\sigma(\tau),\cdots,\sigma(t)\}$ should be exclusive to the above one. In the former case, the following calculation is possible:

$$\begin{aligned}
\sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_{t+1}(\sigma) &= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_t(\sigma)\{(1-Q_{t+1})+Q_{t+1}\} \\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_t(\sigma) \qquad (12)
\end{aligned}$$

from Eq. (8). We also have the similar result of

$$\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0}Q_{t+1}(\sigma)=\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0}Q_t(\sigma)$$

in the latter case. As a result, $\sum_{\sigma\in A_{t+1}}Q_{t+1}(\sigma)=\sum_{\sigma\in A_t}Q_t(\sigma)$ is given. Similarly, we have the following calculation

$$\begin{aligned}
&\sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}C_{t+1}(\varphi,\sigma)Q_{t+1}(\sigma) \\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}\left(\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma) \\
&\quad \times\{(1-Q_{t+1})+Q_{t+1}\} \\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}\left(\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma)
\end{aligned}$$

in the former case but we have $\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0} C_t(\varphi,\sigma)Q_t(\sigma)$ in the latter case. We can unify both expressions into

$$\sum_{\sigma\in A_{t+1}}\left(|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma).$$

We apply the above to the first item of Eq. (11) to get the following result:

$$\begin{aligned}R(\varphi,\omega)=&\sum_{t=\tau}^{T-1}\sum_{\sigma\in A_{t+1}}\left\{(V(t)-V(t+1))+|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)\right\}\\ &\times Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)+\sum_{\sigma\in A_T}V(T)Q_T(\sigma)P_T(\varphi,\omega,\sigma)-\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\end{aligned}\quad(13)$$

Considering the monotonically non-increasingness of target value, the linearity of $C_t(\varphi,\sigma)$ for φ and the concavity of $P_t(\varphi,\omega,\sigma)$ for φ, which are seen from condition (2), Eq. (10) and (9), respectively, the payoff $R(\varphi,\omega)$ becomes strictly concave for φ. We take expectation for the payoff $R(\varphi,\omega)$ by a mixed strategy of target π to get

$$\begin{aligned}R(\varphi,\pi)&=\sum_{\omega\in\Omega}\pi(\omega)R(\varphi,\omega)\\ &=\sum_{\omega\in\Omega}\pi(\omega)\left[\sum_{t=\tau}^{T-1}\sum_{\sigma\in A_{t+1}}\left\{(V(t)-V(t+1))+|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)\right\}\right.\\ &\quad\left.\times Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)+\sum_{\sigma\in A_T}V(T)Q_T(\sigma)P_T(\varphi,\omega,\sigma)\right]-\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma).\end{aligned}\quad(14)$$

The expression is linear for π and strictly concave for φ. We often use the detection probability of target as a criterion in the search problem and we can obtain its expectation $P(\varphi,\pi)$ by applying parameters $c_0(i,t)=0$ and $V(t)=1$ to Eq. (14), as follows.

$$\begin{aligned}P(\varphi,\pi)&=\sum_{\omega\in\Omega}\pi(\omega)Q_T(\sigma)P_T(\varphi,\omega,\sigma)\\ &=1-\sum_{\omega\in\Omega}\pi(\omega)\sum_{\sigma\in A_T}\left[\prod_{t=\tau}^{T}\left(1-\sigma(t)Q_t-\delta_{\sigma(t),-1}\right)\right]\exp\left(-\sum_{t=\tau}^{T}|\sigma(\omega(t))|\varphi(\omega(t),t)\right)\end{aligned}\quad(15)$$

4 Maximization of the Expected Payoff In this section, our purpose is to propose a computational algorithm to derive an optimal distribution of searching resource of maximizing the expected payoff given a target mixed strategy π.

4.1 Necessary and sufficient conditions for optimal solution The following maximization problem has a unique optimal solution because the expected payoff of Eq. (14) is strictly concave for φ, as shown in Section 3.

$$(P1)\ \max_{\varphi}\ R(\varphi,\pi)\quad s.t.\ \sum_{i\in K}\varphi(i,t)\le\Phi(t),\ t\in\widehat{\boldsymbol{T}},\quad \varphi(i,t)\ge 0,\ i\in\boldsymbol{K},\ t\in\widehat{\boldsymbol{T}}$$

We can derive necessary and sufficient conditions for the optimal solution of Problem $(P1)$ as its Karush-Kuhn-Tucker (KKT) conditions, using Lagrangean multipliers $\{\lambda(t),t\in\widehat{\boldsymbol{T}}\}$

and $\{\mu(i,t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, and a Lagrangean function

$$L(\varphi;\lambda,\mu) = R(\varphi,\pi) + \sum_{t\in\widehat{\boldsymbol{T}}} \lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) + \sum_{i\in K,}\sum_{t\in\widehat{\boldsymbol{T}}} \mu(i,t)\varphi(i,t).$$

As the result, we have the following conditions:

$$\frac{\partial L}{\partial\varphi(i,t)} = \frac{\partial R}{\partial\varphi(i,t)} - \lambda(t) + \mu(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{16}$$

$$\lambda(t) \geq 0,\ t \in \widehat{\boldsymbol{T}} \tag{17}$$

$$\mu(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{18}$$

$$\sum_{i\in K}\varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}} \tag{19}$$

$$\varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{20}$$

$$\lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) = 0,\ t \in \widehat{\boldsymbol{T}} \tag{21}$$

$$\mu(i,t)\varphi(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. \tag{22}$$

We express $\partial R/\partial\varphi(i,t)$ as a function of variable $\varphi(i,t)$ in an explicit manner and have

$$\frac{\partial R}{\partial\varphi(i,t)} = B(i,t)\exp(-\alpha_i\varphi(i,t)) + C(i,t). \tag{23}$$

In the expression above, we use the following notation:

$$\begin{aligned} B(i,t) \equiv \sum_{\omega\in\Omega_{it}} \pi(\omega) \Bigg[&\sum_{\zeta=t}^{T-1} \sum_{\sigma\in A_{\zeta+1}} \left\{ -\Delta V(\zeta) + |\sigma(\zeta+1)| \sum_{i\in K} c_0(i,\zeta+1)\varphi(i,\zeta+1) \right\} \\ &\times |\sigma(t)|\alpha_i Q_{\zeta+1}(\sigma)\exp\left(-\sum_{\xi=\tau,\xi\neq t}^{\zeta} |\sigma(\xi)|\alpha_{\omega(\xi)}\varphi(\omega(\xi),\xi) \right) \\ &+ \sum_{\sigma\in A_T} V(T)|\sigma(t)|\alpha_i Q_T(\sigma)\exp\left(-\sum_{xi=\tau,\xi\neq t}^{T} |\sigma(\xi)|\alpha_{\omega(\xi)}\varphi(\omega(\xi),\xi) \right) \Bigg] \end{aligned} \tag{24}$$

$$C(i,t) \equiv \sum_{\sigma\in A_t} |\sigma(t)|c_0(i,t)Q_t(\sigma)\sum_{\omega\in\Omega}\pi(\omega)P_{t-1}(\varphi,\omega,\sigma) - \sum_{\sigma\in A_T}|\sigma(t)|c_0(i,t)Q_T(\sigma) \tag{25}$$

with definitions of $\Omega_{it} \equiv \{\omega \in \Omega | \omega(t) = i\}$ and $\Delta V(\zeta) \equiv V(\zeta+1) - V(\zeta)$.

For an instance σ with $\sigma(t) = 0$ at time t, the distribution plan of searching resouce, $\{\varphi(i,t), i \in \boldsymbol{K}\}$, cannot be executed at that time practically and then we count or enumerate only $\sigma(t)$ of $|\sigma(t)| = 1$ to calculate $B(i,t)$ and $C(i,t)$.

Anyway we can verify $C(i,t) < 0$ because of Eq. (12) and the following transformation.

$$\begin{aligned} &\sum_{\sigma\in A_T} |\sigma(t)|c_0(i,t)Q_T(\sigma) = |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_{T-1}} Q_{T-1}(\sigma) = \cdots \\ &= |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_t} Q_t(\sigma) > |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_t} Q_t(\sigma)\sum_{\omega}\pi(\omega)P_{t-1}(\varphi,\omega,\sigma). \end{aligned}$$

Let us consider the properties of optimal distribution of searching resource.

First, the complementary slackness condition (21) tells us that an equation $\sum_{i\in K}\varphi(i,t)=\Phi(t)$ holds if $\lambda(t)>0$. Next we are going to derive necessary and sufficient conditions for $\varphi(i,t)>0$ or $\varphi(i,t)=0$.

If $\varphi(i,t)>0$, we have $\mu(i,t)=0$ from Eq. (22) and then $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)=\lambda(t)$ from (16) and (23). It also indicates $B(i,t)+C(i,t)>\lambda(t)$.

In the case of $B(i,t)=0$ in Definition (24), there is no path running through Cell i at t or $\Omega_{it}=\emptyset$ and an optimal distribution should be $\varphi(i,t)=0$ by the following reason. If $\varphi(i,t)>0$, we have $C(i,t)=\lambda(t)$. But the equation contradicts $C(i,t)<0$ and $\lambda(t)\geq 0$.

Now let us assume $B(i,t)>0$. If $\varphi(i,t)=0$, it follows that $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)=B(i,t)+C(i,t)\leq\lambda(t)$ from Eqs. (16) and (18). Conversely, we have $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)<B(i,t)+C(i,t)\leq\lambda(t)$ for any $\varphi(i,t)>0$ if $B(i,t)+C(i,t)\leq\lambda(t)$ but the fact contradicts the inequality we derived from $\varphi(i,t)>0$ just before. Therefore, $\varphi(i,t)=0$ is equivalent to the condition $B(i,t)+C(i,t)\leq\lambda(t)$ in the case of $B(i,t)>0$. Noting that $B(i,t)+C(i,t)\leq\lambda(t)$ includes the condition $B(i,t)=0$, necessary and sufficient condition for $\varphi(i,t)=0$ is $B(i,t)+C(i,t)\leq\lambda(t)$.

According to the discussion so far, we classify the conditions that make an optimal distribution of resource $\varphi(i,t)$ positive or zero into the following two cases:

(i) If and only if $B(i,t)+C(i,t)>\lambda(t)$, an optimal distribution $\varphi^*(i,t)>0$ is given by

$$\varphi^*(i,t)=\frac{1}{\alpha_i}\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}.$$

(ii) If and only if $B(i,t)+C(i,t)\leq\lambda(t)$, $\varphi(i,t)=0$.

We have a formula about an optimal distribution in both cases of (i) and (ii).

$$\varphi^*(i,t)=\frac{1}{\alpha_i}\left[\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}\right]^+, \tag{26}$$

where we use notation $[x]^+\equiv\max\{0,x\}$. The total amount of resouces distributed optimally at time t is given by

$$\sum_{\{i\in K|B(i,t)+C(i,t)>\lambda(t)\}}\frac{1}{\alpha_i}\left[\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}\right]^+ \tag{27}$$

and it is monotonically decreasing for Lagrangean multiplier $\lambda(t)$ within $[0,\overline{\lambda_t}]$ and becomes zero for $\lambda(t)\geq\overline{\lambda_t}$, where $\overline{\lambda_t}\equiv\max_{i\in K}\{B(i,t)+C(i,t)\}$. Here let us make sure that $B(i,t)$ and $C(i,t)$ do not contain $\{\varphi(i,t),\ i\in\boldsymbol{K}\}$ distributed at t.

4.2 A computational algorithm to derive an optimal solution We repeat constructing an optimal distribution $\{\varphi(i,t),i\in\boldsymbol{K}\}$ of a time t for every $t\in\widehat{\boldsymbol{T}}$ while keeping $\{\varphi(i,\zeta),i\in\boldsymbol{K}\}$ unchanged at any other time $\zeta\in\widehat{\boldsymbol{T}}$, from the properties of optimal solution discussed in Section 4.1.

Finding an optimal distribution of searching resource at the time t proceeds as follows. First, we search for an optimal $\lambda(t)$. We apply $\lambda(t)=0$ to Eq. (26) and calculate $\{\varphi(i,t),i\in\boldsymbol{K}\}$. If $\sum_i\varphi(i,t)\leq\Phi(t)$, we have obtained an optimal solution at the time t. Otherwise, we find $\lambda(t)$ satisfying $\sum_{i\in K}\varphi(i,t)=\Phi(t)$ within $[0,\overline{\lambda_t}]$. The search is done by dichotomy using the monotonically decreasingness of the expression (27). We repeat the above procedure for optimal solution until the satisfaction of the KKT conditions (16)$\sim$(22) for every time

$t = \tau, \ldots, T$. We come to an optimal solution $\{\varphi^*(i,t), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}\}$ if the newly-obtained solution does not change from old one for all $t \in \widehat{\boldsymbol{T}}$. Our algorithm always brings a larger expected reward than the previous one after every calculation at each time and therefore it certainly finishes on convergence to the largest reward. Our algorithm is constructed as follows.

An algorithm for an optimal distribution of searching resource: $\Gamma(\pi)$

(S1) Repeat Step (S2)∼(S3) for $t = \tau, \ldots, T$ until the convergence of solution.

(S2) Set $\lambda(t) = 0$ and calculate $\{\varphi(i,t), i \in \boldsymbol{K}\}$ by Eq. (26).

(S3) If $\sum_{i \in K} \varphi(i,t) \leq \Phi(t)$, end and go back to Step (S1). Otherwise, if $\sum_{i \in K} \varphi(i,t) > \Phi(t)$, execute the following algorithm to obtain an optimal multiplier $\lambda^*(t)$ and go back to Step (S1).

- (i) Sort values of $B(i,t) + C(i,t)$ in the increasing order like $B(I_1,t) + C(I_1,t) \leq B(I_2,t) + C(I_2,t) \leq \ldots$ and renumber all cells $i \in \boldsymbol{K}$ in the form of $I_1, I_2, \ldots$. Set parameters $\xi = 1$ and $\lambda_0 = 0$.
- (ii) Apply $\lambda(t) = B(I_\xi, t) + C(I_\xi, t)$ to Eq. (26) and calculate $\{\varphi(i,t), i \in \boldsymbol{K}\}$ as follows:

$$\varphi(i,t) = \begin{cases} 0, & i = 1, \ldots, I_\xi, \\ (1/\alpha_i) \ln \{B(i,t)/ (\lambda(t) - C(i,t))\}, & i = I_{\xi+1}, \ldots, I_K \end{cases} \tag{28}$$

- (iii) If $\sum_{i \in K} \varphi(i,t) \leq \Phi(t)$, find an optimal $\lambda^*(t)$ satisfying

$$\sum_{k=\xi+1}^{K} \frac{1}{\alpha_{I_k}} \ln \frac{B(I_k,t)}{\lambda^*(t) - C(I_k,t)} = \Phi(t)$$

  during an interval $[\lambda_0, \lambda(t)]$ by dichotomy, using the monotonically decreasingness of the left-hand side of the above expression. Calculate an optimal solution $\{\varphi(i,t), i \in \boldsymbol{K}\}$ by substituting the $\lambda^*(t)$ into Eq. (28).

  Otherwise, set $\lambda_0 = \lambda(t)$ and $\xi = \xi + 1$. Go back to (ii).

5 A Search Allocation Game and an Numerical Algorithm for Its Equilibrium

In Section 4, we proposed an algorithm to derive an optimal distribution of searching resource for the maximum expected reward of the searcher given a target strategy π. Here, we model a search game with the expected payoff of Eq. (14) and discuss a computational algorithm to derive an equilibrium of the game.

We know that there is an equilibrium for our search game because the expected payoff has the strict concavity for the searcher's strategy φ and the linearity for the target strategy π [24]. At the equilibrium point, the minimax value of the expected payoff equals its maximin value. That is why we focus on the derivation of the maximin value from here. We can transform the maximin optimization as follows.

$$\max_{\varphi} \min_{\pi} R(\varphi, \pi) = \max_{\varphi} \min_{\pi} \sum_{\omega \in \Omega} \pi(\omega) R(\varphi, \omega) = \max_{\varphi} \min_{\omega \in \Omega} R(\varphi, \omega)$$

Therefore, the problem is equivalent to the following convex problem, which gives an optimal strategy of the searcher φ^*.

$$
\begin{aligned}
(P_M) \qquad & \max_{\varphi,\nu} \ \nu & (29)\\
s.t. \qquad & R(\varphi,\omega) \geq \nu,\ \omega \in \Omega, & (30)\\
& \sum_{i\in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}}, & (31)\\
& \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. & (32)
\end{aligned}
$$

Setting dual variables $\eta(\omega)$, $\lambda(t)$ and $\mu(i,t)$ corresponding to conditions (30), (31) and (32), respectively, and making a Lagrangean function

$$
\begin{aligned}
L(\nu,\varphi;\eta,\lambda,\mu) \ \equiv \ & \nu + \sum_{\omega\in\Omega} \eta(\omega)\left(R(\varphi,\omega) - \nu\right) + \sum_{t\in\widehat{\boldsymbol{T}}} \lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right)\\
& + \sum_{i\in K, t\in\widehat{\boldsymbol{T}}} \mu(i,t)\varphi(i,t),
\end{aligned}
$$

we have the following KKT conditions for an optimal solution:

$$
\begin{aligned}
& R(\varphi,\omega) \geq \nu,\ \omega \in \Omega & (33)\\
& \eta(\omega)\left(R(\varphi,\omega) - \nu\right) = 0,\ \omega \in \Omega & (34)\\
& \eta(\omega) \geq 0,\ \omega \in \Omega & (35)\\
& \frac{\partial L}{\partial \nu} = 1 - \sum_{\omega} \eta(\omega) = 0 & (36)\\
& \frac{\partial L}{\partial \varphi(i,t)} = \sum_{\omega\in\Omega} \eta(\omega)\frac{\partial R(\varphi,\omega)}{\partial \varphi(i,t)} - \lambda(t) + \mu(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} & (37)\\
& \lambda(t) \geq 0,\ t \in \widehat{\boldsymbol{T}} & (38)\\
& \mu(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} & (39)\\
& \sum_{i\in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}} & (40)\\
& \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} & (41)\\
& \lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) = 0,\ t \in \widehat{\boldsymbol{T}} & (42)\\
& \mu(i,t)\varphi(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. & (43)
\end{aligned}
$$

Conditions (37)∼(43) imply the necessary and sufficient conditions of optimal searcher's strategy, which correspond to (16)∼(22). Considering problem $\min_\pi \sum_\omega \pi(\omega)R(\varphi,\omega)$, an optimal target's response π to a searcher's strategy φ must be $\pi(\omega) = 0$ if $R(\varphi,\omega) > \nu$, where $\nu \equiv \min_\omega R(\varphi,\omega)$. We can replace the condition with $\pi(\omega)\left(R(\varphi,\omega) - \nu\right) = 0$. Therefore, we can see that η in conditions (33)∼(36) must be an optimal target strategy π^*.

¿From the discussion so far, we can propose a computational algorithm to derive an equilibrium by combining two algorithms. We take the algorithm $\Gamma(\pi)$ for an optimal searcher's strategy φ^*, discussed in Section 4.2. For an optimal target strategy π^*, we utilize conditions (33)∼(36). In the proposed algorithm, we repeat the derivation of φ^*_π

optimally corresponding to π by $\Gamma(\pi)$ while changing π and finally make π satisfy conditions (33)$\sim$(36). We manipulate π using the property about how $R(\varphi^*_\pi, \omega)$ increases/decreases by $\pi(\omega)$, which is stated in the following lemma.

Lemma 1 (Reference [12]). *We set $\pi_1(k) = \pi(k) + \Delta\pi(k)$ for a $k \in \Omega$ and $\pi_1(\omega) = \pi(\omega)$ for any other path $k \in \Omega$ and apply π_1 to Algorithm $\Gamma(\pi_1)$. The derived optimal solution $\varphi^*_{\pi_1}$ makes payoff $R(\varphi^*_{\pi_1}, k)$ of the path k increase if $\Delta\pi(k) > 0$ and decrease if $\Delta\pi(k) < 0$ from the original value $R(\varphi^*_\pi, k)$.*

Proof: Omitted. □

Now we are ready to propose a new algorithm for an equilibrium of our search game from Lemma 1, in which we reach the equilibrium while changing π.

An algorithm for an equilibrium: Λ

(E1) Initialize π to be $\pi(\omega) = 1/|\Omega|$. Set $l = 0$.

(E2) For given π, run Algorithm $\Gamma(\pi)$ to obtain an optimal solution φ^*_π.
Normalize π such that $\sum_\omega \pi(\omega) = 1$.

(E3) Sort values $\{R(\varphi^*_\pi, \omega), \omega \in \Omega\}$ in the increasing order like $W_1 < W_2 < \cdots < W_M$, where W_k is the k-th smallest value of $R(\varphi^*_\pi, \omega)$. We categorize all paths in some sets of paths based on the value such that $\Omega_k \equiv \{\omega \in \Omega | R(\varphi^*_\pi, \omega) = W_k\}$ for $k = 1, \ldots, M$.
If conditions (33)$\sim$(36) are satisfied for $\eta(\omega) = \pi(\omega)$, end.

(E4) Generate a new π as follows. If l is even, increase $\pi(k)$ by a little bit $\Delta\pi(k) > 0$ for a $k \in \Omega_1$. If odd, add a tiny negative amount $\Delta\pi(k) < 0$ to $\pi(k)$ for a path k of $k \in \arg\max_{\{\omega|\pi(\omega)>0\}} R(\varphi^*_\pi, \omega)$.
Increase l by one, $l = l + 1$, and go back to Step (E2).

In Algorithm Λ, we manipulate the expected payoff in such a way that we push down the payoff a little by decreasing the selection probability $\pi(\omega)$ for a path ω with maximum expected payoff $R(\varphi^*_\pi, \omega)$ and lift it up by increasing the selection probability for a path with minimum payoff. The manipulation leads π to the satisfaction of conditions (33)$\sim$(36).

As seen in some numerical algorithms in a general way, the speed of convergence changes depending on tolerance of error and $\Delta\pi(k)$ in this case. We set $\Delta\pi(k)$ in a similar way to Reference [12], as follows. We assume that $\pi(k)$ changes the expect payoff of path k, $R(\varphi^*_k, k)$, in a linear way, within the maximum expected reward by $\pi(k) = 1$, $\overline{R_k} \equiv R(\varphi^*_k, k)$, and the minimum payoff by $\pi(k) = 0$, $\underline{R_k} \equiv \min_{k \neq \omega \in \Omega} R(\varphi^*_\omega, k)$. The assumption teaches us a function $(\overline{R_k} - \underline{R_k})\pi(k) + \underline{R_k}$ of variable $\pi(k)$ and the increasing/decreasing by $\Delta\pi(k) = \gamma/(\overline{R_k} - \underline{R_k})$ if we want to lift-up/push-down the expected payoff by γ. We embed the control of $\pi(k)$ in Algorithm Λ. Anyway, we adopt the following rule to control γ, assuming that k belongs to $W_{M'}$ in Step (E4) for odd l.

(i) In the case of $M' = 1$, $R(\varphi^*_\pi, \omega)$ coincides for any ω of $\pi(\omega) > 0$ and the algorithm ends.

(ii) In the case of $M' = 2$, increase the expected payoff of path $k \in \Omega_1$ by $\gamma = (W_2 - W_1)/2$ and decrease that of $k \in \Omega_2$ by the same amount with the intention of coincidence of the expected payoffs for both paths.

(iii) In the case of $M' > 2$, increase the expected payoff of path $k \in \Omega_1$ by $\gamma = (W_2 - W_1)$ and decrease it for path $k \in \Omega'_M$ by $\gamma = (W_{M'} - W_{M'-1})$.

π is modified to the following π_1 in Step (E4) and normalized to $\widehat{\pi}$ in (E2).

$$\pi_1(k) = \pi(k) + \Delta\pi(k), \quad \pi_1(\omega) = \pi(\omega)\ (k \neq \omega \in \Omega).$$

Let us see that the modification makes larger expected payoff, namely, $R(\varphi^*_\pi, \pi) > R(\varphi^*_{\widehat{\pi}}, \widehat{\pi})$, as follows.

$$\begin{aligned}
\Delta R &= R(\varphi^*_{\widehat{\pi}}, \widehat{\pi}) - R(\varphi^*_\pi, \pi) = \sum_\omega \widehat{\pi}(\omega) R(\varphi^*_{\widehat{\pi}}, \omega) - R(\varphi^*_\pi, \pi) \\
&= \sum_\omega \frac{\pi_1(\omega)}{1+\Delta\pi(k)} R(\varphi^*_{\widehat{\pi}}, \omega) - R(\varphi^*_\pi, \pi) \\
&= \sum_\omega \frac{1}{1+\Delta\pi(k)} \left\{ R(\varphi^*_{\widehat{\pi}}, \pi) + \Delta\pi(k) R(\varphi^*_{\widehat{\pi}}, k) - (1+\Delta\pi(k)) R(\varphi^*_\pi, \pi) \right\} \\
&= \sum_\omega \frac{1}{1+\Delta\pi(k)} \left\{ \Delta\pi(k) \left(R(\varphi^*_{\widehat{\pi}}, k) - R(\varphi^*_\pi, \pi) \right) - \left(R(\varphi^*_\pi, \pi) - R(\varphi^*_{\widehat{\pi}}, \pi) \right) \right\}
\end{aligned}$$

In Step (E4), we have $R(\varphi^*_\pi, k) < R(\varphi^*_\pi, \pi)$ because the selected $k \in \Omega_1$ has the minimum payoff among all paths. Therefore, we can set $\Delta\pi(k)$ small enough to keep $R(\varphi^*_{\widehat{\pi}}, k) < R(\varphi^*_\pi, \pi)$. We also have $R(\varphi^*_\pi, \pi) \geq R(\varphi^*_{\widehat{\pi}}, \pi)$ and then $\Delta R < 0$ for even l in (E4). Similarly, we can see $R(\varphi^*_{\widehat{\pi}}, k) > R(\varphi^*_\pi, \pi)$ for a path $k \in \Omega_{M'}$. From $\Delta\pi(k) < 0$, we also have $\Delta R < 0$ for odd l. We can verify that, in Step (E4), the changing of π controls $R(\varphi^*_\pi, k)$ in the direction of decreasing the expected payoff.

6 Numerical Examples Let a search space and a time space be $\boldsymbol{K} = \{1, \ldots, 5\}$ and $\boldsymbol{T} = \widehat{\boldsymbol{T}} = \{1, \ldots, 10\}$, respectively. We set other parameters as follows: $\Phi(t) = 1$ $(t \in \widehat{\boldsymbol{T}})$ and $\alpha_i = 0.2$ $(i \in \boldsymbol{K})$. Figure 2 shows four target paths, Ω, by illustrating which cell each path runs through time by time in a $|\boldsymbol{K}| \times |\boldsymbol{T}|$ matrix. Path 3 and 4 always stay at Cell 3 and 2, respectively, but Path 1 and 2 run across several cells symmetrically to each other. Crossing points of paths are effective for search operation because the searcher can cover several paths running there at the same time by distributing searching resources there. We can sort all paths into 2, 1, 3 and 4 in the decreasing order in terms of the number of crossing points.

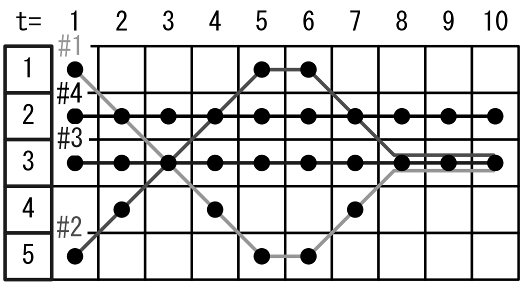

Figure 2: Target paths

We consider the search allocation game with $V(t) = 20$ $(t \in \widehat{\boldsymbol{T}})$ as target value and $c_0(i,t) = 1$ $(i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}})$ as searching cost.

(1) Case of no false contact (Case 1)

In the case of no false contact, we can analyze optimal strategies of players by setting $Q_t = 0$. The value of the game is 5.8. Table 1a shows an optimal distribution plan of searching resource by the searcher and Table 1b indicates the total amount of resources accumulated on each target path and an optimal selection probability of paths by the target.

As seen from Table 1a, the distribution of resource is concentrated in crossing points from cost-effective point of view and the searcher uses up all available resources, $\Phi(t) = 1$, all time points but $t = 1$. At time $t = 1$, the searcher cannot find any crossing point and use a part of $\Phi(t) = 1$. Because of the concentration strategy on crossing points, the most amount of resources are distributed along Path 2 with the largest number of crossing points and the target never takes the path 2, as seen from Table 1b. The target chooses paths 1, 3 and 4, which have similar amount of distributed resources. The searcher decides his distribution strategy of searching resource, taking account of a tradeoff between effective search and target's preference on paths, which is conversely affected by the distribution strategy. On the target side, he tends to avoid the paths on which effective search is easy to be done because of the possession of many crossing points or others, and to make it difficult for the searcher to have easy anticipation on the target's path and concentrate much searching resources on the path.

Table 1a. Optimal distribution of searching resource (Case 1).

Cells \ t	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0.137	1	0	0.718	0.733	1	1	0	0	0
3	0.496	0	1	0.282	0.267	0	0	1	1	1
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
Total	0.634	1	1	1	1	1	1	1	1	1

Table 1b. Accumulated amount of resource and optimal selection of target paths (Case 1)

Paths	1	2	3	4
Amount of accumulated resource	5.0	5.719	5.045	4.589
Selection probability	0.26	0	0.37	0.37

(2) Effect by false contact (Case 2)

In this case, we change some parameters of Case 1 to $Q_t = 0.5$ ($t \in \widehat{\boldsymbol{T}}$) and $t_f = 3$. The false signal makes the search operation less effective and changes the value of the game to 3.3 from 5.8 of Case 1. Equilibrium is shown in Table 2a and 2b. On happening of false contact, the pre-planned distribution of searching resource cannot be done precisely and comes to no use in a sequent investigation process. The fact lessens the value of crossing point on effective search and makes the searcher have the incentive to transfer some resources from crossing points to other points, which is seen from the comparison between Table 1a and Table 2a. At the same time, the target's avoidance to Path 2 and 1 having many crossing points gets weak, comparing with Case 1, and the selection probabilities of Path 1, 3 and 4 become all equal. Whether or not the distribution of searching resource is planned, real false contacts cancel the distribution plan in its sequent investigation process although the searcher does not expend any searching cost. That is the reason why the searcher performs an active behavior to consume more available resources in this case than Case 1. Practically, the searcher exhausts $\Phi(t) = 1$ at time $t = 1$.

Table 2a. Optimal distribution of searching resource (Case 2)

Cells \ t	1	2	3	4	5	6	7	8	9	10
1	0.05	0	0	0	0	0	0	0	0	0
2	0.414	1	0	0.842	0.586	0.549	0.772	0	0	0
3	0.536	0	1	0	0.218	0.366	0.047	1	1	1
4	0	0	0	0.158	0	0	0.181	0	0	0
5	0	0	0	0	0.196	0.085	0	0	0	0
Total	1	1	1	1	1	1	1	1	1	1

Table 2b. Accumulated amount of resource and optimal selection of target paths (Case 2)

Paths	1	2	3	4
Amount of accumulated resource	5.67	5.614	5.168	4.162
Selection probability	0.327	0.019	0.327	0.327

(3) Effect of target value (Case 3)

In this case, we increase target value $V(t)$ to 50 from 20 of Case 2. The value of the game is 15.0 although it was 3.3 in Case 2. Table 3a and 3b show an equilibrium in this case. Since the target value increases, the searcher accelerates the distribution of searching resource, expecting higher reward on detection of target and takes the strategy of exhaustion of available resource (EAR strategy). Therefore, the optimal strategy of this case is very similar to Case 2.

Table 3a. Optimal distribution of searching resource (Case 3)

Cells \ t	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0.573	1	0	0.822	0.529	0.542	0.654	0	0	0
3	0.427	0	1	0	0.285	0.369	0.147	1	1	1
4	0	0	0	0.178	0	0	0.199	0	0	0
5	0	0	0	0	0.186	0.089	0	0	0	0
Total	1	1	1	1	1	1	1	1	1	1

Table 3b. Accumulated amount of resource and optimal selection of target paths (Case 3)

Paths	1	2	3	4
Amount of accumulated resource	5.651	5.477	5.229	4.12
Selection probability	0.327	0.016	0.329	0.329

(4) Effect of distribution cost (Case 4)

We increase cost $c_0(i,t)$ to 4 from 1 of Case 2. The value of the game is 2.7, which is smaller than Case 2. The searcher makes much account of the effectiveness of cell for search and avoids distributing searching resource into ineffective cells other than crossing points. He takes the strategy of partially-using available resource (PAR strategy) of $\Phi(t) = 1$ at time $t = 1, 4, 5, 6$. The focus on crossing points by the searcher decreases the selection probabilities of Path 2 and 1, having many crossing points, but increases those of Path 3 and 4, compared with Case 2.

Table 4a. Optimal distribution of searching resource (Case 4)

Cells \ t	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0.374	1	0	0.697	0.379	0.465	0.435	0	0	0
3	0.139	0	1	0	0.256	0.222	0.223	1	1	1
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
Total	0.514	1	1	0.697	0.687	0.658	1	1	1	1

Table 4b. Accumulated amount of resource and optimal selection of target paths (Case 4)

Paths	1	2	3	4
Amount of accumulated resource	5	5.132	4.84	3.35
Selection probability	0.117	0	0.442	0.442

From studies in Case 1∼4, we enumerate some properties on optimal strategy of player.

1. The distribution plan of searching resource is certainly executed in the case of no false contact. In the false contact model, however, the plan is not necessarily done and the usage of resource has some uncertainty. The fact lessens the effectiveness of search in crossing points, leads the searcher to pay attention to other points for search and mitigates the target's avoidance to paths with many crossing points. The cancellation of the distribution plan means the no-expense of searching cost as well as the no-use of searching resource and the searcher has more activeness to use resource in the presence of false contacts. The value of the game becomes smaller, of course.

2. The increase of target value brings larger value of the game and makes the searcher more active for search. We could enumerate other influences by the increase of target value on the game.

   (1) The searching cost is relatively getting smaller compared with the target value and then the searcher tends to use more resource at the time when he takes the PAR strategy in the case of lower target value. The searcher's tendency causes more resources being distributed in non-crossing points. Because of that, the target is going to increase the selection probability of the paths with many crossing points and to decrease the probability of other paths.

   (2) In the case that the searcher already takes the EAR strategy for available resource every time, the amount of used resource cannot be increased and optimal strategies of players do not change a lot even though the target value gets larger.

3. In the case that the distribution cost of resource increases, the value of the game decreases and the searcher has less incentive to distribute his searching resource. If such a situation causes the decrease of the amount of distributed resources, the decreasing would be mainly adopted to non-crossing points and the target takes paths with many crossing points with less probability and other paths with more probability.

7 Conclusion This paper deals with a search game with false contacts in a precise manner. The false contacts are thought to happen independent of detection of true targets and their occurrence probability might be constant in the search operation. The false contact is inevitable to step in the search operation and then it is a most serious phenomenon we should take account of to analyze search operations using any sensor. Nevertheless, there have been just a few researches on the topics because the false contacts are difficult to deal with in terms of modeling and solution.

In this paper, we deal with the false contact as an event with fixed occurrence probability and enumerate all instances with the false contacts as well as the detection of target. At first, we formulate a maximization problem with the objective of the expected reward on the searcher's side into a nonlinear programming problem and propose a numerical algorithm to derive an optimal distribution of searching resource in a search space. Then we analyze the properties of optimal target strategy of taking several paths moving in the space. For the search problem on both sides of the searcher and the target, i.e. the search allocation game, we embed the properties of optimal target strategy in the algorithm proposed for optimal searcher's strategy to construct a repetition algorithm to derive an equilibrium point for the game. We define the payoff of the game as the target value gained by its detection minus the searching cost expended to execute a distribution plan of searching resource, which is a general criterion including the detection probability of target. As long as we take a direction of dealing with the false contact in a precise and practical manner, the dependency of occurrence of the false contacts on cell, time and the amount of distributed searching resource would be future topics to handle.

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Ryusuke Hohzaki
Department of Computer Science
National Defense Academy
1-10-20 Hashirimizu, Yokosuka,
239-8686, Japan
E-mail: hozaki@cc.nda.ac.jp