GLOBAL EXISTENCE AND EXPONENTIAL ATTRACTOR OF SOLUTIONS OF FIX-CAGINALP EQUATION

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Abstract. We consider the Fix-Caginalp equation with the Neumann boundary condition in $\mathbf{R}^n$ with $n = 1, 2, 3$. We obtain a global solution by the existence of the Lyapunov function. After, we construct a dynamical system corresponding to the equation. By the existence of the Lyapunov function, the ω-limit set is included in the set of its stationary solution. We treat its dynamical properties such as a global attractor, absorbing set, exponential attractor and so on. It is important to obtain the estimate independent of the initial value. Finally, we construct an exponential attractor.

1 Introduction In this paper, we consider the following Fix-Caginalp equation:

$$\begin{cases} \tau\phi_t = \epsilon^2\Delta\phi + \phi - \phi^3 + 2u & x \in \Omega,\ t > 0, \\ u_t + \frac{l}{2}\phi_t = \kappa\Delta u & x \in \Omega,\ t > 0, \\ \frac{\partial\phi}{\partial\nu} = \frac{\partial u}{\partial\nu} = 0 & x \in \partial\Omega,\ t > 0, \\ \phi(x,0) = \phi_0(x) & x \in \Omega, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1}$$

where τ, l, κ and ϵ are positive constants, ν is the outer unit normal vector and Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ for $n = 1, 2, 3$. An equation (1) was proposed by Caginalp in [4] to describe the phase transitions with finite thickness interfaces. The unknown functions ϕ and u represent the phase function and the reduced temperature, respectively. The positive constants τ, l, κ and ϵ represent the relaxation time, the latent heat, the thermal diffusivity and a length scale which is a measure of the strength of the bonding at the microscopic level, respectively. In [12], they consider the historic background of the model and the derivation of a more general thermodynamically consistent model. At first in [4], he proved a global existence of a solution under the restriction $\frac{\epsilon^2}{\tau} < \kappa$. After in [7], [2], [3] and [16], they dropped the restriction and proved the global existence under the other boundary conditions

$$\phi(x,t) = \phi_{\partial\Omega}(x), \quad u(x,t) = u_{\partial\Omega}(x),$$

$$\frac{\partial\phi}{\partial\nu}(x,t) = 0, \quad u(x,t) = u_{\partial\Omega}(x)$$

and

$$\phi(x,t) = \phi_{\partial\Omega}(x), \quad \frac{\partial u}{\partial\nu}(x,t) = 0$$

for $x \in \partial\Omega,\ t > 0$, where $\phi_{\partial\Omega}(x)$ and $u_{\partial\Omega}(x)$ are given functions on $\partial\Omega$. In [7], they consider the stationary problem with the Neumann boundary condition, derive the existence and

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non-existence of nontrivial solutions and the multi-existence of trivial solutions according to the values of constants l, ϵ and $\int_\Omega u dx + \frac{l}{2}\int_\Omega \phi dx$ and deal with their stabilities. If $n = 1$, the stationary problem with the Dirichlet boundary condition is considered in [5] and [9]. They show that there exist exactly $2m + 1$ solutions with m being an integer determined by ϵ^2 and Ω. In [7], they consider the asymptotic behaviour of solution of (1). For results with initial data in different settings of spaces, see [7], [1] and [3]. Lately in [13], they consider non-local stationary problem and get some results on multiple existence, stability and bifurcation of the solution. For a system of reaction-diffusion equations in a bounded domain $\Omega \subset \mathbf{R}^2$, the existence of a global attractor and exponential attractor is proved in [11]. Their key fact is that its dynamical system has the squeezing property. Although the global existence for $(\phi_0, u_0) \in H^1(\Omega) \times L^2(\Omega)$ is known by [7] and [16], we treat more general space $H^\gamma(\Omega) \times H^\gamma(\Omega)$. For the definition of function space and notion of dynamical system, see Section 2 in this paper or [15], [6], [8], [14], [9]. In [16], he proves the dynamical properties with the Dirichlet boundary condition instead of the Neumann boundary condition. Since we can use the Poincaré inequality, the estimates of the Dirichlet boundary condition case are easier. In particular, since the solution (ϕ, u) with the Dirichlet boundary condition has the global dissipative property, we don't have to consider a space H_k mentioned in Theorem 4 in this paper in order to construct a global attractor. The purpose of this paper is to establish the existence of a global solution, the properties of ω-limit set and the exponential attractor in the dynamical system introduced by the Fix-Caginalp equation. The first theorem is concerned with the global existence.

Theorem 1 *Let $\Omega \subset \mathbf{R}^n (n = 1, 2, 3)$ be a bounded domain with smooth boundary $\partial\Omega$. We suppose that $\phi_0, u_0 \in H^\gamma(\Omega)$ for $\underline{\gamma} < \gamma < \overline{\gamma}$, where $\left(n, \underline{\gamma}, \overline{\gamma}\right) = \left(1, 0, \frac{1}{4}\right), \left(2, 0, \frac{1}{2}\right), \left(3, \frac{1}{2}, \frac{2}{3}\right)$. Then, the problem (1) admits a unique global solution (ϕ, u) such that*

$$\phi, u \in C\left((0,\infty); H^1(\Omega)\right) \cap C\left([0,\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,\infty); H^{-1}(\Omega)\right).$$

The associated nonlinear semigroup $T(t)$

$$T(t)\left(\phi_0(\cdot), u_0(\cdot)\right) = \left(\phi(\cdot,t), u(\cdot,t)\right)$$

defines a dynamical system in $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

To obtain the a priori estimate for H^1 norm, we use the Lyapunov function

$$L(\phi, u)(t) = \frac{1}{2}\int_\Omega u^2 dx + \frac{l\epsilon^2}{8}\int_\Omega |\nabla\phi|^2 dx + \frac{l}{4}\int_\Omega W(\phi)dx + \frac{\kappa\delta}{2}\int_\Omega |\nabla u|^2 dx$$

for $\delta < \frac{4\tau}{l}$, where

$$W(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2.$$

In the second theorem, we obtain the regularity of solution.

Theorem 2 *Under the same assumption as Theorem 1,*

$$\phi, u \in C^\infty\left((0,+\infty); C^\infty(\overline{\Omega})\right).$$

For any $\eta > 0$, the orbit $t \in [\eta, +\infty) \mapsto (\phi(\cdot,t), u(\cdot,t))$ is compact in $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

Combining the estimates obtained in Theorems 1 and 2 with the existence of the Lyapunov function, we consider the structure of ω-limit set in the third theorem. At first, by E we denote the set of stationary solution corresponding to (1). Since $\phi(t), u(t) \in H^1(\Omega)$ for $t > 0$, we assume that $\phi_0, u_0 \in H^1(\Omega)$. As proved in Theorem 1, it is also easy to show that the dynamical system is defined on $H^1(\Omega) \times H^1(\Omega)$.

Theorem 3 *We suppose that* $\phi_0, u_0 \in H^1(\Omega)$. *Then,* $\omega(\phi_0, u_0)$ *is nonempty, compact, invariant and connected in* $H^1(\Omega) \times H^1(\Omega)$. *And* $\omega(\phi_0, u_0)$ *is a single point and it holds that* $\omega(\phi_0, u_0) \subset E$.

We construct an exponential attractor in $H^1(\Omega) \times H^1(\Omega)$ in the last theorem. However, the solution (ϕ, u) of (1) does not have the global dissipative property. Thus, we restrict the initial function to

$$H_k = \left\{(\phi_0, u_0) \in H^1(\Omega) \times H^1(\Omega) \mid L(\phi_0, u_0) \leq k\right\}$$

for fixed $k > 0$ and reduce a dynamical system to its subdynamical system $\{T(t) : H_k \to H_k\}$.

Theorem 4 *Under the same assumption as Theorem 3,* $T(t)$ *is dissipative in* H_k. *The dynamical system* $T(t)$ *has a global attractor* $\mathcal{A} \subset H_k$. *Then, there exists a compact absorbing and positively invariant set* $\mathcal{X} \subset H_k$ *such that its subdynamical system* $\{T(t) : \mathcal{X} \to \mathcal{X}\}$ *admits an exponential attractor* $\mathcal{E}$ *in* $H^1(\Omega) \times H^1(\Omega)$.

This paper is composed of 6 sections. In Section 2, we introduce the notions and theories of an abstract evolution equation and dynamical system. We also refer to the function space involved in this paper. In Section 3, we apply the existence theorem in Section 2 and establish the local solution of (1). In Section 4, we derive the a priori estimates and extend the local solution globally in time. In Section 5, we consider a nonlinear mapping from the initial function to the solution of (1) and define the dynamical system. The obtained estimates in Section 4 lead us to the proof of Theorems 1, 2 and 3. In section 6, we construct an exponential attractor and prove Theorem 4. Now that we restrict to H_k and have the Lyapunov function, our result follows at once.

2 Preliminaries We introduce the results and related facts in an abstract evolution equation. These results are mentioned in mainly [15] and [9], [8], [6]. Let X be a Banach space with the norm $\|\cdot\|$. Let A be a densely defined, closed linear operator in X. We assume that the spectrum of A is contained in an open sectorial domain such that

$$\sigma(A) \subset \Sigma_\omega \equiv \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \omega\}, \qquad \omega_A < \omega < \frac{\pi}{2} \tag{2}$$

and

$$\left\|(\lambda - A)^{-1}\right\| \leq \frac{M_\omega}{|\lambda|}, \qquad \lambda \notin \Sigma_\omega, \omega_A < \omega < \frac{\pi}{2} \tag{3}$$

for $\omega_A \in [0, \frac{\pi}{2})$, where $M_\omega > 0$ is a constant depending on A and ω. We call A a sectorial operator of X with angle $0 \leq \omega_A < \frac{\pi}{2}$. We consider the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} U_t + AU = F(U) & t > 0, \\ U(0) = U_0 \end{cases} \tag{4}$$

in X. Here, F is a nonlinear operator from $\mathcal{D}(A^\eta)$ into X, where $0 < \eta < 1$ and satisfies a Lipschitz condition of the form

$$\begin{aligned} \|F(U) - F(V)\| \quad \leq \quad & \Phi\left(\left\|A^\beta U\right\| + \left\|A^\beta V\right\|\right) \times \\ & \left\{\|A^\eta (U - V)\| + (\|A^\eta U\| + \|A^\eta V\|)\left\|A^\beta (U - V)\right\|\right\} \end{aligned} \tag{5}$$

for $U, V \in \mathcal{D}(A^\eta)$ with $0 < \beta \leq \eta < 1$, where $\Phi(\cdot)$ is some increasing continuous function. We have the following global existence theorem.

Theorem 5 (Theorem 4.1 in [15]) *Let (2), (3) and (5) with $0 < \beta \le \eta < 1$ be satisfied. Then, for any $U_0 \in \mathcal{D}(A^\beta)$, (4) admits a unique local solution U in*

$$U \in C\left((0, T_{U_0}]; \mathcal{D}(A)\right) \cap C\left([0, T_{U_0}]; \mathcal{D}(A^\beta)\right) \cap C^1\left((0, T_{U_0}]; X\right),$$

where T_{U_0} denotes the maximal existence time depending only on the norm $\left\|A^\beta U_0\right\|$. Moreover, it holds that

$$\left\|A^\beta U\right\| + t^{1-\beta}\left\|U_t\right\| + t^{1-\beta}\left\|AU\right\| \le C_{U_0},$$

where C_{U_0} is a positive constant depending only on $\left\|A^\beta U_0\right\|$.

Here, we note that $\mathcal{D}(A^\beta) = X$ for $\beta = 0$. We can take $\beta = 0$ in the condition (5) throughout theorems in this section.

Theorem 6 (Corollary 4.1 in [15]) *Under the assumption of Theorem 5, we suppose that any local solution U satisfies the estimate*

$$\left\|A^\beta U(t)\right\| \le C_{U_0},$$

for $0 \le t \le T_{U_0}$ with some positive constant C_{U_0} depending only on $\left\|A^\beta U_0\right\|$ and independent of T_{U_0}. Then, (4) admits a unique global solution U for all $t > 0$.

Let $K(R)$ be a bounded ball in the space $\mathcal{D}(A^\beta)$

$$K(R) = \left\{U \in \mathcal{D}(A^\beta) \mid \left\|A^\beta U\right\| \le R\right\}$$

for $0 < R < \infty$. Then, for all $U_0 \in K(R)$, there exists a local solution of (4) on some interval $[0, T_{U_0}]$. There exists the time $T_R > 0$ such that $[0, T_R] \subset [0, T_{U_0}]$ for all $U_0 \in K(R)$. We have the theorem of the continuous dependence.

Theorem 7 (Theorem 4.3 and Corollary 4.2 in [15]) *Under the assumption of Theorem 5, let U and V be the solutions of (4) for the initial functions U_0 and V_0 in $K(R)$, respectively. Then, we have*

$$t^\eta\left\|A^\eta\left(U(t) - V(t)\right)\right\| + t^\beta\left\|A^\beta\left(U(t) - V(t)\right)\right\| + \left\|U(t) - V(t)\right\| \le L_R\left\|U_0 - V_0\right\|$$

and

$$t^{\eta-\beta}\left\|A^\eta\left(U(t) - V(t)\right)\right\| + \left\|A^\beta\left(U(t) - V(t)\right)\right\| \le L_R\left\|A^\beta\left(U_0 - V_0\right)\right\|$$

for $0 < t \le T_R$, where L_R is a positive constant depending only on R.

We assume that there exists an increasing continuous function $p(\cdot) > 0$ such that any local solution satisfies

$$\left\|A^\beta U(t)\right\| \le p(\left\|A^\beta U_0\right\|)$$

for $t \in [0, T_{U_0}]$ and $U_0 \in \mathcal{D}(A^\beta)$. Theorem 6 implies that there exists a global solution on $[0, +\infty)$ with the estimate

$$\left\|A^\beta U(t)\right\| \le p(\left\|A^\beta U_0\right\|) \tag{6}$$

for $t \in [0, +\infty)$ and $U_0 \in \mathcal{D}(A^\beta)$. We define a nonlinear operator $T(t) : \mathcal{D}(A^\beta) \to \mathcal{D}(A^\beta)$ by $T(t)U_0(\cdot) = U(\cdot, t)$. Let $\mathcal{M}$ be a subset of $\mathcal{D}(A^\beta)$, $\mathcal{M}$ being a metric space with the distance $d(U, V) = \left\|A^\beta\left(U - V\right)\right\|$ for $U, V \in \mathcal{M}$. A family of nonlinear operators $T(t)$ for $t \ge 0$ from $\mathcal{M}$ to itself is said to be a continuous semigroup on $\mathcal{M}$ provided that
(SG.1) $T(0)$ is an identity mapping on $\mathcal{M}$,
(SG.2) $T(t)T(s) = T(t + s)$ for $t, s \ge 0$,
(SG.3) $T(t)$ is continuous from $[0, +\infty) \times \mathcal{M}$ to $\mathcal{M}$.
To show the property (SG.3), we combine Theorem 7 with the estimate (6). We apply the estimate on the larger ball $K_{p(R)} \supset K_R$ because $\cup_{0 \le t < \infty} T(t)K_R \subset K_{p(R)}$.

Theorem 8 (Proposition 6.2 in [15]) *For any* $0 < R < \infty$*, it holds that*

$$\left\| A^{\beta} \left(T(t)U_0 - T(t)V_0 \right) \right\| \leq L_{p(R)}^{n+1} \left\| A^{\beta} \left(U_0 - V_0 \right) \right\|$$

for $t \in [nT_{p(R)}, (n+1)T_{p(R)}]$ *with* $n \in \mathbf{N} \cup \{0\}$ *and* $U_0, V_0 \in K_R$*, where* $L_{p(R)}^{n+1} > 0$ *is a constant depending only on* n *and* $p(R)$*.*

Henceforth, we write $X = \mathcal{D}(A^{\beta})$. We denote the totality of trajectories starting from the points in $\mathcal{M}$ by the triplet $(T(t), \mathcal{M}, X)$ and call it a dynamical system. A set $\Sigma \subset \mathcal{M}$ is said to be positively invariant under $T(t)$ if $T(t)\Sigma \subset \Sigma$ for all $t \geq 0$. A set $\Sigma \subset \mathcal{M}$ is said to be negatively invariant under $T(t)$ if $\Sigma \subset T(t)\Sigma$ for all $t \geq 0$. A set Σ is invariant under $T(t)$ if it satisfies both conditions. A set $A \subset \mathcal{M}$ is said to attract a set $B \subset \mathcal{M}$ under $T(t)$ if

$$\sup_{v \in T(t)B} \inf_{u \in A} \|v - u\| \to 0$$

as $t \to +\infty$. $T(t)$ is said to be dissipative if there exists a bounded set $C \subset \mathcal{M}$ such that attracts every point of $\mathcal{M}$ under $T(t)$. A set $\mathcal{A} \subset \mathcal{M}$ of $(T(t), \mathcal{M}, X)$ is said to be a global attractor if $\mathcal{A}$ is a maximal compact invariant set and attracts every bounded set $B \subset \mathcal{M}$. A set $D \subset \mathcal{M}$ is said to be an absorbing set if for every bounded set $B \subset \mathcal{M}$, there exists t_0 such that $\cup_{t \geq t_0} T(t)B \subset D$ holds. We take $t_1 \geq t_0$ so that $\cup_{t \geq t_1} T(t)D \subset D$ holds. Let $\mathcal{X} = \overline{\cup_{t \geq t_1} T(t)D} \subset D$. $\mathcal{E}$ is said to be an exponential attractor of $(T(t), \mathcal{X}, X)$, provided that
(EA.1) $\mathcal{A} \subset \mathcal{E} \subset \mathcal{X}$ holds, where $\mathcal{A}$ is a global attractor,
(EA.2) $\mathcal{E}$ is compact in X,
(EA.3) $\mathcal{E}$ is positively invariant under $T(t)$,
(EA.4) $\mathcal{E}$ has a finite fractal dimension $d_F(\mathcal{E})$,
(EA.5) $\sup_{u \in T(t)\mathcal{X}} \inf_{v \in \mathcal{E}} \|u - v\| \leq c_0 e^{-c_1 t}$, where c_0 and c_1 are positive constants. Here, if we denote by $N_r(\mathcal{E})$ the smallest number of $r-$balls necessary to cover $\mathcal{E}$, we define a fractal dimension by

$$d_F(\mathcal{E}) = \limsup_{r \to 0} \frac{\log N_r(\mathcal{E})}{\log \frac{1}{r}}.$$

Then, we have

Theorem 9 (Theorem 3.1 in [6]) *Let* $F(U)$ *satisfy the Lipschitz condition*

$$\|F(U) - F(V)\| \leq C_{\mathcal{X}} \left\| A^{\frac{1}{2}} \left(U - V \right) \right\|$$

for $U, V \in \mathcal{X}$*, where* $C_{\mathcal{X}} > 0$ *depends only on* $\mathcal{X}$*. Moreover, we assume that the mapping* $S(t, U_0) = T(t)U_0$ *satisfies the Lipschitz condition*

$$\|S(s, U_0) - S(t, V_0)\| \leq C_{\mathcal{X},T} \left(\|U_0 - V_0\| + |t - s| \right)$$

for $U_0, V_0 \in \mathcal{X}$ *and* $s, t \in [0, T]$ *with any* $T > 0$*, where* $C_{\mathcal{X},T}$ *depends only on* $\mathcal{X}$ *and* T*. Then, the flow* $\{T(t)\}$ *admits an exponential attractor* $\mathcal{E}$*.*

Finally, we introduce the function space treated in this paper. For $p \in \mathbf{N}$, $H^p(\Omega)$ denotes the usual Sobolev space with the norm

$$\|w\|_{H^p} = \left(\sum_{|\alpha| \leq p} \|D^{\alpha} w\|_2^2 \right)^{\frac{1}{2}}$$

for $w \in H^p(\Omega)$, where $\|\cdot\|_p$ denotes the standard L^p norm in Ω, α is a multi index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}.$$

For $0 \le s_0 < s < s_1 < +\infty$, $H^s(\Omega)$ is the interpolation space between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$, denoted $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta, s = (1-\theta)\, s_0 + \theta s_1$ with $\theta \in [0,1]$. Then, the interpolation inequality

$$\|\cdot\|_{H^s} \le C \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta}$$

holds according to Theorem 1.15 in [15]. Moreover, we denote

$$H_N^m(\Omega) = \left\{ u \in H^m(\Omega) \mid \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega \right\}$$

for $m > \frac{3}{2}$. By $\mathcal{D}(\Omega)$, we denote the space of all infinitely differentiable functions on Ω with compact supports. $H_0^s(\Omega)$ is defined as the closure of the set $\mathcal{D}(\Omega)$ in the space $H^s(\Omega)$. $H^{-s}(\Omega)$ is defined as the dual space of $H_0^s(\Omega)$.

3 Local solution We prove the local existence and uniqueness of the solution by the theories of an abstract evolution equation. We show that the nonlinear term in (1) satisfies the condition (5).

Proposition 1 (Local existence in H^γ) *Suppose that $\phi_0, u_0 \in H^\gamma(\Omega)$ for $\underline{\gamma} < \gamma < \overline{\gamma}$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^\gamma_{\phi_0,u_0}]; H^1(\Omega)\right) \cap C\left([0, T^\gamma_{\phi_0,u_0}]; H^\gamma(\Omega)\right) \cap C^1\left((0, T^\gamma_{\phi_0,u_0}]; H^{-1}(\Omega)\right),$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are defined in Theorem 1. In this paper, $T^s_{\phi_0,u_0}$ denotes the maximal existence time depending only on the norms $\|u_0\|_{H^s}$ and $\|\phi_0\|_{H^s}$ of initial functions.

Proof of Proposition 1: (1) can be written into

$$\begin{cases} U_t + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0 \equiv \begin{pmatrix} \phi_0 \\ u_0 \end{pmatrix}, \end{cases}$$

where

$$U = \begin{pmatrix} \phi \\ u \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}, \quad F = \begin{pmatrix} \frac{1}{\tau}\left\{\left(\epsilon^2+1\right)\phi - \phi^3 + 2u\right\} \\ \left(\kappa - \frac{l}{\tau}\right) u + \frac{l}{2\tau}\left(\phi^3 - \phi\right) \end{pmatrix},$$

$$A_1 = -\frac{\epsilon^2}{\tau}(\Delta - 1), \quad A_2 = -\kappa(\Delta - 1) \quad \text{and} \quad B = \frac{l\epsilon^2}{2\tau}\Delta.$$

The two operators A_1 and A_2 are positive definite self-adjoint operators of $H^{-1}(\Omega)$ with domains $\mathcal{D}(A_1) = \mathcal{D}(A_2) = H^1(\Omega)$. We regard B as a linear and bounded operator from $H^1(\Omega)$ to $H^{-1}(\Omega)$. If necessary, we put $w(x,t) = pu(x,t)$ for small $p > 0$. Then, the second equation in (1) is converted into

$$w_t + \frac{lp}{2}\phi_t = \kappa \Delta w.$$

For sufficiently small $p > 0$, we can suppose that

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ pB & A_2 \end{pmatrix}$$

and hence A are strictly positive operators of $X \equiv H^{-1}(\Omega) \times H^{-1}(\Omega)$. Theorems 2.1 and 2.16 in [15] imply that A is a sectorial operator with angle $0 \le \omega_A < \frac{\pi}{2}$ in X. Then, it holds that

$$\mathcal{D}(A^{\beta}) = H^{\gamma}(\Omega) \times H^{\gamma}(\Omega)$$

for $\frac{1}{2} < \beta < 1$, where $\gamma = 2\beta - 1$ (for details, see Theorems 12.1 and 16.7 in [15]). Under our setting, we can apply Theorem 5 in Section 2 to (1). In fact, by the next lemma, we show that the nonlinear term in (1) satisfies the condition (5). We set

$$(n, \underline{\beta}, \overline{\beta}, \overline{\alpha}) = \left(n, \frac{\underline{\gamma}+1}{2}, \frac{\overline{\gamma}+1}{2}, \overline{\alpha}\right) = \begin{cases} (1, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}) & \text{for } n = 1, \\ (2, \frac{1}{2}, \frac{3}{4}, 1) & \text{for } n = 2, \\ (3, \frac{3}{4}, \frac{5}{6}, 1) & \text{for } n = 3. \end{cases}$$

Lemma 1 *Let $n = 1, 2, 3$. Then, there exist α and β satisfying $0 < \underline{\beta} < \beta < \overline{\beta} < \alpha < \overline{\alpha} \le 1$ such that*

$$\left\| (\phi - \psi)^3 \right\|_{H^{-1}} \le C \left\| A_1^{\beta} (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^{\alpha} (\phi - \psi) \right\|_{H^{-1}}$$

for $\phi, \psi \in H^{\alpha}(\Omega)$, where C is a positive constant depending only on α, β and Ω.

Proof of Lemma 1: In the case of $n = 1, 2$, we note that

$$\|w\|_q \le C \|w\|_{H^1}$$

for $w \in H^1(\Omega)$, where $q > 1$ and C is a positive constant depending only on q and Ω. Henceforth, we denote a positive embedding constant depending only on q and Ω by C. We take $0 < p < 2$ and $4 < q$ with $\frac{2}{2+p} + \frac{2}{q} = 1$. For $n = 1$, we have

$$\begin{aligned} \left\| (\phi - \psi)^3 \right\|_{H^{-1}} &= \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \le 1} \left| \int_{\Omega} (\phi - \psi)^3 w dx \right| \\ &\le \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \le 1} \|w\|_q \|\phi - \psi\|_{2+p}^2 \|\phi - \psi\|_q \\ &\le C \|\phi - \psi\|_{H^{\frac{p}{4+2p}}}^2 \|\phi - \psi\|_{H^{\frac{1}{2+p}}} \\ &\le C \left\| A_1^{\frac{4+3p}{4(2+p)}} (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^{\frac{3+p}{2(2+p)}} (\phi - \psi) \right\|_{H^{-1}}. \end{aligned}$$

Here, $\frac{1}{2} < \frac{4+3p}{4(2+p)} < \frac{5}{8} < \frac{3+p}{2(2+p)} < \frac{3}{4}$. For $n = 2$, we have

$$\begin{aligned} \left\| (\phi - \psi)^3 \right\|_{H^{-1}} &\le C \|\phi - \psi\|_{H^{\frac{p}{2+p}}}^2 \|\phi - \psi\|_{H^{\frac{2}{2+p}}} \\ &\le C \left\| A_1^{\frac{1+p}{2+p}} (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^{\frac{4+p}{2(2+p)}} (\phi - \psi) \right\|_{H^{-1}}. \end{aligned}$$

Here, $\frac{1}{2} < \frac{1+p}{2+p} < \frac{3}{4} < \frac{4+p}{2(2+p)} < 1$. In the case of $n = 3$, we note that

$$\|w\|_6 \le C \|w\|_{H^1}$$

for $w \in H^1(\Omega)$. We take $\frac{3}{2} < p < 3$ and $\frac{18}{5} < q < 6$ with $\frac{5}{6+p} + \frac{1}{q} = \frac{5}{6}$. We have

$$\begin{aligned}\left\|(\phi-\psi)^3\right\|_{H^{-1}} &\le \sup_{w\in H_0^1(\Omega),\|w\|_{H^1}\le 1} \|w\|_6 \|\phi-\psi\|^2_{\frac{2}{5}(6+p)} \|\phi-\psi\|_q \\ &\le C\|\phi-\psi\|^2_{H^{\frac{3(1+p)}{2(6+p)}}} \|\phi-\psi\|_{H^{\frac{9-p}{6+p}}} \\ &\le C\left\|A_1^{\frac{5(3+p)}{4(6+p)}}(\phi-\psi)\right\|^2_{H^{-1}} \left\|A_1^{\frac{15}{2(6+p)}}(\phi-\psi)\right\|_{H^{-1}}.\end{aligned}$$

Here, $\frac{3}{4} < \frac{5(3+p)}{4(6+p)} < \frac{5}{6} < \frac{15}{2(6+p)} < 1$. □

For $U = \begin{pmatrix}\phi \\ u\end{pmatrix}, V = \begin{pmatrix}\psi \\ v\end{pmatrix} \in \mathcal{D}(A^\alpha)$ with $\overline{\beta} < \alpha < \overline{\alpha}$, we have

$$F(U)-F(V) = \begin{pmatrix} \frac{1}{\tau}\left\{\left(\epsilon^2+1-3\phi\psi\right)(\phi-\psi)-(\phi-\psi)^3+2(u-v)\right\} \\ \left(\kappa-\frac{l}{\tau}\right)(u-v)+\frac{l}{2\tau}\left\{(\phi-\psi)^3+(3\phi\psi-1)(\phi-\psi)\right\}\end{pmatrix}$$

and concentrate on the estimetes

$$\|\phi-\psi\|_{H^{-1}}, \quad \left\|(\phi-\psi)^3\right\|_{H^{-1}}, \quad \|\phi\psi(\phi-\psi)\|_{H^{-1}}, \quad \|u-v\|_{H^{-1}}.$$

Now by the estimates as obtained in Lemma 1, we can apply Theorem 5 to our setting. □

Remark 1 (Local existence in L^2) *In the case of $n=1$, We can take $\gamma=0$ in Proposition 1. Now that it holds that $H^{\frac{1}{2}+r}(\Omega) \subset C(\overline{\Omega})$ for $r>0$, we have*

$$\left\|(\phi-\psi)^3\right\|_{H^{-1}} \le \sup_{w\in H_0^1(\Omega),\|w\|_{H^1}\le 1} \|w\|_C \|\phi-\psi\|_2^2 \|\phi-\psi\|_C \le C\|\phi-\psi\|_2^2 \|\phi-\psi\|_{H^{\frac{1}{2}+r}},$$

where $r \in (0,\frac{1}{2})$ and $\|\cdot\|_C$ denotes the norm of the space of continuous functions in Ω. Hence, for $\phi_0, u_0 \in L^2(\Omega)$, (1) admits a unique local solution (ϕ,u) such that

$$\phi, u \in C\left((0,T^0_{\phi_0,u_0}];H^1(\Omega)\right) \cap C\left([0,T^0_{\phi_0,u_0}];L^2(\Omega)\right) \cap C^1\left((0,T^0_{\phi_0,u_0}];H^{-1}(\Omega)\right).$$

Proposition 2 (Local existence in H^1) *Suppose that $\phi_0, u_0 \in H^1(\Omega)$. Then, (1) admits a unique local solution (ϕ,u) such that*

$$\phi, u \in C\left((0,T^1_{\phi_0,u_0}];H^2_N(\Omega)\right) \cap C\left([0,T^1_{\phi_0,u_0}];H^1(\Omega)\right) \cap C^1\left((0,T^1_{\phi_0,u_0}];L^2(\Omega)\right).$$

Proof of Proposition 2: In Theorem 5, we take

$$X = L^2(\Omega)\times L^2(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^1(\Omega)\times H^1(\Omega) \quad \mathcal{D}(A) = H^2_N(\Omega)\times H^2_N(\Omega) \quad \beta=\eta=\frac{1}{2}.$$

We have

$$\left\|(\phi-\psi)^3\right\|_2 = \|\phi-\psi\|_6^3 \le C^3\|\phi-\psi\|^3_{H^1}$$

for $\phi, \psi \in H^1(\Omega)$. Hence, we can apply Theorem 5 to our setting. □

Proposition 3 (Local existence in H^2) *Suppose that $\phi_0, u_0 \in H^2_N(\Omega)$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^2_{\phi_0,u_0}]; H^3_N(\Omega)\right) \cap C\left([0, T^2_{\phi_0,u_0}]; H^2_N(\Omega)\right) \cap C^1\left((0, T^2_{\phi_0,u_0}]; H^1(\Omega)\right).$$

Proof of Proposition 3: In Theorem 5, we take

$$X = H^1(\Omega) \times H^1(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^2_N(\Omega) \times H^2_N(\Omega) \quad \mathcal{D}(A) = H^3_N(\Omega) \times H^3_N(\Omega) \quad \beta = \eta = \frac{1}{2}.$$

Since it holds that

$$\|w\|_C \leq C \|w\|_{H^2}$$

for $w \in H^2_N(\Omega)$, we have

$$\left\|\nabla (\phi - \psi)^3\right\|_2 = 3 \left\|(\phi - \psi)^2 \nabla (\phi - \psi)\right\|_2 \leq 3C^2 \|\phi - \psi\|^2_{H^2} \|\phi - \psi\|_{H^1} \leq 3C^2 \|\phi - \psi\|^3_{H^2}$$

for $\phi, \psi \in H^2_N(\Omega)$, which proves the proposition. □

Proposition 4 (Local existence in H^3) *Suppose that $\phi_0, u_0 \in H^3_N(\Omega)$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^3_{\phi_0,u_0}]; H^4_N(\Omega)\right) \cap C\left([0, T^3_{\phi_0,u_0}]; H^3_N(\Omega)\right) \cap C^1\left((0, T^3_{\phi_0,u_0}]; H^2_N(\Omega)\right).$$

Proof of Proposition 4: In Theorem 5, we take

$$X = H^2_N(\Omega) \times H^2_N(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^3_N(\Omega) \times H^3_N(\Omega) \quad \mathcal{D}(A) = H^4_N(\Omega) \times H^4_N(\Omega) \quad \beta = \eta = \frac{1}{2}.$$

The following estimate shows the proposition.

$$\begin{aligned} \left\|\Delta (\phi - \psi)^3\right\|_2 &\leq 6 \left\|(\phi - \psi) |\nabla (\phi - \psi)|^2\right\|_2 + 3 \left\|(\phi - \psi)^2 \Delta (\phi - \psi)\right\|_2 \\ &\leq 6C^2 \|\phi - \psi\|_{H^3} \|\phi - \psi\|_{H^2} \|\phi - \psi\|_{H^1} + 3C^2 \|\phi - \psi\|^3_{H^2} \\ &\leq 9C^2 \|\phi - \psi\|^3_{H^3} \end{aligned}$$

for $\phi, \psi \in H^3_N(\Omega)$. □

4 Global solution We derive the a priori estimates to obtain the global solution. The tools are the Lyapunov function and energy method.

Lemma 2 *For $\phi_0, u_0 \in H^1(\Omega)$ and $t \in [0, T^1_{\phi_0,u_0}]$,*

$$L(\phi, u)(t) = \frac{1}{2} \int_\Omega u^2 dx + \frac{l\epsilon^2}{8} \int_\Omega |\nabla \phi|^2 dx + \frac{l}{4} \int_\Omega W(\phi) dx + \frac{\kappa\delta}{2} \int_\Omega |\nabla u|^2 dx$$

is the Lyapunov function for (1), where $\delta < \frac{4\tau}{l}$ and $W(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2$.

Proof of Lemma 2: We have only to prove that $L(\phi,u)(t)$ is monotone decreasing with respect to t. Now that we have $(\phi(t),u(t)) \in H^1(\Omega)\times H^1(\Omega)$ for $t\in[0,T^1_{\phi_0,u_0}]$ from Proposition 2, $L(\phi,u)(t)<\infty$ because of the inclusion $H^1(\Omega)\subset L^4(\Omega)$. Note that

$$l\tau a^2+2l\delta ab+4\delta b^2 = l\left(\sqrt{\tau}a+\frac{\delta}{\sqrt{\tau}}b\right)^2+\delta\frac{4\tau-l\delta}{\tau}b^2\geq 0$$

for $a,b\in\mathbf{R}$ and $\delta<\frac{4\tau}{l}$. We have

$$\begin{aligned}
L(\phi,u)(t)-L(\phi,u)(t') &= \int_{t'}^{t}\frac{d}{dt}L(\phi,u)(s)ds\\
&= \int_{t'}^{t}\int_\Omega uu_t dxds+\frac{l\epsilon^2}{4}\int_{t'}^{t}\int_\Omega \nabla\phi\cdot\nabla\phi_t dxds+\frac{l}{4}\int_{t'}^{t}\int_\Omega\left(\phi^2-1\right)\phi\phi_t dxds\\
&\quad +\kappa\delta\int_{t'}^{t}\int_\Omega\nabla u\cdot\nabla u_t dxds\\
&= \int_{t'}^{t}\int_\Omega u\left(\kappa\Delta u-\frac{l}{2}\phi_t\right)dxds-\frac{l\epsilon^2}{4}\int_{t'}^{t}\int_\Omega\Delta\phi\phi_t dxds\\
&\quad +\frac{l}{4}\int_{t'}^{t}\int_\Omega\left(\phi^2-1\right)\phi\phi_t dxds-\delta\int_{t'}^{t}\int_\Omega u_t\left(u_t+\frac{l}{2}\phi_t\right)dxds\\
&= -\kappa\int_{t'}^{t}\int_\Omega|\nabla u|^2 dxds-\frac{1}{4}\int_{t'}^{t}\int_\Omega\left(l\tau\phi_t^2+2l\delta\phi_t u_t+4\delta u_t^2\right)dxds\leq 0
\end{aligned}$$

for $0\leq t'<t\leq T^1_{\phi_0,u_0}$. In particular, we have

$$\kappa\int_0^t\|\nabla u\|_2^2ds+\frac{\delta(4\tau-l\delta)}{4\tau}\int_0^t\|u_t\|_2^2ds\leq L(\phi_0,u_0)-L(\phi,u)(t)\leq L(\phi_0,u_0). \tag{7}$$

On the other hand, since

$$l\tau a^2+2l\delta ab+4\delta b^2=\delta\left(2b+\frac{l}{2}a\right)^2+l\frac{4\tau-l\delta}{4}a^2\geq 0$$

for $a,b\in\mathbf{R}$ and $\delta<\frac{4\tau}{l}$, it also holds that

$$\kappa\int_0^t\|\nabla u\|_2^2ds+\frac{l(4\tau-l\delta)}{16}\int_0^t\|\phi_t\|_2^2ds\leq L(\phi_0,u_0)-L(\phi,u)(t)\leq L(\phi_0,u_0). \tag{8}$$

□

Proposition 5 (Global existence in H^1) *Suppose that $\phi_0,u_0\in H^1(\Omega)$. Then, (1) admits a unique global solution (ϕ,u) such that*

$$\phi,u\in C\left((0,+\infty);H^2_N(\Omega)\right)\cap C\left([0,+\infty);H^1(\Omega)\right)\cap C^1\left((0,+\infty);L^2(\Omega)\right).$$

Proof of Proposition 5: By Proposition 2, there exists a unique local solution (ϕ,u) in the same function space. We have only to derive the a priori estimate thanks to Theorem

6. From Lemma 2, it holds that

$$\begin{aligned}
&\frac{1}{2}\|u\|_2^2+\frac{l\epsilon^2}{8}\|\nabla\phi\|_2^2+\frac{l}{16}\|\phi\|_2^2+\frac{l}{16}\int_\Omega\left(\phi^2-\frac{3}{2}\right)^2dx+\frac{\kappa\delta}{2}\|\nabla u\|^2-\frac{5l}{64}|\Omega|\\
&= L(\phi,u)(t)\\
&\le L(\phi_0,u_0)\\
&\le \frac{1}{2}\|u_0\|_2^2+\frac{l\epsilon^2}{8}\|\phi_0\|_{H^1}^2+\frac{l}{16}\|\phi_0\|_4^4+\frac{l}{16}|\Omega|+\frac{\kappa\delta}{2}\|\nabla u_0\|^2.
\end{aligned}$$

The Sobolev embedding theorem implies that the right-hand side is finite, which completes the proof of Proposition 5. □

Proposition 6 (Global existence in H^2) *Suppose that $\phi_0, u_0 \in H_N^2(\Omega)$. Then, (1) admits a unique global solution (ϕ, u) such that*

$$\phi,u\in C\left((0,+\infty);H_N^3(\Omega)\right)\cap C\left([0,+\infty);H_N^2(\Omega)\right)\cap C^1\left((0,+\infty);H^1(\Omega)\right).$$

Proof of Proposition 6: As mentioned in Proposition 5, we derive the a priori estimates for H^2 norm. In this paper, we denote by $C_{H^s}>0$ the constant depending only on the norms $\|u_0\|_{H^s}$ and $\|\phi_0\|_{H^s}$ of initial functions, the measure $|\Omega|$ and physical constants $\tau, l, \kappa, \epsilon$. We have the following two inequalities from (1):

$$\begin{aligned}
\frac{\tau}{2}\frac{d}{dt}\|\phi_t\|_2^2+\epsilon^2\|\nabla\phi_t\|_2^2+3\int_\Omega\phi^2\phi_t^2dx &= \int_\Omega\phi_t\left(\tau\phi_t-\epsilon^2\Delta\phi+\phi^3\right)_t dx\\
&= \|\phi_t\|_2^2+2\int_\Omega u_t\phi_t dx \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\kappa\|\nabla u_t\|_2^2+\frac{l}{\tau}\|u_t\|_2^2-\frac{\epsilon^2 l}{2\tau}\int_\Omega\nabla u_t\cdot\nabla\phi_t dx\\
&\quad+\frac{l}{2\tau}\int_\Omega u_t\left(\phi_t-3\phi^2\phi_t\right)dx\\
&= \int_\Omega u_t\left\{u_{tt}-\kappa\Delta u_t+\frac{l}{\tau}u_t+\frac{\epsilon^2 l}{2\tau}\Delta\phi_t+\frac{l}{2\tau}\left(\phi_t-3\phi^2\phi_t\right)\right\}dx\\
&= \frac{l}{2\tau}\int_\Omega u_t\left(-\tau\phi_t+\epsilon^2\Delta\phi+\phi-\phi^3+2u\right)_t dx=0. \qquad (10)
\end{aligned}$$

By integrating (9) over $(0,t)$ with respect to t, we have

$$\frac{\tau}{2}\|\phi_t\|_2^2+\epsilon^2\int_0^t\|\nabla\phi_t\|_2^2ds\le\frac{\tau}{2}\|(\phi_0)_t\|_2^2+2\int_0^t\|\phi_t\|_2^2ds+\int_0^t\|u_t\|_2^2ds, \qquad (11)$$

which implies that $\phi_t\in L^2(\Omega)$ by (7) and (8). Hence by (1), we have

$$\|\Delta\phi\|_2\le C_{H^2}\quad\text{and}\quad\|\phi_t\|_2\le C_{H^2}. \qquad (12)$$

Next by integrating (10) over $(0,t)$ with respect to t, we have

$$\begin{aligned} &\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|(u_0)_t\|_2^2 + \kappa\int_0^t \|\nabla u_t\|_2^2\, ds + \frac{l}{\tau}\int_0^t \|u_t\|_2^2\, ds \\ &-\frac{\epsilon^2 l}{2\tau}\int_0^t\int_\Omega \nabla u_t\cdot\nabla\phi_t dxds + \frac{l}{2\tau}\int_0^t\int_\Omega u_t\left(\phi_t - 3\phi^2\phi_t\right)dxds = 0. \end{aligned} \tag{13}$$

Here, it holds that

$$\begin{aligned} \int_0^t\int_\Omega \nabla u_t\cdot\nabla\phi_t dxds &= \frac{2}{l}\int_0^t\int_\Omega \nabla u_t\cdot\nabla\left(\kappa\Delta u - u_t\right)dxds \\ &= -\frac{\kappa}{l}\left(\|\Delta u\|_2^2 - \|\Delta u_0\|_2^2\right) - \frac{2}{l}\int_0^t \|\nabla u_t\|_2^2\, ds. \end{aligned}$$

From (12), $\|\phi\|_{H^2}$ is bounded, which implies $\phi\in C(\overline{\Omega})$ from the Sobolev embedding theorem. Then, it holds that

$$\left|\int_0^t\int_\Omega u_t\left(\phi_t - 3\phi^2\phi_t\right)dxds\right| \le C_{H^2}\int_0^t\left(\|u_t\|_2^2 + \|\phi_t\|_2^2\right)ds.$$

Thus (13) becomes

$$\begin{aligned} &\frac{1}{2}\|u_t\|_2^2 + \frac{\epsilon^2\kappa}{2\tau}\|\Delta u\|_2^2 + \left(\kappa + \frac{\epsilon^2}{\tau}\right)\int_0^t \|\nabla u_t\|_2^2\, ds \\ \le\ &\frac{1}{2}\|(u_0)_t\|_2^2 + \frac{\epsilon^2\kappa}{2\tau}\|\Delta u_0\|_2^2 + \frac{lC_{H^2}}{2\tau}\int_0^t\left(\|u_t\|_2^2 + \|\phi_t\|_2^2\right)ds. \end{aligned} \tag{14}$$

Finally, we obtain

$$\|\Delta u\|_2 \le C_{H^2} \quad \text{and} \quad \|u_t\|_2 \le C_{H^2} \tag{15}$$

by (7) and (8). After all, (12) and (15) imply the conclusion of proposition. □

Proposition 7 (Global existence in H^3) *Suppose that $\phi_0, u_0 \in H_N^3(\Omega)$. Then, (1) admits a unique global solution (ϕ, u) such that*

$$\phi, u \in C\left((0,+\infty); H_N^4(\Omega)\right)\cap C\left([0,+\infty); H_N^3(\Omega)\right)\cap C^1\left((0,+\infty); H_N^2(\Omega)\right).$$

Proof of Proposition 7: We derive the a priori estimates for H^3 norm. We have

$$\begin{aligned} \frac{\tau}{2}\frac{d}{dt}\int_\Omega |\nabla\phi_t|^2\, dx &= \int_\Omega \nabla\phi_t\cdot\nabla\left(\epsilon^2\Delta\phi + \phi - \phi^3 + 2u\right)_t dx \\ &\le \int_\Omega\left(2|\nabla\phi_t|^2 + |\nabla u_t|^2\right)dx + 3\int_\Omega \Delta\phi_t\phi^2\phi_t dx \\ &= \int_\Omega\left(2|\nabla\phi_t|^2 + |\nabla u_t|^2\right)dx + \frac{3}{\epsilon^2}\int_\Omega \phi^2\phi_t\left(\tau\phi_t - \phi + \phi^3 - 2u\right)_t dx \\ &\le 2\|\nabla\phi_t\|_2^2 + \|\nabla u_t\|_2^2 + C_{H^2}\|\phi_{tt}\|_2^2 + C_{H^2}\|\phi_t\|_2^2 + C_{H^2}\|u_t\|_2^2 \end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_t|^2\,dx &= \int_\Omega \nabla u_t\cdot\nabla\left(\kappa\Delta u-\frac{l}{2}\phi_t\right)_t dx\\
&\le \frac{l}{2}\int_\Omega \Delta u_t\phi_{tt}dx\\
&= \frac{l}{2\kappa}\int_\Omega\left(u_t+\frac{l}{2}\phi_t\right)_t\phi_{tt}dx\\
&\le \frac{l}{4\kappa}\|u_{tt}\|_2^2+\frac{l(l+1)}{4\kappa}\|\phi_{tt}\|_2^2.
\end{aligned}
$$

We integrate these inequalities with respect to t and obtain

$$
\begin{aligned}
\frac{\tau}{2}\int_\Omega|\nabla\phi_t|^2\,dx \quad&\le\quad \frac{\tau}{2}\|\nabla(\phi_0)_t\|_2^2+C_{H^2}\int_0^t\|\phi_{tt}\|_2^2\,ds\\
&+\int_0^t\left(2\|\nabla\phi_t\|_2^2+\|\nabla u_t\|_2^2+C_{H^2}\|\phi_t\|_2^2+C_{H^2}\|u_t\|_2^2\right)ds
\end{aligned}
$$

and

$$
\frac{1}{2}\int_\Omega|\nabla u_t|^2\,dx\le\frac{1}{2}\|\nabla(u_0)_t\|_2^2+\frac{l}{4\kappa}\int_0^t\|u_{tt}\|_2^2\,ds+\frac{l(l+1)}{4\kappa}\int_0^t\|\phi_{tt}\|_2^2\,ds.
$$

Now we have only to estimate $\int_0^t\|u_{tt}\|_2^2\,ds$ and $\int_0^t\|\phi_{tt}\|_2^2\,ds$ for $t>0$ owing to (7), (8), (11) and (14). It holds that

$$
\begin{aligned}
&\tau\int_0^t\int_\Omega\phi_{tt}^2dxds=\int_0^t\int_\Omega\phi_{tt}\left(\epsilon^2\Delta\phi+\phi-\phi^3+2u\right)_t dxds\\
&\le\quad \frac{\epsilon^2}{2}\|\nabla(\phi_0)_t\|_2^2+\frac{1}{2}\|\phi_t\|_2^2+\int_0^t\int_\Omega\sqrt{\frac{1}{\tau}}\left(3\phi^2|\phi_t|+2|u_t|\right)\cdot\sqrt{\tau}\,|\phi_{tt}|\,dxds\\
&\le\quad \frac{\epsilon^2}{2}\|\nabla(\phi_0)_t\|_2^2+\frac{1}{2}\|\phi_t\|_2^2+\frac{9}{\tau}\|\phi\|_\infty^4\int_0^t\|\phi_t\|_2^2\,ds+\frac{4}{\tau}\int_0^t\|u_t\|_2^2\,ds\\
&\quad+\frac{\tau}{2}\int_0^t\int_\Omega\phi_{tt}^2dxds.
\end{aligned}
$$

Hence, we have

$$
\int_0^t\|\phi_{tt}\|_2^2\,ds\le\frac{\epsilon^2}{\tau}\|\nabla(\phi_0)_t\|_2^2+C_{H^2}
$$

from (7), (8) and (12). Next, we have

$$
\begin{aligned}
\int_0^t\int_\Omega u_{tt}^2dxds \quad&=\quad \int_0^t\int_\Omega u_{tt}\left(\kappa\Delta u-\frac{l}{2}\phi_t\right)_t dxds\\
&\le\quad -\frac{\kappa}{2}\int_0^t\frac{d}{ds}\|\nabla u_t\|_2^2\,ds+\int_0^t\int_\Omega|u_{tt}|\cdot\frac{l}{2}|\phi_{tt}|\,dxds\\
&\le\quad \frac{\kappa}{2}\|\nabla(u_0)_t\|_2^2+\frac{1}{2}\int_0^t\int_\Omega u_{tt}^2dxds+\frac{l^2}{8}\int_0^t\int_\Omega\phi_{tt}^2dxds
\end{aligned}
$$

and

$$
\int_0^t\|u_{tt}\|_2^2\,ds\le\kappa\|\nabla(u_0)_t\|_2^2+\frac{l^2\epsilon^2}{4\tau}\|\nabla(\phi_0)_t\|_2^2+C_{H^2},
$$

which yields the desired estimates. □

5 Dynamical system For $(\phi_0, u_0) \in H^\gamma(\Omega) \times H^\gamma(\Omega)$, we show that (1) has a global solution

$$\phi, u \in C\left((0,+\infty); H^1(\Omega)\right) \cap C\left([0,+\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,+\infty); H^{-1}(\Omega)\right).$$

By $T(t)$, we denote a nonlinear semigroup $(\phi_0, u_0) \mapsto (\phi(t), u(t))$ acting on $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

Proof of Theorem 1 By Proposition 1, we have a local solution ϕ, u in $[0, T^\gamma_{\phi_0,u_0}]$ with the estimate

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^\gamma}$$

for $t \in [0, T^\gamma_{\phi_0,u_0}]$ by Theorem 5. Let any small $t_1 \in (0, T^\gamma_{\phi_0,u_0})$ be fixed. Then, it holds that $\phi(t_1), u(t_1) \in H^1(\Omega)$. By Proposition 5, there exists a global solution

$$\phi, u \in C\left((t_1,+\infty); H^2_N(\Omega)\right) \cap C\left([t_1,+\infty); H^1(\Omega)\right) \cap C^1\left((t_1,+\infty); L^2(\Omega)\right)$$

with the estimate

$$\|\phi(t)\|_{H^1} + \|u(t)\|_{H^1} \le C_{H^1} \tag{16}$$

for $t \ge t_1$ with initial functions $\phi_0 = \phi(t_1), u_0 = u(t_1)$. Then, we have

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^1}$$

for $t \ge t_1$. Again, according to Theorem 5,

$$t_1^{1-\beta}\left(\|\phi(t_1)\|_{H^1} + \|u(t_1)\|_{H^1}\right) \le C_{H^\gamma}.$$

Finally, we have

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^\gamma}$$

for $t \ge 0$. By Theorems 6 and 8, we can extend a time local solution globally in the space

$$\phi, u \in C\left((0,+\infty); H^1(\Omega)\right) \cap C\left([0,+\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,+\infty); H^{-1}(\Omega)\right)$$

and have a continuous mapping $T(t)$ from $[0,+\infty) \times H^\gamma(\Omega)$ to $H^\gamma(\Omega)$, which shows that $T(t)$ defines a dynamical system in $H^\gamma(\Omega) \times H^\gamma(\Omega)$. □

Proof of Theorem 2 For any $\eta > 0$, we have $\phi(\eta), u(\eta) \in H^1(\Omega)$. By the same argument as proof of Theorem 1, we have a global solution

$$\phi, u \in C\left((\eta,+\infty); H^2_N(\Omega)\right) \cap C\left([\eta,+\infty); H^1(\Omega)\right) \cap C^1\left((\eta,+\infty); L^2(\Omega)\right)$$

with the estimate (16) for $t \ge \eta$ with initial functions $\phi_0 = \phi(\eta), u_0 = u(\eta)$. Hence, the compactness of the orbit in $H^\gamma(\Omega) \times H^\gamma(\Omega)$ follows. Differentiating (1) with respect to t successively and making similar energy estimates to the proof of Proposition 7, we have the uniform boundedness of the orbit $\cup_{t\ge\eta} T(t)(\phi_0, u_0)$ in $H^m_N(\Omega) \times H^m_N(\Omega)$ for any small $\eta > 0$ and $m = 4, 5, \cdots$. We use the standard bootstrap argument to prove that

$$(\phi, u) \in C^\infty\left((0,+\infty); C^\infty(\overline{\Omega})\right) \times C^\infty\left((0,+\infty); C^\infty(\overline{\Omega})\right).$$

□

Proof of Theorem 3 We have a unique global solution $\phi, u \in H^1(\Omega)$ and Lyapunov function $L(\phi, u)(t)$. Therefore, the ω-limit set $\omega(\phi_0, u_0)$ of ϕ_0 and u_0 is nonempty, compact, invariant and connected in $H^1(\Omega) \times H^1(\Omega)$ according to Theorem 4.3.3 in [9]. And it holds that $\omega(\phi_0, u_0) \subset E$ by Theorem 4.3.4 in [9]. For any $\eta > 0$, we have $\phi(\eta), u(\eta) \in H^2_N(\Omega)$ by Proposition 2. By the estimates in Proposition 7, $\cup_{t\geq\eta}T(t)(\phi_0, u_0)$ is precompact in $H^2_N(\Omega) \times H^2_N(\Omega)$. As mentioned in Proposition 1, A is supposed to be a positive operator in $L^2(\Omega) \times L^2(\Omega)$ with domain $H^2_N(\Omega) \times H^2_N(\Omega)$. The similar computation to Lemma 2 shows that

$$-\frac{d}{dt}L(\phi, u)(t) \geq \frac{l(4\tau - l\delta)}{32}\int_\Omega \phi_t^2 dx + \frac{\delta(4\tau - l\delta)}{8\tau}\int_\Omega u_t^2 dx.$$

Hence, we can apply Theorem 1.1 in [10] to deduce that $\omega(\phi_0, u_0)$ is a single point in E. By the second equation in (1), (ϕ, u) satisfies

$$\frac{d}{dt}\int_\Omega \left(u + \frac{l}{2}\phi\right) dx = \kappa \int_\Omega \Delta u dx = 0.$$

Hence, we have

$$\int_\Omega \left(u + \frac{l}{2}\phi\right) dx = \int_\Omega \left(u_0 + \frac{l}{2}\phi_0\right) dx = m$$

for some $m \in \mathbf{R}$. The stationary solution $\Phi = \Phi(x)$ is satisfies

$$\begin{cases} \epsilon^2\Delta\Phi + \Phi - \Phi^3 + \frac{2}{|\Omega|}\left(m - \frac{l}{2}\int_\Omega \Phi dx\right) = 0 & x \in \Omega, \\ \frac{\partial\Phi}{\partial\nu} = 0 & x \in \partial\Omega \end{cases}$$

because the stationary solution satisfies $\Delta U = 0$ in Ω and $U = U(x)$ is constant in Ω. □

6 Exponential attractor First, we derive the estimate for H^3 norm to obtain an absorbing set in H^3. Next, we construct an exponential attractor in $H^1 \times H^1$.

Proof of Theorem 4: If $(\phi_0, u_0) \in H_k$, then we have

$$\|\phi\|_{H^1} + \|u\|_{H^1} \leq \sqrt{\left(k + \frac{5l}{64}|\Omega|\right)\left\{\min\left(\frac{l\epsilon^2}{8}, \frac{l}{16}\right)^{-\frac{1}{2}} + \min\left(\frac{1}{2}, \frac{\kappa\delta}{2}\right)^{-\frac{1}{2}}\right\}} \tag{17}$$

for all $t \geq 0$ by Proposition 5. By Theorem 5, Propositions 2 and 3, we have $\phi(\frac{t_1}{2}), u(\frac{t_1}{2}) \in H^2_N(\Omega)$ and $\phi(t_1), u(t_1) \in H^3_N(\Omega)$ for small $t_1 > 0$ with the estimate

$$\left(\frac{t_1}{2}\right)^{\frac{1}{2}}\left(\left\|\phi\left(\frac{t_1}{2}\right)\right\|_{H^2} + \left\|u\left(\frac{t_1}{2}\right)\right\|_{H^2}\right) \leq C_{H^1} \leq C_k$$

with initial functions $\phi_0 = \phi(0), u_0 = u(0)$ by (17) and

$$\left(\frac{t_1}{2}\right)^{\frac{1}{2}}\left(\|\phi(t_1)\|_{H^3} + \|u(t_1)\|_{H^3}\right) \leq C_{H^2}$$

with initial functions $\phi_0 = \phi\left(\frac{t_1}{2}\right), u_0 = u\left(\frac{t_1}{2}\right)$, where $C_k > 0$ is a constant depending only on the fixed k, the measure $|\Omega|$ and physical constants $\tau, l, \kappa, \epsilon$. Hence, we have

$$\|\phi(t)\|_{H^3} + \|u(t)\|_{H^3} \leq C_k$$

for all $t > t_1$ by Proposition 7. For any bounded set $B \subset H_k$, we have

$$\cup_{t\geq t_1}T(t)B \subset \mathcal{B} \equiv \{(\phi, u) \in H_k \mid \|\phi\|_{H^3} + \|u\|_{H^3} \leq C_k\}$$

for some $C_k > 0$. In particular, $T(t)\mathcal{B} \subset \mathcal{B}$ for all $t \geq t_1$. This set $\mathcal{B}$ shows us the existence of an absorbing set in H_k, which implies that the dynamical system $T(t)$ is dissipative in H_k. We apply Theorem 1.1 in [14] to guarantee the existence of global attractor $\mathcal{A} \subset H_k$. Let $\mathcal{X} = \overline{\cup_{t \geq t_1} T(t)\mathcal{B}}$. Then, $\mathcal{X}$ is a compact, invariant and absorbing set in $H^1(\Omega) \times H^1(\Omega)$. From now on, we consider the subdynamical system $T(t) : \mathcal{X} \to \mathcal{X}$. To construct an exponential attractor, we apply Theorem 9. Let $U = T(t)U_0 = \begin{pmatrix} \phi \\ u \end{pmatrix} \in \mathcal{X}, V = T(t)V_0 = \begin{pmatrix} \psi \\ v \end{pmatrix} \in \mathcal{X}$ and $s, t \in [0, T]$ for any $T > 0$. The first inequality follows at once from Propositions 2 and 3. Next, we prove the second inequality. We have

$$\begin{aligned} \|U(t) - V(s)\|_{H^1} &\leq \|U(t) - V(t)\|_{H^1} + \|V(t) - V(s)\|_{H^1} \\ &\leq \|U(t) - V(t)\|_{H^1} + \int_s^t \left\| \frac{dV}{dt}(p) \right\|_{H^1} dp \\ &\leq \|U(t) - V(t)\|_{H^1} + \int_s^t \left(\|AV\|_{H^1} + \|F(V)\|_{H^1} \right) dp \end{aligned}$$

for $s \leq t$. Since it holds the estimate in Theorem 8 and $AV, F(V) \in H^1(\Omega) \times H^1(\Omega)$ for $V(t) \in \mathcal{X}$,

$$\|U(s) - V(t)\|_{H^1} \leq C_k \|U_0 - V_0\|_{H^1} + C_k |t - s|,$$

which completes the proof of Theorem 4. □

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