A NEW CHARACTERIZATION OF ACG^*-FUNCTIONS

Surinder Pal Singh and Inder. K. Rana

Received November 23, 2012; revised April 19, 2013

Abstract. In this paper, we propose an equivalent form of the restricted generalized absolute continuity, which characterizes the Denjoy integral of real valued functions.

1 Introduction The notion of absolute continuity and its relation with generalized forms of integrable functions has been studied by various authors, see e.g. [3, 4]. An extension of absolute continuity is the restricted generalized absolute continuity, denoted by ACG^*.

We characterize the class of ACG^*-functions in a much simpler manner, waiving off the continuity hypothesis and the oscillations therein and call them ACG_V-functions. The continuity will be intrinsically involved in ACG_V-functions, giving more insights to the fact that ACG^*-functions belong to the class of ACG_δ-functions. This further gives an alternative proof of the fact that every real valued Denjoy integrable function is Henstock-Kurzweil integrable.

We shall also discuss some questions arising from the standard Radon-Nikodym theorem, reflecting the essence of these extensions of absolute continuity.

2 Preliminaries Let $I = [a, b]$ be a compact real interval and μ be the Lebesgue measure on I. For a subinterval $J \subset I$, let $Sub(J)$ denotes the collection of subintervals of J and $\mathcal{F}(J)$ be the algebra generated by $Sub(J)$. Finally, let $\mathcal{F}$ denotes the algebra $\mathcal{F}(I)$.

Given a function $F : I \to \mathbb{R}$, we define the corresponding finitely additive set function on $\mathcal{F}$, still denoted by F, such that

$$F(\cup_{i=1}^{p}[c_i, d_i]) = \sum_{i=1}^{p}(F(d_i) - F(c_i))$$

holds for each non-overlapping finite collection $\{[c_i, d_i] : i = 1, 2, \dots, p\}$ of subintervals in $[a, b]$. Similarly, given a set function F on $\mathcal{F}$, we define the corresponding point function $F(x) = F([a, x])$, for each $x \in [a, b]$.

Throughout this paper, F will denote a set function on $\mathcal{F}$ as well as the corresponding point function on I and vice versa. Whenever we would have to deal only with the point functions we shall denote them by small letters, mostly by f. The oscillation of F on an interval $J \subset I$ is defined as

$$\omega(F, J) = sup\{|F(K)| : K \in Sub(J)\}.$$

Definition 1 (ACG^***-functions**).

(i) A function $F : I \to \mathbb{R}$ is called AC^* over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{p}\omega(F, J_i) < \varepsilon$ holds for each non-overlapping collection $\{J_i : i = 1, 2, \dots, p\}$ of intervals in I such that both end points of each J_i belong to the set E and $\sum_{i=1}^{p}\mu(J_i) < \delta$.

2010 *Mathematics Subject Classification.* 26A39, 28B05.

Key words and phrases. Henstock-Kurzweil integral, charges and generalized absolute continuity .

(ii) *The function F is said to be generalized absolutely continuous in the restricted sense (namely ACG^*) over some set $E \subset I$ if F is continuous and E can be written as a countable union of sets E_n on each of which F is AC^*.*

It is easy to see that if a function F is absolutely continuous over some interval $J \subset I$, then F is AC^* over J. Thus every absolutely continuous function is ACG^*. Before we define another important generalization of absolute continuity, we need to fix certain notions.

A collection $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ of point-interval pairs is said to be a *partial division* in I if I_i's are mutually disjoint compact intervals in I and $t_i \in I_i$, for each i. If further, $\cup_{i=1}^{p} I_i = I$, it is called a *division* of I.

A positive valued function $\delta : I \to (0, \infty)$ is called *a gauge on* I and a partial division $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ of I is called δ*-fine* if for every i, we have $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$.

Now we present the class of ACG_δ-functions and the Henstock-Kurzweil integral, as follows:

Definition 2 (ACG_δ**-functions).**

(i) *A function F is said to be AC_δ over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a positive number $\eta > 0$ and a gauge $\delta : E \to (0, \infty)$ such that $\sum_{i=1}^{p} |F(J_i)| < \varepsilon$ holds for each δ-fine partial division $\{(t_i, J_i) : i = 1, 2, \ldots, p\}$ in I such that each tag $t_i \in E$ and $\sum_{i=1}^{p} \mu(J_i) < \eta$.*

(ii) *The function F is said to be ACG_δ over some set $E \subset I$ if E can be written as a union of countable collection of sets $\{E_n : n \in \mathbb{N}\}$ on each of which F is AC_δ.*

Definition 3 (**The Henstock-Kurzweil integral**). *A function $f : I \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable (or simply HK-integrable), with $A \in \mathbb{R}$ as its integral, if for every $\varepsilon > 0$ there is a gauge $\delta : I \to (0, \infty)$ such that the inequality*

$$\left| \sum_{i=1}^{p} f(t_i)\mu(I_i) - A \right| < \varepsilon$$

is satisfied for all δ-fine divisions $\{(t_i, I_i) : i = 1, \ldots, p\}$ of I.

The Henstock-Kurzweil integral of f over I is denoted by $(HK) \int_I f d\mu$. A function $F : \mathcal{F} \to \mathbb{R}$ is called the *primitive* of f if $F(J) = (HK) \int_J f d\mu$, for each $J \in \mathcal{F}$.

It is well known that the Henstock-Kurzweil integral on real line generalizes the notions of Riemann, Lebesgue and improper integrals. By keeping the tags independent of the subintervals, the McShane integral is defined which is equivalent to the Lebesgue integral.

Further, by replacing the subintervals in HK-integration with measurable sets or closed sets or by making the tags free, we are restricted to the McShane integration. For more details about such integrals, see [2, 5].

3 Absolute Continuity of Charges and the HK-integral .

Let $f : I \to R$ be a Henstock-Kuzweil integrable function with primitive F, that is

$$F(A) = (HK) \int_A f d\mu \text{ for all } A \in \mathcal{F}. \tag{1}$$

Then F is finitely additive on $\mathcal{F}$ and is called the *charge associated with* f.

The notion of absolute continuity for charges can be defined similar to that of measures (abbreviated as $F \ll \mu$). Let Ω denote the Lebesgue σ-algebra on I. Recall the Radon-Nikodym theorem for measures.

Theorem 4 (Radon-Nikodym)**.** *A measure $F : \Omega \longrightarrow \mathbb{R}$ is absolutely continuous with respect to μ (abbreviated as $F \ll \mu$) if and only if there exists a Lebesgue integrable function $f : I \to \mathbb{R}$ satisfying*

$$F(E) = \int_E f d\mu \text{ for all } E \in \Omega.$$

A corresponding result for charges is not known, see [1] for more details. Since the primitive of an HK-integrable function need not be countably additive, consider the following questions:

(i) Let F be a charge such that $F \ll \mu$. Does there exist an HK-integrable function f such that (1) holds true?

(ii) Let $F : \mathcal{F} \longrightarrow R$ be the charge associated with an HK-integrable function $f : I \longrightarrow R$. Is $F \ll \mu$?

The first question is answered affirmatively and the second one negatively, even for functions on metric measure spaces, in [7]. Further, some Radon-Nikodym type properties for the Henstock-Kurzweil integral have been proved in [6].

Thus the primitives of Henstock-Kurzweil integrable functions may not be absolutely continuous. However, these primitives happen to be the class of ACG^*-functions, see [2, Theorem 11.4].

Now we define some other extensions of the absolute continuity and prove them to be equivalent to the restricted generalized absolute continuity, ACG^*.

Definition 5 (ACG_V^***-functions**)**.**

(i) A function F is is said to be AC_V^* over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying $\sum_{i=1}^{p} \omega(F, J_i) < \varepsilon$ for each non-overlapping finite collection of intervals $\{J_i : i = 1, 2, \ldots, p\}$ in I such that at least one point of each J_i belongs to the set E and $\sum_{i=1}^{p} \mu(J_i) < \delta$.

(ii) The function F is said to be ACG_V^* over some set $E \subset I$ if E can be written as a countable union of sets E_n on each of which F is AC_V^*.

Definition 6 (ACG_V**-functions**)**.**

(i) A function F is said to be AC_V over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying $\sum_{i=1}^{p} |F(J_i)| < \varepsilon$ for each non-overlapping finite collection of intervals $\{J_i : i = 1, 2, \ldots, p\}$ in I such that at least one point of each J_i belong to the set E and $\sum_{i=1}^{p} \mu(J_i) < \delta$.

(ii) The function F is said to be ACG_V over some set $E \subset I$ if E can be written as a countable union of sets E_n on each of which F is AC_V.

It should be noted that we don't assume the continuity of F in both of the above definitions. Also, as earlier, it is easy to see that if F is absolutely continuous on I then it is ACG_V as well as ACG_V^* over I.

The main motivation behind these generalizations was the following result from [3, page 27]:

Lemma 7. *Let F be a continuous function on I and $A \subset I$ be a closed set then F is AC^* over A if and only if F is AC_V over A.*

A close observation to the proof of this lemma shows that the analogous result holds true even if AC_V is replaced with AC_V^*. Now we prove the main result of this paper.

Theorem 8. *For any function $F : I \to \mathbb{R}$, the following are equivalent:*

(i) F is ACG^* over I

(ii) F is ACG^*_V over I

(iii) F is ACG_V over I

(iv) F is ACG_δ over I.

Proof. Let F be an ACG^* function. Then F is continuous and I can be written as a countable union of sets E_n on each of which F is AC^*. Using [3, Theorem 6.7], we may assume that each E_n is a closed set. Further an application of Lemma 7 implies that F is AC^*_V and AC_V over each E_n. This proves that F is ACG^*_V as well as ACG_V.

It is easy to observe that if F is an ACG^*_V function over I then F is an ACG_V function over I.

Now let F be an ACG_V function. Let E_n be the countable collection of sets on each of which F is AC_V and $\cup_n E_n = I$. Fix some $n \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Choose $\delta > 0$ corresponding to $\varepsilon/2$, as per the definition of F being AC_V over E_n. Then we choose a gauge $\eta : I \to (0, \infty)$ such that $\eta(t) = b - a$ for all $t \in I$. Let $\{(t_i, [c_i, d_i] : 1, \ldots, p\}$ be a η-fine partial division in I which is tagged in E_n and $\sum_i (d_i - c_i) < \delta$. We refine the partitions $\{[c_i, d_i]\}$ by further splitting the intervals $[c_i, d_i]$ into further two subintervals $[c_i, t_i]$ and $[t_i, d_i]$. The resulting two partitions in I are suitable for the AC_V case with E_n. Thus we have

$$\sum_{i=1}^{p} |F(d_i) - F(c_i)| \leq \sum_{i=1}^{p} |F(d_i) - F(t_i)| + \sum_{i=1}^{p} |F(t_i) - F(c_i)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that if F is ACG_V over I then F is ACG_δ over I.

Finally, for the proof of the fact ACG_δ-functions are ACG^*, see [2, page 339]. □

Thus we see that the continuity of F is intrinsically assumed in ACG^*_V, ACG_V and ACG_δ, as these function classes are equivalent to the class of ACG^*-functions. The class of ACG_V-functions seems simplest to handle since it is relieved from continuity, oscillations, gauges and their corresponding fine partitions.

Remarks: There are various extensions to the classes of ACG^*-functions over finite dimensional Euclidean spaces. Lee Tuo-Yeong has proved some of them to be equivalent to the primitives of HK-integrable functions, see [4].

If we directly extend the notion of ACG_δ-functions over $\mathbb{R}^m$, the following question remains open:

What is the relation between the class of real valued ACG_δ-functions over finite dimensional Euclidean spaces and the primitives of Henstock-Kurzweil integrable functions?

References

[1] C. Fefferman, *A Radon-Nikodym theorem for finitely additive set functions*, Pacific J. Math. 23 (1967), 35–45.

[2] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., Providence, RI, 1994.

[3] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, (1989).

[4] Tuo-Yeong Lee, *Some full descriptive characterizations of the Henstock-Kurzweil integral in the Euclidean space*, Czechoslovak Math. J. 55(130) (2005), no. 3, 625–637.

[5] Surinder Pal Singh and Inder K. Rana *Some alternative approaches to the McShane integral*, Real Anal. Exchange 35 (2009/2010) no.1, 229-233.

[6] Surinder Pal Singh and Inder K. Rana *The Hake's theorem and variational measures.*, Real Analysis Exchange Vol 37 No 2, (2011/2012), 477-488.

[7] Surinder Pal Singh and Inder K. Rana *Absolute continuity of charges on metric measure spaces.*, *communicated*.

Communicated by *Eiichi Nakai*

Surinder Pal Singh, Department of Mathematics, Guru Nanak Dev University, Amritsar, India, email: spsingh.math@gmail.com
Inder K. Rana, Department of Mathematics, IIT Bombay, India, email: ikr@math.iitb.ac.in