TIGHTLY BORDERED CONVEX AND CO-CONVEX SETS

Douglas S. Bridges

Received July 30, 2012 ; revised May 24, 2013

Abstract. We investigate conditions under which a convex or co-convex set in a normed space is tightly bordered, in the sense that a point of the set that is bounded away from its boundary lies in the interior of the set. The investigation lies entirely within a constructive framework.

1 Introduction We say that a subset S of a metric space is **tightly bordered**, or has a **tight border,** if $x \in S^{\circ}$ for each $x \in S$ with $\rho(x, \partial S) > 0$.[1] Every open set is tightly bordered. With classical logic, the law of excluded middle (**LEM**) leads to every subset of a metric space being tightly bordered. In constructive mathematics,[2] things are not so simple: if $x \in S$ and $\rho(x, \partial S) > 0$, then it is absurd that $x \notin S^{\circ}$; but this information does not, of itself, enable us to compute $r > 0$ such that the ball $B(x, r)$ is contained in S. This is part of a more general difficulty in constructive geometry and analysis: namely, placing a point in a set (a positive conclusion) when all we know is that it cannot fail to belong therein (negative information). This situation really can arise in constructive practice, so it makes sense to try to provide conditions on the set S that ensure, constructively, that it is tightly bordered. We discuss such conditions in this paper, which can be regarded as a continuation of work begun in [9]. That work arose naturally in a constructive study of the Dirichlet problem (for more on which, see [6]); our present study was motivated by an ongoing search for the 'right' definition of a differential manifold in constructive analysis.

Our visual intuition suggests that when we are dealing with a convex set or a co-convex set—that is, the complement of a convex one—in a normed space, we might be able to prove the tightness of the border.[3] In fact, as the Brouwerian examples in the final section of this paper show, even for such relatively special sets, we cannot expect to do that without additional hypotheses. Our main purpose is to discuss, in Sections 2 and 3, conditions under which a convex set C or its complement

$$\sim C \equiv \{x \in X : \forall_{y \in C} (\|x - y\| > 0)\}$$

is tightly bordered. In particular, we show that if, in a Banach space, a convex set C has inhabited interior and $C \cup \sim C$ is dense in X, then both C and $\sim C$ are tightly bordered (Propositions 5 and 13). In the course of our discussion, we also deal with a number of classically trivial, but constructively significant, geometric properties of convex and co-convex sets.

[1] We do not require that ∂S be located: that is, that the distance from any point of X to ∂S exist. Instead, we are using Richman's convention about distance expressions (see [9]), under which, for example, the expression $\rho(x, \partial S) > 0$ means that there exists $r > 0$ such that $\rho(x, y) > r$ for each $y \in \partial S$.

[2] That is, roughly, mathematics with intuitionistic, rather than classical, logic, and with an appropriate foundation such as those presented in [1, 2, 11]. For more on this type of constructive mathematics in practice, see [3, 4, 7, 8].

[3] For convex sets we can often establish results that hold more generally in classical, but not in constructive, analysis. For example, every convex subset C of $\mathbf{R}^n$ with positive Lebesgue measurable is located—that is, $\rho(x, C) \equiv \inf\{\|x - y\| : y \in C\}$ exists for each $x \in \mathbf{R}^n$; see [5].

2 Tightly bordered convex sets Although we assume that the reader has access to one or more of such books on constructive analysis as [3, 4, 7, 8, 13], it is convenient for all if we quote two results.

Lemma 1 *let C be a convex subset of a normed space X, let $\xi \in C^\circ$, and let $r > 0$ be such that the ball $B(\xi, r)$ is contained in C. Let $z \neq \xi$, $0 < t < 1$, and $z' = t\xi + (1-t)z$. If $B(z, tr)$ intersects C, then $B(z', t^2 r) \subset C$* ([8], Lemma 5.1.1).

Note that for points x, y in a normed space,

$$[x, y] \equiv \{tx + (1-t)y : 0 \leqslant t \leqslant 1\}.$$

We adopt other natural notations for 'intervals' joining x and y without further comment.

Proposition 2 *Let C be a subset of a Banach space such that $C \cup {\sim}C$ is dense, let $x \in C$ and $y \in {\sim}C$, and let $\varepsilon > 0$. Then there exists $z \in \partial C$ such that $\rho(z, [x, y]) < \varepsilon$* ([9], Proposition 8).

Lemma 1 and Proposition 2 are two of several results in convex geometry that will be found throughout the paper. Here is the next one.

Proposition 3 *Let C be a convex subset of a Banach space X such that C° is inhabited. Then $\overline{C}^\circ = C^\circ$.*

Proof. Construct $\xi \in C$ and $r > 0$ such that $B(\xi, r) \subset C$. Let $x \in \overline{C}^\circ$. In trying to prove that $x \in C^\circ$, we may assume that $\|x - \xi\| > r$. Pick s such that $0 < s < r$ and $\overline{B}(x, s) \subset \overline{C}$. Let

$$0 < t < \frac{s}{\|x - \xi\|} \quad \text{and} \quad z = \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $0 < t < 1$ and $x = t\xi + (1-t)z$. Moreover, $\|x - z\| = t\|x - \xi\| < s$, so $z \in \overline{C}$. Hence $B(z, tr)$ intersects C, and therefore, by Lemma 1, $B(x, t^2 r) \subset C$. Thus $x \in C^\circ$. ∎

Proposition 4 *Let C be a convex subset of a Banach space X such that $C \cup {\sim}C$ is dense in X and $\overline{C}^\circ = C^\circ$. Then C is tightly bordered.*

Proof. Let x be a point of C with $\rho(x, \partial C) > 0$. Choose r such that $0 < 2r < \rho(x, \partial C)$, and consider any $y \in B(x, r)$. Suppose that $y \in {\sim}C$. Applying Proposition 2, we can find $z \in \partial C$ and $t \in [0, 1]$ such that $\|z - (1-t)x - ty\| < r$. Then

$$\begin{aligned} \|x - z\| &\leqslant \|x - (1-t)x - ty\| + \|z - (1-t)x - ty\| \\ &< \|x - y\| + r < 2r, \end{aligned}$$

which contradicts our choice of r. It follows that $y \notin {\sim}C$. Since y is arbitrary, we conclude that $B(x, r) \cap {\sim}C$ is empty, and hence, by the density of $C \cup {\sim}C$, that $B(x, r) \subset \overline{C}$. Thus $x \in \overline{C}^\circ = C^\circ$. ∎

Proposition 5 *Let C be a convex subset of a Banach space X such that C° is inhabited and $C \cup {\sim}C$ is dense in X. Then C is tightly bordered. Moreover, if ξ is an interior point of C and $\|x - \xi\| < \rho(\xi, \partial C)$, then $x \in C^\circ$.*

Proof. Propositions 3 and 4 together show that C is tightly bordered. Given $\xi \in C^{\circ}$ and $x \in X$ with $\|x - \xi\| < \rho(\xi, \partial C)$, pick $r > 0$ such that $\|x - \xi\| + 3r < \rho(\xi, \partial C)$ and $B(\xi, r) \subset C$. In proving that $x \in C^{\circ}$, we may assume that $\|x - \xi\| > r/2$. Let

$$t = \frac{r}{\|x - \xi\| + r} \quad \text{and} \quad z \equiv \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $0 < t < 1$, $x = t\xi + (1-t)z$,

$$\|\xi - z\| = \frac{1}{1-t}\|x - \xi\| < \|x - \xi\| + r,$$

and $z \neq \xi$. Now choose $\zeta \in C \cup {\sim}C$ such that $\|z - \zeta\| < tr$. If $\zeta \in {\sim}C$, then by Proposition 2, there exist $\eta \in [\xi, \zeta]$ and $y \in \partial C$ such that $\|\eta - y\| < tr$; in that case,

$$\begin{aligned} \|\xi - y\| &\leqslant \|\xi - \eta\| + \|\eta - y\| \\ &\leqslant \|\xi - \zeta\| + tr \\ &\leqslant \|\xi - z\| + \|z - \zeta\| + r \\ &\leqslant \|x - \xi\| + r + tr + r < \rho(\xi, \partial C), \end{aligned}$$

a contradiction. Hence $\zeta \in C \cap B(z, tr)$. Applying Lemma 1, we now see that $B(x, t^2 r) \subset C$, so $x \in C^{\circ}$. ∎

Recall that the **metric complement** of a set S in a metric space X is the set

$$-S \equiv \{x \in X : \rho(x, S) > 0\},$$

and (from [9]) that S is **coherent** if $-{\sim}S \subset S$.

Proposition 6 *Let C be a coherent, convex subset of a normed space X such that $C \cup {\sim}C$ is dense in X. Then C is tightly bordered.*

Proof. Let $x \in C$ with $\rho(x, \partial C) > 0$, and choose $r > 0$ such that $\rho(x, \partial C) \geqslant 3r$. Given $y \in B(x, r)$, suppose there exists $z \in {\sim}C$ such that $\|y - z\| < r$. Since $C \cup {\sim}C$ is dense in X, we can apply Proposition 2 to produce $b \in [x, z]$ such that $\rho(b, \partial C) < r$; whence

$$\begin{aligned} \rho(x, \partial C) &\leqslant \|x - b\| + \rho(b, \partial C) \\ &\leqslant \|x - z\| + r \\ &\leqslant \|x - y\| + \|y - z\| + r < 3r, \end{aligned}$$

a contradiction from which we conclude that $\rho(y, {\sim}C) \geqslant r$. It follows from the coherence of C that $y \in C$. Hence $B(x, r) \subset C$ and therefore $x \in C^{\circ}$. ∎

We shall return to coherence towards the end of the next section.

3 Tightly bordered co-convex sets

When can we be sure that the *complement* of a convex subset of a normed space is tightly bordered? Our answer depends on some additional results on convex geometry—in particular, one on boundary crossings (Proposition 11), improving Proposition 5.1.5 of [8].

Proposition 7 *The interior of a convex subset C of a normed space X is convex. If also C° is inhabited, then it is dense in C. Moreover, $-C^{\circ} = -C$.*

Proof. If $x, y \in C^\circ$, then there exists $r > 0$ such that $B(x,r) \subset C$ and $B(y,r) \subset C$. Given t with $0 \leqslant t \leqslant 1$, we see that $z \equiv tx + (1-t)y$ belongs to C. In order to prove that $z \in C^\circ$, we may assume that $\|z - x\| > r/2$ and $\|z - y\| > r/2$, in which case $0 < t < 1$. Since $y \in B(y, tr) \cap C$, we now see from Lemma 1 that $B(z, t^2 r) \subset C$; whence $z \in C^\circ$. Thus C° is convex.

Now assume that there is a point $\xi \in C^\circ$. For each $x \in C$, either $\|x - \xi\|$ is so small that $x \in C^\circ$ or else $x \neq \xi$. In the latter case, Lemma 1 tells us that for each $t \in (0,1)$, the point $t\xi + (1-t)x$ belongs to C°. Letting $t \to 0$, we see that $x \in \overline{C^\circ}$.

Finally, since $-C \subset -C^\circ$ and C° is dense in C, it readily follows that $-C^\circ = -C$. ∎

Proposition 8 *Let X be a normed space, and C a convex subset of X with inhabited interior. Then $-C$ is dense in both $\sim C$ and $\sim C^\circ$.*

Proof. By Proposition 7, C° is convex, and $-C^\circ = -C$. Applying Lemma 5.1.4 of [8] to C°, we find that $-C$ is dense in $\sim C^\circ$. But $-C \subset \sim C \subset \sim C^\circ$, so $-C$ is also dense in $\sim C$. ∎

We digress briefly in order to establish the conclusion of the preceding proposition under different hypotheses. This requires us to state the following **ridiculously useful lemma** (Lemma 5.1.3 of [8]):

> *Let X be a normed space, let x_1, x_2 be distinct points of X, and let $x_3 = \lambda x_1 + (1-\lambda)x_2$ with $\lambda \neq 0, 1$. For all $\alpha, \beta > 0$, if $\|x - x_1\| \leqslant \alpha/|\lambda|$ and $\|y - x_2\| \leqslant \beta/|1-\lambda|$, then $\|\lambda x + (1-\lambda)y - x_3\| \leqslant \alpha + \beta$.*

Lemma 9 *Let C be an inhabited, convex subset of a finite-dimensional Banach space X, and let $x \in -(-C)$. Then $\neg\neg(x \in C)$.*

Proof. Translating if necessary, we may assume that $0 \in C$. Let n be the dimension of X; if $n = 0$, then the conclusion is trivial; so we may assume that $n \geqslant 1$. Fix $r > 0$ such that $B(x,r) \subset -(-C)$. In order to derive a contradiction, assume that $x \notin C$. Suppose that C contains n linearly independent vectors $x_1, \ldots, x_n$. Then if contains a nondegenerate ball $B(y,t)$ in the interior of the (convex) simplex with vertices $0, x_1, \ldots, x_n$. Since $x \notin C$, we have $\|x - y\| \geqslant t$. Let $z = \lambda x + (1-\lambda)y$, where

$$\lambda = 1 + \frac{r}{2\|x-y\|}$$

Then $\|z - x\| = r/2$ and

$$y = \frac{\lambda}{\lambda - 1}x - \frac{1}{\lambda - 1}z.$$

Pick s such that

$$0 < s < \min\left\{t, \frac{r}{2(\lambda - 1)}\right\},$$

and apply the ridiculously useful lemma with $x_1 = x, x_2 = z, \alpha = 0$, and $\beta = s$. We find that if $\|z - \zeta\| < s(\lambda - 1)$, then

$$\left\|\left(\frac{\lambda}{\lambda - 1}x - \frac{1}{\lambda - 1}\zeta\right) - y\right\| < s < t,$$

so

$$\frac{\lambda}{\lambda - 1}x - \frac{1}{\lambda - 1}\zeta \in C.$$

Also,

$$\begin{aligned}\|\zeta - x\| &\leqslant \|\zeta - z\| + \|z - x\| \\ &< s(\lambda - 1) + \frac{r}{2} < r,\end{aligned}$$

so $\zeta \in -(-C)$. If $\zeta \in C$, then, since

$$x = \frac{1}{\lambda}\zeta + \left(1 - \frac{1}{\lambda}\right)\left(\frac{\lambda}{\lambda - 1}x - \frac{1}{\lambda - 1}\zeta\right),$$

the convexity of C yields $x \in C$, contradicting our assumption that $x \notin C$. We conclude that $\zeta \notin C$ for each $\zeta \in B(z, s(\lambda - 1))$; whence $z \subset -C$, which is also absurd, since $B(z, r/2) \subset B(x, r) \subset -(-C)$. It follows from all this that that C cannot contain n linearly independent vectors in X.

Next, suppose that for some k with $1 \leqslant k \leqslant n$, we have proved that C cannot contain k linearly independent vectors in X. Suppose that C contains $k - 1$ linearly independent vectors, and let V be the finite-dimensional subspace of X spanned by those vectors. If there exists $y \in C \cap -V$, then C contains k linearly independent vectors, a contradiction. Hence $C \subset \overline{V} = V$. Since $k - 1 < n$, there exists a point $z \in B(x, r) \cap -V$; then $z \in -C \cap B(x, r)$, which is impossible. This completes the inductive proof that C cannot contain k linearly independent vectors for any k with $1 \leqslant k \leqslant n$. It now follows that $C = \{0\}$, so we can find an element y of $B(x, r)$ with positive norm. Then $y \in -C \cap B(x, r)$, a final contradiction that ensures that $\neg(x \notin C)$. ∎

Proposition 10 *Let C be an inhabited, located, convex subset of a finite-dimensional Banach space, such that $\sim C$ is inhabited. Then $\sim C$ is dense in $-C$.*

Proof. Let $x \in \sim C$. If $\rho(x, -C) > 0$, then $x \in -(-C)$, so, by Lemma 9, $\neg\neg(x \in C)$, a contradiction. Hence $\rho(x, -C) = 0$. ∎

We now return to our main path, with the promised improvement on Proposition 5.1.1 of [8].

Proposition 11 *Let X be a Banach space, C a convex subset of X such that $C \cup \sim C$ is dense in X, and ξ an interior point of C. Let $z \in -C$, and for each $t \in [0, 1]$ write*

$$z_t \equiv (1 - t)\xi + tz.$$

Then the following hold:

(i) $\gamma(\xi, z) \equiv \inf\{t \in [0, 1] : z_t \in C\}$ *exists, and* $0 < \gamma(\xi, z) < 1$.

(ii) $z_{\gamma(\xi,z)}$ *is the unique intersection of* $[\xi, z]$ *with* ∂C.

(iii) *If* $\gamma(\xi, z) < t \leqslant 1$, *then* $z_t \in C^\circ$.

(iv) *If* $0 \leqslant t < \gamma(\xi, z)$, *then* $z_t \in -C$.

Moreover, the mapping $(\xi, z) \rightsquigarrow z_{\gamma(\xi,z)}$ is continuous at each point of $C^\circ \times -C$.

Proof. By Proposition 7, C° is convex and dense in C, and $-C^\circ = -C$. On the other hand, Proposition 8 shows that $-C$ is dense in $\sim C$. Hence $C^\circ \cup -C^\circ$ is dense in $C \cup \sim C$ and therefore in X. Moreover, since Proposition 8 also gives $-C^\circ$ dense in $\sim C^\circ$, we have

$$\partial C^\circ = \overline{C^\circ} \cap \overline{\sim C^\circ} = C \cap \overline{-C^\circ} = C \cap \overline{-C} = C \cap \overline{\sim C} = \partial C.$$

Applying Proposition 5.1.5 of [8] to C°, and again using both the density of C° in C and the identity $-C^\circ = -C$, we now see that $\gamma(\xi, z)$ exists and satisfies (i)–(iv), and that $\gamma : C^\circ \times -C \to \partial C$ is pointwise continuous. ∎

One more lemma and we are ready to deal with co-convex sets and tight borders.

Lemma 12 *Let X be a Banach space, and C a convex subset of X with inhabited interior such that $C \cup \sim C$ is dense in X. Then $\partial(\sim C) = \partial C$.*

Proof. It is clear that $\partial C \subset \partial(\sim C)$. For the reverse inclusion, first fix ξ in C° and $r > 0$ such that $B(\xi, r) \subset C$. Given $v \in \partial(\sim C)$, we have $v \neq \xi$. Set $z \equiv 2v - \xi$ and note that $v \in (\xi, z)$. Taking $x_1 = v, x_2 = z, x_3 = \xi, \lambda = 2$, and $\alpha = \beta = r/2$ in the ridiculously useful lemma, we see that for each $y \in B(z, r/2)$ and each $u \in B(v, r/4)$,

$$\|(2u - y) - \xi\| < \frac{r}{2} + \frac{r}{2} = r$$

and therefore $2u - y \in C$. It follows that if also $y \in C$, then

$$u = \frac{1}{2}\left((2u - y) + y\right) \in C$$

for each $u \in B(v, r/4)$. But then $\rho(v, \sim C) \geqslant r/4$, so $v \notin \partial(\sim C)$, a contradiction from which we conclude that $y \notin C$ for each $y \in B(z, r/2)$. Hence $\rho(z, C) \geqslant r/2$, and so $z \in -C$. By Proposition 11, there exists a unique $t \in (0, 1)$ such that $w \equiv (1 - t)\xi + tz$ belongs to ∂C, $y \in C^\circ$ for all $y \in [\xi, w)$, and $y \in -C$ for all $y \in (w, z]$.

Given $\varepsilon > 0$, pick a point ζ in the open segment (ξ, v) such that $0 < \|\zeta - v\| < \varepsilon$ and $\zeta \neq w$. Since $\zeta \in (\xi, z)$, either $\zeta \in (\xi, w)$ or $\zeta \in (w, z)$. In the latter case, since $v \in (\zeta, z)$, we have $v \in (w, z)$ and so $v \in -C$; but this is absurd, since it puts v in $(\sim C)^\circ$ and thereby contradicts the choice of v as an element of $\partial(\sim C)$. Hence, in fact, $\zeta \in (\xi, w)$ and therefore $\zeta \in C^\circ$. Since ε is arbitrarily small and, by definition of $\partial(\sim C)$, there are points of $\sim C$ arbitrarily close to v, it follows that $v \in \partial C$. Thus $\partial(\sim C) \subset \partial C$. ∎

Proposition 13 *Let X be a Banach space, and C a convex subset of X with inhabited interior such that $C \cup \sim C$ is dense in X. Then $\sim C$ is tightly bordered.*

Proof. Consider any $x \in \sim C$ with $\rho(x, \partial(\sim C)) > 0$. By Lemma 12, $\rho(x, \partial C) > 0$. Let $0 < r < \rho(x, \partial C)$, and apply Proposition 8 to produce $z \in -C \cap B(x, r)$. Given $\xi \in B(x, r)$, suppose that $\xi \in C^\circ$. By Proposition 11, there exists a unique point y in $[\xi, z] \cap \partial C$. But then y belongs to the convex set $B(x, r)$, so $\rho(x, \partial C) < r$—a contradiction. Hence $\xi \notin C^\circ$, so $\xi \in -C^\circ$ (since C° is an open set) and therefore, by Proposition 7, $\xi \in -C$. It now follows that $B(x, r) \subset -C \subset \sim C$ whence $x \in (\sim C)^\circ$. ∎

The hypothesis that $C \cup \sim C$ be dense in X appears in most of the preceding results. One situation in which it arises is when C is located; another is given by the next proposition.

Proposition 14 *Let C be a coherent, convex subset of a normed space such that $\sim C$ is located. Then $C \cup \sim C$ is dense in X.*

Proof. Given x in X and $\varepsilon > 0$, we have either $\rho(x, \sim C) < \varepsilon$ or $\rho(x, \sim C) > 0$. In the latter case, $x \in -\sim C$ and therefore, by coherence, $x \in C$. Since $\varepsilon > 0$ is arbitrary, we conclude that $C \cup \sim C$ is dense in X. ∎

As a partial converse to Proposition 14, we have:

Proposition 15 *Let C be a convex subset of a normed space X such that C° is inhabited and $C \cup \sim C$ is dense in X. Then C is coherent; in fact, if $\rho(x, \sim C) > 0$, then $x \in C^\circ$.*

Proof. Fix $\xi \in C^\circ$. Given $x \in -\sim C$, pick $r > 0$ such that $B(\xi, 2r) \subset C^\circ$ and $\rho(x, \sim C) > 2r$. In order to prove that $x \in C$, we may assume that $\|x - \xi\| > r$. Compute t such that $0 < t < 1$ and

$$\frac{t}{1-t}\|x - \xi\| < r,$$

and let

$$z = \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $x = t\xi + (1-t)z$ and $\|x - z\| < r$. Hence

$$B(z, tr) \subset B(z, r) \subset B(x, 2r) \subset -\sim C.$$

Since $C \cup \sim C$ is dense in X, it follows that $B(z, tr)$ intersects C. Lemma 1 now shows us that $B(x, t^2 r) \subset C$; whence $x \in C^\circ$. ∎

We conclude this section with two more results about borders of convex subsets. The first of these will be used in the Brouwerian examples in Section 4.

Proposition 16 *Let C be an inhabited, located, convex subset of a Hilbert space H. Then for each $x \in \sim C$, $\rho(x, \partial C)$ exists and equals $\rho(x, C)$.*

Proof. Replacing C by $\overline{C}$, we may assume that C is closed in H. Let $x \in \sim C$. By a well-known extension of Theorem 4.3.1 of [8], there exists a unique $z \in C$ such that $\|x - z\| = \rho(x, C)$. If $\rho(x, \partial C) < \|x - z\|$, then there exists $\zeta \in C$ such that $\|x - \zeta\| < \rho(x, C)$, which is absurd; hence $\rho(x, \partial C) \geqslant \|x - z\|$. It remains to prove that $z \in \partial C$; for that, it will suffice to show that for each $\varepsilon > 0$, there exists $\zeta \in \sim C$ with $\|z - \zeta\| < 3\varepsilon$. Either $\|x - z\| < 3\varepsilon$, in which case we can take $\zeta = x$, or else, as we assume, $\|x - z\| > 2\varepsilon$. Letting

$$t = \frac{2\varepsilon}{\|x - z\|} \text{ and } y = tx + (1-t)z,$$

we see that $0 < t < 1$ and $\|y - z\| = 2\varepsilon$. Since C is located, $C \cup \sim C$ is dense in H; so there exists $\zeta \in C \cup \sim C$ such that $\|y - \zeta\| < \varepsilon$. Then

$$\begin{aligned}\|x - \zeta\| &\leqslant \|x - y\| + \|y - \zeta\| \\ &< (1-t)\|x - z\| + \varepsilon \\ &= \|x - z\| - 2\varepsilon + \varepsilon \\ &< \|x - z\| = \rho(x, C),\end{aligned}$$

from which it follows that $\zeta \in \sim C$. Also,

$$\|z - \zeta\| \leqslant \|y - \zeta\| + \|y - z\| < 3\varepsilon.$$

This completes the proof that $z \in \partial C$. ∎

Corollary 17 *Let C be an inhabited, located, convex subset of a Hilbert space with $\sim C$ inhabited. Then $\partial(\sim C) = \partial C$.*

Proof. Clearly, $\partial C \subset \partial(\sim C)$. To prove the reverse inequality, consider $x \in \partial(\sim C)$, and first observe that, by definition of $\partial(\sim C)$, there are points of $\sim C$ arbitrarily close to x. Since C is convex and located in the Hilbert space, there exists $z \in \overline{C}$ such that $\|x - z\| = \rho(x, C)$. Given $\varepsilon > 0$, we have either $x \neq z$ or $\|x - z\| < \varepsilon$. In the first case, $x \in -C \subset (\sim C)^{\circ}$, which is impossible since $x \in \partial(\sim C)$. Thus we have $\|x - z\| < \varepsilon$, and so, since $z \in \overline{C}$, there exists $y \in C$ with $\|x - y\| < \varepsilon$. Hence, $\varepsilon > 0$ being arbitrary, there are points of C arbitrarily close to x, which ensures that $x \in \partial C$. Thus $\partial(\sim C) \subset \partial C$, as required. ■

4 Limiting Brouwerian counterexamples In this section we present two Brouwerian counterexamples.[4] The first shows why we needed some of the hypotheses for the results in Sections 2 and 3.

Brouwerian Example 1 [LEM]. *A convex subset C of $\mathbf{R}$ that has inhabited interior, has tightly bordered complement, but for which none of the following properties can be derived:*

(i) *$C \cup \sim C$ is dense in $\mathbf{R}$.*

(ii) *∂C is located in $\mathbf{R}$.*

(iii) *$\partial(\sim C) = \partial C$.*

(iv) *C is tightly bordered.*

Let P be any proposition such that $\neg\neg P$ holds, and define

$$C \equiv [0,1] \cup \{x \in [0,2] : P\}.$$

This set is convex and contains $1/2$ in its interior. If $C \cup \sim C$ is dense in $\mathbf{R}$, then we can choose $x \in C \cup \sim C$ with $|x - 3/2| < 1/2$. If $x \in \sim C$, then $\neg P$, which is absurd; so $x \in C$ and therefore P holds. If ∂C is located, then either $\rho(3/4, \partial C) > 1/4$ or $\rho(3/4, \partial C) < 1/2$. The first case is ruled out, since it implies that $2 \notin \partial C$ and hence that $\neg P$ holds. Thus there exists $x \in \partial C$ with $x > 1/2$, and therefore P holds. Next we observe that $2 \in \partial(\sim C)$: indeed, $(2, \infty) \subset \sim C$, and, since $\neg\neg P$ holds, $(1,2) \subset \sim\sim C$. However, if $2 \in \partial C$, then $C \cap (1,2]$ is inhabited, so P holds. On the other hand, if $x \in \sim C$ and $\rho(x, \partial(\sim C)) > 0$, then either $x < 0$ or $x > 2$, so $x \in \sim C$. Thus $\sim C$ is tightly bordered.

Finally, if $x \in \partial C$ and $|x - 1| < 1$, then we must have $\neg P$, a contradiction; whence $\rho(1, \partial C) \geqslant 1$; it follows that if C is tightly bordered, then $1 \in C^{\circ}$, so $C \cap (1,2]$ is inhabited and therefore P holds.

For the remaining three Brouwerian examples, each connected with the hypothesis in Proposition 5 that C has inhabited interior, we remind the reader of two essentially nonconstructive classical principles:

> **The limited principle of omniscience, LPO:** For each binary sequence $(a_n)_{n \geqslant 1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$,
>
> **Markov's principle, MP:** For each binary sequence $(a_n)_{n \geqslant 1}$, if it is impossible that $a_n = 0$ for all n, then there exists n such that $a_n = 1$.

[4] "A Brouwerian counterexample is not a counterexample in the usual sense; it is *evidence* that a statement does not admit of a constructive proof" ([7], page 3).

LPO is equivalent to the statement

$$\forall_{x\in\mathbf{R}}\,(x=0\vee|x|>0)\,.$$

MP, which is weaker than **LPO**, is equivalent to

$$\forall_{x\in\mathbf{R}}\,(\neg\,(x=0)\Rightarrow x\neq 0)\,,$$

where '$x\neq 0$' means '$|x|>0$'.

Brouwerian Example 2 [LPO]. *An inhabited, balanced, convex subset C of $\mathbf{R}$ such that C and ∂C are compact, $\sim C$ is open and located, both C and $\sim C$ are tightly bordered, but we cannot determine that C° is either empty or inhabited.*

Take any nonnegative, small real number a and let $C=[-a,a]$. Then C is inhabited by 0 and is compact; ∂C $(=\{-a,a\})$ is located; $\sim C=-C$ is open and are located. However, if C° is inhabited, then $a\neq 0$; and if $C^\circ=\varnothing$, then $a=0$.

We can turn this into a Markovian example—one in which the derivability of the property under examination leads to that of Markov's principle—by choosing $a\geqslant 0$ such that $\neg\neg(a=0)$. Then C° is open if and only if $a\neq 0$. Thus we have:

Brouwerian Example 3 [MP]. *An inhabited, balanced, convex subset C of $\mathbf{R}$ such that C and ∂C are compact, $\sim C$ is open and located, both C and $\sim C$ are tightly bordered, C° cannot be empty, but we cannot determine that C° is inhabited.*

We now give a much more complicated Brouwerian example, showing that if we replace $\mathbf{R}$ by a Hilbert subspace of $l_2(\mathbf{R})$, then we can replace **MP** by **LPO** in Brouwerian Example 3:

Brouwerian Example 4 [LPO]. *A balanced, compact, tightly bordered, convex subset C of a Hilbert space such that ∂C is compact, $\sim C$ is located and open (and hence tightly bordered), C° cannot be empty, but we cannot determine that C° is inhabited.*

Let $(a_n)_{n\geqslant 1}$ be a binary sequence with $a_1=1$ and at most one other term equal to 1. Let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of unit vectors in the Hilbert space $l_2(\mathbf{R})$, and let H be the linear subspace

$$\left\{\sum_{n=1}^{\infty}a_n\,\langle x,e_n\rangle\,e_n : x\in l_2(\mathbf{R})\right\}.$$

We first prove that H is closed in $l_2(\mathbf{R})$ and is therefore a Hilbert space. Let $\left(x^{(n)}\right)_{n\geqslant 1}$ be a sequence in H that converges to a limit $x^\infty\in l_2(\mathbf{R})$. For each n, there exists $z_n\in l_2(\mathbf{R})$ such that $x^{(n)}=\sum_{k=1}^{\infty}a_k\,\langle z_n,e_k\rangle\,e_k$. Thus for each $k\geqslant 1$, $a_k\,\langle z_n,e_k\rangle\to\langle x^\infty,e_k\rangle$ as $n\to\infty$, and so

$$\begin{aligned}a_k\,\langle x^\infty,e_k\rangle &= a_k\lim_{n\to\infty}a_k\,\langle z_n,e_k\rangle\\ &= \lim_{n\to\infty}a_k^2\,\langle z_n,e_k\rangle=\lim_{n\to\infty}a_k\,\langle z_n,e_k\rangle=\langle x^\infty,e_k\rangle\,.\end{aligned}$$

Hence $x^\infty=\sum_{k=1}^{\infty}a_k\,\langle x^\infty,e_k\rangle\,e_k\in H$.

Call a pair (λ^+,λ^-) of nonnegative sequences **acceptable** if there exists ν such that

- $\rhd$ $\lambda_n^+=\lambda_n^-=0$ for all $n\geq\nu$, and

▷ $\sum_{n=1}^{\infty}(\lambda_n^+ + \lambda_n^-) = \sum_{n=1}^{\nu}(\lambda_n^+ + \lambda_n^-) = 1$
(where, for example, λ_n^+ is the nth term of λ^+).

Let C be the closure in H of the set S of all points of the form

$$\sum_{n=1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n e_n,$$

where the sequence pair (λ^+, λ^-) is admissible. It is straightforward to prove the following facts:

(a) C is a balanced, convex subset of H that contains 0.

(b) If $x \in C$, then $|\langle x, e_1\rangle| \leqslant 1$.

(c) If $a_n = 0$ for all $n \geqslant 2$, then $H = \mathbf{R}e_1$, $C = \{te_1 : -1 \leqslant t \leqslant 1\}$, which is both compact and tightly bordered, $\partial C = \{-e_1, e_1\}$, and $\sim C$ is both located and open in H.

(d) If there exists $N \geqslant 2$ such that $a_N = 1$, then $H = \mathrm{span}\{e_1, e_N\}$, C is the closed convex hull of $\left\{\pm e_1, \pm N^{-1}e_N\right\}$ and is compact; ∂C is the compact closure of the parallelogram with vertices $\pm e_1, \pm N^{-1}e_N$; and $\sim C$ is both located and open in H. Moreover, by elementary Euclidean geometry, $B(0, r_N) \subset C$, where

$$r_N = N^{-1}\cos\left(\tan^{-1}\frac{1}{N}\right),$$

and $\rho(x, \partial C) \leqslant r_N$ for each $x \in C$.

(e) It is impossible that C has empty interior.

To prove that C is totally bounded, fix $\varepsilon > 0$ and let F be a finite ε-approximation to the set $C \cap \mathbf{R}e_1$. Pick N such that $\sum_{n=N+1}^{\infty} n^{-2} < \varepsilon^2$. If $a_n = 1$ for some n with $2 \leqslant n \leqslant N$, then (as noted at (d) above) C is compact. So we may assume that $a_n = 0$ whenever $2 \leqslant n \leqslant N$. Given an acceptable sequence pair (λ^+, λ^-), define

$$x = \sum_{n=1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n e_n \in S. \tag{1}$$

Since $\langle x, e_1\rangle e_1 \in C \cap \mathbf{R}e_1$, there exists $y \in F$ with $\|\langle x, e_1\rangle e_1 - y\| < \varepsilon$. Then

$$\begin{aligned}\|x - y\| &\leqslant \|\langle x, e_1\rangle e_1 - y\| + \left\|\sum_{n=N+1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n\right\| \\ &< \varepsilon + \left(\sum_{n=N+1}^{\infty} n^{-2}\right)^{1/2} < 2\varepsilon.\end{aligned}$$

Thus F is a finite 2ε-approximation to S. Since $\varepsilon > 0$ is arbitrary, we see that S is totally bounded; whence C is totally bounded and hence, being complete, compact.

To prove that C is tightly bordered, let $x \in C$ and $0 < r < \rho(x, \partial C)$. Since $e_1 \in \partial C$, we have

$$0 \neq e_1 - x = (1 - \langle x, e_1\rangle)\, e_1 - \sum_{n=2}^{\infty} a_n \langle x, e_n\rangle\, e_n$$

so either $\langle x, e_1 \rangle \neq 1$ or there exists $n \geqslant 2$ with $a_n \neq 0$. Since in the latter event C is tightly bordered and therefore $x \in C^\circ$, we may assume that $\langle x, e_1 \rangle \neq 1$; similarly, since $-e_1 \in \partial C$, we may assume that $\langle x, e_1 \rangle \neq -1$. Thus either $|\langle x, e_1 \rangle| > 1$, in which case $x \notin C$, a contradiction; or else, as must be the case, $-1 < \langle x, e_1 \rangle < 1$. Now choose an integer $N > 1/r$. If $a_n = 1$ for some $n > N$, then $\rho(x, \partial C) \leqslant r_n < r_N < r$, a contradiction; hence $a_n = 0$ for all $n > N$. It follows that

– either there exists n with $2 \leqslant n \leqslant N$ and $a_n = 1$, in which case C is tightly bordered and so $x \in C^\circ$;

– or else $a_n = 0$ for all $n \geqslant 2$, when $x = \langle x, e_1 \rangle e_1 \in \{te_1 : -1 < t < 1\} = C^\circ$.

Thus C is tightly bordered.

Turning now to $\sim C$, we first observe that since C is located and therefore $C \cup \sim C$ is dense in H, in order to prove that $\sim C$ is located, it will suffice to prove that $\rho(x, \sim C)$ exists for each $x \in C$. Given such x and $\varepsilon > 0$, choose a positive integer N such that $r_N < \varepsilon/2$. If $a_n = 1$ for some n with $2 \leqslant n \leqslant N$, then $\sim C$ is clearly located, being the outside of a parallelogram; so we may assume that $a_n = 0$ for $2 \leqslant n \leqslant N$. It follows that C is a subset of the closed convex hull of $\{\pm e_1, \pm N^{-1} e_N\}$. Given $x \in X$, and writing

$$T \equiv \{te_1 : -1 < t < 1\},$$

we have either $\rho(x, T) > 0$ or $\rho(x, T) < \varepsilon/2$. In the first case, there exists $m > N$ such that $a_m = 1$, so $\sim C$ is located. In the second case, pick $y \in T$ such that $\|x - y\| < \varepsilon/2$. Then $y + N^{-1} e_m \in \sim C$ and

$$\left\| x - \left(y + N^{-1} e_m \right) \right\| \leqslant \|x - y\| + N^{-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Putting all this together, we see that for each $x \in C$ and each $\varepsilon > 0$, either $\rho(x, \sim C)$ exists or there exists $y \in \sim C$ such that $\|x - y\| < \varepsilon$. Hence $\sim C$ is located. It follows from Proposition 11 of [9] that ∂C is located; since ∂C is a closed subset of the compact set C, it is therefore compact.

To prove that $\sim C$ is open, let $x \in \sim C$. Since $C \cap \mathbf{R} e_1$ is closed, located, and convex, there exists $z \in C \cap \mathbf{R} e_1$ such that $\rho(x, C \cap \mathbf{R} e_1) = \|x - z\|$; then $x \neq z$, so

$$0 < d \equiv \rho(x, C \cap \mathbf{R} e_1).$$

Either $\rho(x, \mathbf{R} e_1) > 0$ or $\rho(x, \mathbf{R} e_1) < d/2$. In the first case, choose a positive integer $N > 1/\rho(x, \mathbf{R} e_1)$. We may assume that $a_n = 0$ for $2 \leqslant n \leqslant N$; so C is a subset of the closed convex hull of $\{\pm e_1, \pm N^{-1} e_N\}$. If $\|x - y\| < \rho(x, \mathbf{R} e_1) - 1/N$, then $\rho(y, \mathbf{R} e_1) > 1/N$, so $y \in \sim C$. Hence $\rho(x, C) \geqslant \rho(x, \mathbf{R} e_1) - 1/N$, and therefore $x \in -C = (\sim C)^\circ$. This leaves us with the case $\rho(x, \mathbf{R} e_1) < d/2$, in which, if $|\langle x, e_1 \rangle| < 1$, then

$$\rho(x, \mathbf{R} e_1) = \|x - \langle x, e_1 \rangle e_1\| \geqslant \rho(x, C \cap \mathbf{R} e_1),$$

a contradiction. Hence $|\langle x, e_1 \rangle| \geqslant 1$ and therefore either $\langle x, e_1 \rangle \leqslant -1$ or $\langle x, e_1 \rangle \geqslant 1$. We illustrate with the latter alternative. We have

$$\begin{aligned} |\langle x, e_1 \rangle| - 1 &= \|\langle x, e_1 \rangle e_1 - e_1\| \\ &\geqslant \|x - e_1\| - \|x - \langle x, e_1 \rangle e_1\| \geqslant d - \frac{d}{2} = \frac{d}{2}, \end{aligned}$$

so $|\langle x, e_1\rangle| \geqslant 1 + d/2$. It follows that if $\|x - y\| < d/2$, then

$$\begin{aligned}|\langle y, e_1\rangle| &\geqslant |\langle x, e_1\rangle| - |\langle x, e_1\rangle - \langle y, e_1\rangle| \\ &\geqslant 1 + \frac{d}{2} - \|x - y\| > 1\end{aligned}$$

and therefore $y \notin C$. Hence $B(x, d/2) \subset -C$ and therefore $x \in -C = (\sim C)^\circ$. This completes the proof that $\sim C$ is open.

Finally, suppose that C° is inhabited; then, since C is convex and balanced, $0 \in C^\circ$. Pick $r > 0$ such that $B(0, r) \subset C$ and compute N such that $r_N < r$. If $a_n = 1$ for some $n \geqslant N$, then $\rho(0, \partial C) = r_N$ and there exist points of $\sim C$ within r of 0, a contradiction. Hence $a_n = 0$ for all $n \geqslant N$. By testing $a_2, \ldots, a_{N-1}$, we can show that either $a_n = 0$ for all n or else there exists $n < N$ such that $a_n = 1$.

5 A Final Remark In several of our results, we have used the hypothesis that $C \cup \sim C$ is located, where C is an inhabited convex subset of the ambient normed space X. Could it be that that hypothesis actually implies that C is located? If $X = \mathbf{R}$, then the answer is 'yes'. To see this, first translate C to ensure that it contains 0, and set $a_1 = 0$. Fixing $x > 0$, pick $b_1 \in C \cup \sim C$ such that $0 < b_1 - x < x/2$. If $b_1 \in C$, then $x \in [a_1, b_1] \subset C$; so we may assume that $b_1 \in \sim C$. Let $c_1 = (b_1 - a_1)/2$, and pick $y_1 \in C \cup \sim C$ such that $y_1 > 0$ and $|c_1 - y_1| < \min\{2^{-1}\varepsilon, c_1/6\}$. If $y_1 \in C$, set $a_2 = y_1$ and $b_2 = b_1$. If $y_1 \in \sim C$, set $a_2 = a_1$ and $b_2 = y_1$. At this stage, we have $a_2 \in C$ and $b_2 \in \sim C$ such that $0 < a_2 < \frac{2}{3}(b_1 - a_1)$. Continuing on in this way, we construct an increasing sequence $(a_n)_{n \geqslant 1}$ in C and a decreasing sequence $(b_n)_{n \geqslant 1}$ in $\sim C$ such that $0 < b_n - a_n \to 0$ as $n \to \infty$. These sequences have a common limit $a_\infty \in \partial C$, and $\rho(x, C)$ exists and equals $x - a_\infty$. The case $x < 0$ is handled similarly. Since $(-\infty, 0) \cup (0, \infty)$ is dense in $\mathbf{R}$, we conclude that C is located.

However, when we move from one to two dimensions, we have a Brouwerian counterexample to the locatedness of C. Given any proposition P, take $X = \mathbf{R}^2$ and

$$C = ([0, 1] \cup \{x \in [0, 2] : P\}) \times \{0\} \subset \mathbf{R}^2.$$

Then $\sim C$ contains the dense subset $\{(x, y) \in \mathbf{R}^2 : y \neq 0\}$ of $\mathbf{R}^2$ and so is itself dense in $\mathbf{R}^2$. But if C is located, then (cf. Brouwerian Example 1 above) we can easily derive $P \vee \neg P$.

References

[1] P. Aczel and M. Rathjen: *Notes on Constructive Set Theory*, Report No. 40, Institut Mittag-Leffler, Royal Swedish Academy of Sciences, 2001.

[2] P. Aczel and M. Rathjen: *Constructive Set Theory*, in preparation.

[3] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[4] E. Bishop and D.S. Bridges: *Constructive Analysis*, Grundlehren der Math. Wiss. **279**, Springer Verlag, Heidelberg, 1985.

[5] D.S. Bridges: 'Locatedness, convexity, and Lebesgue measurability', Quart. J. Math. Oxford (2) **39**, 411–421, 1988.

[6] D.S. Bridges, M. McKubre-Jordens: 'Solving the Dirichlet problem constructively', J. Logic Anal. **5**(3), 1-22, 2013.

[7] D.S. Bridges and F. Richman: *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes **97**, Cambridge Univ. Press, Cambridge, U.K., 1987.

[8] D.S. Bridges, L.S. Vîţă: *Techniques of Constructive Analysis*, Universitext, Springer New York, 2006.

[9] D.S. Bridges, F. Richman, Wang Yuchuan: 'Sets, complements and boundaries', Proc. Koninklijke Nederlandse Akad. Wetenschappen (Indag. Math.) **7**(4), 425–445, 1996.

[10] D.S. Bridges, A. Calder, W.H. Julian, R. Mines, F. Richman, 'Locating metric complements in $\mathbb{R}^n$', in *Constructive Mathematics* (F. Richman, ed.), Springer Lecture Notes in Math. **873**, 241–249, 1981.

[11] P. Martin-Löf: 'An intuitionistic theory of types', in *Twenty-five Years of Constructive Type Theory* (G. Sambin, J. Smith, eds), 127–172, Oxford Logic Guides **36**, Clarendon Press, Oxford, 1998.

[12] F. Richman, D.S. Bridges, A. Calder, W.H. Julian, R. Mines: 'Compactly generated Banach spaces', Arch. Math. **36**, 239–243, 1981.

[13] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics: An Introduction* (two volumes), North Holland, Amsterdam, 1988.

Communicated by *Mariko Yasugi*

Author's address: Department of Mathematics & Statistics, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand.
email: d.bridges@math.canterbury.ac.nz