NORMAL REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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Abstract. We first show that there exist no real hypersurfaces M^{2n-1} which are Kenmotsu manifolds with respect to the almost contact metric structure (ϕ, ξ, η, g) on M induced from the Kähler structure of a complex $n(\geqq 2)$-dimensional nonflat complex space form $\widetilde{M}_n(c)$. Next, weakening this condition, we classify normal real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$ and give some necessary and sufficient conditions for a real hypersurface M to be normal from the viewpoint of submanifold theory.

1 Introduction We denote by $\widetilde{M}_n(c)$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$, namely it is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive or negative, which is called an n-dimensional *nonflat complex space form* of constant holomorphic sectional curvature c.

In order to bridge between submanifold theory and contact geometry, we study real hypersurfaces M^{2n-1} isometrically immersed into $\widetilde{M}_n(c)$. We take and fix a unit normal vector field $\mathcal{N}$ locally on M. It is well-known that every real hypersurface M^{2n-1} of $\widetilde{M}_n(c)$ admits an almost contact metric structure (ϕ, ξ, η, g) from the Kähler structure (J, g) of the ambient space $\widetilde{M}_n(c)$. Making use of such a structure, many geometers have studied real hypersurface in nonflat complex space forms (cf. [14]). On the other hand, contact geometry has been developed also by many geometers (for examples, see [3, 4, 8]).

In this paper, we pay particular attention to normal real hypersurfaces M in $\widetilde{M}_n(c)$, that is, M satisfies $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$, where $d\eta$ is given by $d\eta(X, Y) = (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\}$ and $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Note that normal almost contact metric manifolds in contact geometry correspond to complex manifolds in complex differential geometry.

The purpose of this paper is to prove the following:

Theorem. *For connected real hypersurfaces M^{2n-1} isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, the following statements* (1) *and* (2) *hold with respect to the almost contact metric structure (ϕ, ξ, η, g) on M induced from the Kähler structure of the ambient space $\widetilde{M}_n(c)$.*

(1) *There exist no real hypersurfaces M which are Kenmotsu manifolds.*

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(2) *The following six statements* $2_a), 2_b), 2_c), 2_d), 2_e)$ *and* $2_f)$ *are mutually equivalent.*

$2_a)$ *M is locally congruent to a hypersurface of type (A).*

$2_b)$ *M is a normal almost contact metric manifold.*

$2_c)$ *Every geodesic $\gamma = \gamma(s)$ on M has constant first curvature $\kappa_\gamma := \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ along γ, where $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\widetilde{M}_n(c)$.*

$2_d)$ *M is locally congruent to a naturally reductive Riemannian homogeneous manifold and expressed as an orbit of a subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$, namely M is a homogeneous real hypersurface of $\widetilde{M}_n(c)$.*

$2_e)$ *M is a Hopf hypersurface and the shape operator A of M is ϕ-invariant, i.e., A satisfies $g(A\phi X, \phi Y) = g(AX, Y)$ for all vecors X and Y orthogonal to the characteristic vector ξ on M.*

$2_f)$ *M is locally congruent to a GO-space, and a homogeneous real hypersurface of $\widetilde{M}_n(c)$.*

Due to this fact, we can see that normal real hypersurfaces are nice examples of real hypersurfaces having many geometric properties in $\widetilde{M}_n(c)$ but they are not Kenmotsu manifolds.

2 Definitions in contact geometry It is well-known that an almost contact metric manifold (M, ϕ, ξ, η, g) satisfies

$$\phi\xi = 0,\ \eta(\phi X) = 0,\ \eta(\xi) = 1,\ \phi^2 X = -X + \eta(X)\xi,\ g(X, \xi) = \eta(X),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for vector fields X and Y on M.

We can define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\Big(X,\, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi,\, \eta(X)\frac{d}{dt}\Big),$$

where f is a smooth function on $M\times\mathbb{R}$. Then the almost complex structure J is integrable if and only if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be normal if the almost complex structure J on $M \times \mathbb{R}$ is integrable. We can see that an almost contact metric manifold M is normal if and only if

$$(\phi\nabla_X\phi)Y - (\nabla_{\phi X}\phi)Y - (\nabla_X\eta)(Y)\cdot\xi = 0 \quad \text{for all } X, Y \in TM, \tag{2.1}$$

where ∇ denotes the Riemannian connection to the Riemannian metric g of M (see page 171 in [18]). An almost contact metric manifold (M, ϕ, ξ, η, g) is called a *Kenmotsu manifold* if M satisfies the following two equalities:

$$(\nabla_X\phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi \quad \text{and} \quad \nabla_X\xi = X - \eta(X)\xi \tag{2.2}$$

for vector fields X and Y on M. It follows from (2.1) and (2.2) that every Kenmotsu manifold is normal. We next recall the definition of Sasakian manifolds. An almost contact metric manifold (M, ϕ, ξ, η, g) is called a *Sasakian manifold* if M satisfies the following equation:

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X \quad \text{for all } X, Y \in TM. \tag{2.3}$$

It follows from (2.1) and (2.3) that every Sasakian manifold is a normal almost contact metric manifold. A Sasakian manifold M is called a *Sasakian space form* if every ϕ-sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ associated to a unit vector $u(\in TM)$ orthogonal o ξ does not depend on the choice of u, where R is the curvature tensor of M. Sasakian manifolds and Sasakian space forms are analogues to Kähler manifolds and complex space forms, respectively.

3 Preliminaries on real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$ Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of an $n(\geqq 2)$-dimensional nonflat complex space form $\widetilde{M}_n(c)$ with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{3.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{3.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\widetilde{M}_n(c)$. We denote by V_λ the eigenspace associated with the principal curvature λ, namely we set $V_\lambda = \{v \in TM | Av = \lambda v\}$.

On M it is well-known that an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the structure (J, g) of the ambient space $\widetilde{M}_n(c)$, which is defined by

$$g(\phi X, Y) = g(JX, Y), \ \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

It follows from (3.1), (3.2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{3.3}$$

$$\nabla_X \xi = \phi AX. \tag{3.4}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z, W) = (c/4)\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\ &+ g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W). \end{aligned} \tag{3.5}$$

We have the Codazzi equation given by

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \tag{3.6}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. Every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact means that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in a nonflat complex space form.

Lemma A ([11, 7])**.** *Let M be a Hopf hypersurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following hold.*

(1) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where δ is the principal curvature associated with ξ. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*

(2) *The principal curvature δ associated with ξ is locally constant.*

We here recall the following real hypersurfaces which are the simplest examples of Hopf hypersurfaces.
When $c > 0$,

(A_1) a geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$,

(A_2) a tube of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic complex submanifold $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$ in $\mathbb{C}P^n(c)$.

When $c < 0$,

(A_0) a horosphere HS in $\mathbb{C}H^n(c)$,

$(A_{1,0})$ a geodesic sphere $G(r)$ of radius $r\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$,

$(A_{1,1})$ a tube of of radius $r\,(0 < r < \infty)$ around a totally geodesic complex hypersurface $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$,

(A_2) a tube of radius $r\,(0 < r < \infty)$ around a totally geodesic complex submanifold $\mathbb{C}H^\ell(c)$ with $1 \leqq \ell \leqq n-2$.

Unifying these real hypersurfaces in $\widetilde{M}_n(c), n \geqq 2$, we call them *hypersurfaces of type* (A). The following shows the importance of hypersurfaces of type (A) in the theory of real hypersurfaces in $\widetilde{M}_n(c)$ (for example, see [14]).

Theorem A. *For every real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, the length of the derivative of the shape operator A of M satisfies $\|\nabla A\|^2 \geqq (c^2/4)(n-1) > 0$ at its each point. In particular, $\|\nabla A\|^2 = (c^2/4)(n-1)$ holds on M if and only if M is locally congruent to a hypersurface of type (A).*

The following gives a characterization of hypersurfaces of type (A) in $\widetilde{M}_n(c)$.

Theorem B. *Let M be a connected real hypersurface of a nonflat complex space $\widetilde{M}_n(c), n \geqq 2$. Then the following conditions are mutually equivalent*:

(1) *M is locally congruent to a hypersurface of type* (A);

(2) *$\phi A = A\phi$ holds on M, where ϕ is the structure tensor of M and A is the shape operator of M in $\widetilde{M}_n(c)$*;

(3) *The shape operator A of M in $\widetilde{M}_n(c)$ satisfies*

$$g((\nabla_X A)Y, Z) = (c/4)\big(-\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)\big) \tag{3.7}$$

for arbitrary vectors X, Y and Z on M.

It is well-known that every hypersurface of type (A) is a homogeneous real hypersurface in $\widetilde{M}_n(c)$, namely it is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$. For other homogeneous real hypersurfaces in $\widetilde{M}_n(c)$, see the classification theorems of all homogeneous real hypersurfaces in a nonflat complex space form (cf. [16, 5]).

In the rest of this section, we recall the notion of ruled real hypersurfaces, which are typical examples non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. A real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ is *ruled* if the holomorphic distribution $T^0M = \{X \in TM | X \perp \xi\}$ is integrable and each of its leaves is locally congruent to a totally geodesic complex hypersurface $M_{n-1}(c)$ of the ambient space $\widetilde{M}_n(c)$. By this definition we find that a real hypersurface M is ruled if and only if $\widetilde{\nabla}_X Y \in T^0M$ for all $X, Y \in T^0M$, where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}_n(c)$. This, together with (3.1) and (3.4), shows that a real hpersurface M is ruled if and only if $g(AX, Y) = 0$ for all $X, Y \in T^0M$.

The construction of ruled real hypersurfaces is as follows. We take an arbitrary real smooth curve $\gamma = \gamma(s)$ defined on some open interval I on $\mathbb{R}$ in $\widetilde{M}_n(c)$ and consider the totally geodesic complex hypersurface, say $M^{(s)}_{n-1}(c)$ of $\widetilde{M}_n(c)$ through the point $\gamma(s)$ in such a way that the tangent space $T_{\gamma(s)}M^{(s)}_{n-1}$ at the point $\gamma(s)$ is orthogonal to the real plane spanned by $\dot\gamma(s)$ and $J\dot\gamma(s)$ for each point $\gamma(s)$. Then the real hypersurface M given by $M = \bigcup_{s \in \mathrm{I}} M^{(s)}_{n-1}$ is a ruled real hypersurface in $\widetilde{M}_n(c)$. Note that in general ruled real hypersurfaces M have singular points, i.e., M is not smooth at those points. So, in order to remove such singular points, we consider ruled real hypersurfaces locally. Moreover, we remark that the set M_* defined by $M_* = \{p \in M | \xi_p \text{ is not a principal curvature vector}\}$ is an open dense subset of a ruled real hypersurface M. When we treat ruled real hypersurfaces M, we study the open dense subset M_* of M. At the end of this section we review the following fundamental of Hopf hypersurfaces.

Proposition 1. *For each Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ the holomorphic distribution T^0M is not integrable.*

4 Naturally reductive homogeneous Riemannian manifolds

We recall the following characterization of homogeneous Riemannian manifolds.

Lemma B ([1])**.** *A complete and simply connected Riemannian manifold M is homogeneous if and only if there exits a tensor field T of type $(1,2)$ on M such that*

(i) $g(T_X Y, Z) + g(Y, T_X Z) = 0,$

(ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z),$

(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$

for X, Y and $Z \in TM$. Here g, ∇ and R denote the Riemannian metric, the Riemannian connection and the Riemannian curvature tensor of M, respectively.

We here review the definition of a naturally reductive homogeneous Riemannian manifold. Let $M = G/K$ be a Riemannian homogeneous space with Riemannian metric g, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. We call $M = G/K$ *reductive* if there is an Ad_K-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ satisfying

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad \mathfrak{k} \cap \mathfrak{m} = 0,$$

which is called a *reductive decomposition.* A Riemannian homogeneous space M is said to be *naturally reductive* if it is naturally reductive with respect to some transitive Lie subgroup of isometry group. Here, $M = G/K$ is *naturally reductive* with respect to G if there is a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ such that

$$g([X, Z]_{\mathfrak{m}}, Y) + g(Z, [X, Y]_{\mathfrak{m}}) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

Note that $[\ ,\ ]_{\mathfrak{m}}$ denotes the canonical projection onto $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. This notion gives us some geometric properties. For example, it is known that every geodesic $\gamma = \gamma(s)$ on each naturally reductive Riemannian homogeneous space M is a homogeneous curve, namely the curve γ is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}(M)$ of M. In fact, a geodesic $\gamma = \gamma(s)$ with $\gamma(0) = o$ is an orbit of the one-parameter subgroup generated by $X := \dot\gamma(0) \in \mathfrak{m}$, where we canonically identify $\mathfrak{m}$ and the tangent space T_oM at the origin o (for details, see [9]). A Riemannian manifold all of whose geodesics are homogeneous curves is called a *geodesic orbit space* or a *GO-space*. Naturally reductive homogeneous spaces are GO-spaces, but the converse does not hold. We refer to, for examples, [2, 17].

The following is a characterization of naturally reductive homogeneous Riemannian manifolds, which is derived from the viewpoint of Lemma B.

Lemma C ([19])**.** *A complete and simply connected Riemannian manifold M is naturally reductive homogeneous if and only if there exits a tensor field T of type $(1, 2)$ on M such that*

(i) $g(T_XY, Z) + g(Y, T_XZ) = 0,$

(ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_XY, Z) - R(Y, T_XZ),$

(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_XY},$

(iv) $T_XX = 0$

for X, Y and $Z \in TM$. Here g, ∇ and R denote the Riemannian metric, the Riemannian connection and the Riemannian curvature tensor of M, respectively.

We call T a *naturally reductive homogeneous structure* on M.

5 Proof of Theorem We shall verify Statement (1). We suppose that there exists a real hypersurface M^{2n-1} which is a Kenmotsu manifold isometrically immersed into $\widetilde{M}_n(c)$. Then by the first equality in (2.2) and (3.3) we have

$$\eta(Y)\phi X + g(X, \phi Y)\xi = -\eta(Y)AX + g(AX, Y)\xi. \tag{5.1}$$

Putting $X = Y = \xi$ in (5.1), we see that $A\xi = g(A\xi, \xi)\xi$, so that M is a Hopf hypersurface in $\widetilde{M}_n(c)$. So we can take a nonzero vector X in such a way that $AX = \lambda X$ and $g(X, \xi) = 0$. For such a vector X and $Y = \xi$, from (5.1) we find that $\phi X = -\lambda X$, which is a contradiction. Hence we get Statement (1).

Next, we investigate Statement (2). We shall show that Condition 2_a) is equivalent to one of Conditions 2_b), 2_c), 2_d), 2_e) and 2_f) one by one.

We suppose Condition 2_b). It follows from (3.3) and (3.4) that Equation (2.1) is equivalent to

$$\eta(Y)(\phi A - A\phi)X + g((A\phi - \phi A)X, Y)\xi = 0 \quad \text{for all } X, Y \in TM. \tag{5.2}$$

Setting $X = Y = \xi$ in (5.2), we see $\phi A\xi = 0$, so that our real hypersurface M is a Hopf hypersurface in $\widetilde{M}_n(c)$. Then, putting $Y = \xi$ in (5.2), we know that $(\phi A - A\phi)X = 0$ for any $X \in TM$, so that M is locally congruent to a hypersurface of type (A) (see Theorem B). Thus we obtain Condition 2_a).

Conversely, it follows from $\phi A = A\phi$ that Equation (5.2) holds. Hence we find that Condition 2_a) implies Condition 2_b).

We suppose Condition 2_c). Note that Condition 2_c) is equivalent to the following equation:

$$g((\nabla_X A)X, X) = 0 \quad \text{for each } X \in TM. \tag{5.3}$$

Let M be a real hypersurface satisfying (5.3) of $\widetilde{M}_n(c)$. We may easily check that (5.3) is equivalent to

$$g((\nabla_X A)Y, Z) + g((\nabla_Y A)Z, X) + g((\nabla_Z A)X, Y) = 0 \tag{5.4}$$

for any X, Y and Z tangent to M. On the other hand, by virtue of Codazzi equation (3.6) we have

$$\begin{aligned} &g((\nabla_Z A)X, Y) - g((\nabla_X A)Z, Y) \\ &\quad = (c/4)(\eta(Z)g(\phi X, Y) - \eta(X)g(\phi Z, Y) - 2\eta(Y)g(\phi Z, X)). \end{aligned} \tag{5.5}$$

Exchanging X and Y, we get

$$\begin{aligned} &g((\nabla_Z A)Y, X) - g((\nabla_Y A)Z, X) \\ &\quad = (c/4)(\eta(Z)g(\phi Y, X) - \eta(Y)g(\phi Z, X) - 2\eta(X)g(\phi Z, Y)). \end{aligned} \tag{5.6}$$

Summing up (5.4), (5.5) and (5.6), we obtain (3.7). Therefore M is locally congruent to a hypersurface of type (A) (see Theorem B). Hence we have Condition 2_a).

Since (5.3) is derived directly from (3.7), the converse is obvious. Then we can see that Condition 2_a) implies Condition 2_c).

We suppose Condition 2_d). Let M be a Riemannian manifold satisfying Condition 2_d). We take an arbitrary geodesic $\gamma = \gamma(s)$ on M. Then the curve γ is a homogeneous curve on M because M is a naturally reductive homogeneous Riemannian manifold. This, together with the assumption that M is homogeneous in $\widetilde{M}_n(c)$ through an equivariant isometric immersion $\iota : M \to \widetilde{M}_n(c)$, implies that the curve $\iota \circ \gamma$ is a homogeneous curve in the ambient space $\widetilde{M}_n(c)$. Hence all the curvatures of the curve $\iota \circ \gamma$ in the sense of Frenet formula are constant along $\iota \circ \gamma$. So, in particular the first curvature $\kappa_\gamma := \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ is constant along γ, where we identify $\iota \circ \gamma$ with γ. This, combined with (3.1), yields that $|g(A\dot\gamma, \dot\gamma)|$ is constant along γ. Thus, by the continuity of the function $g(A\dot\gamma, \dot\gamma)$ we find that $g(A\dot\gamma, \dot\gamma)$ is constant along each geodesic γ on M. Then our real hypersurface M satisfies (5.3). Therefore, by the above discussion we can see that M is locally congruent to a hypersurface of type (A). Hence we obtain Condition 2_a).

Conversely, we suppose Condition 2_a). For a hypersurface M of type (A) in $\widetilde{M}_n(c)$, we take the universal cover $\widetilde{M}$ of M. We define the following tensor T of type $(1, 2)$ on $\widetilde{M}$ as follows:

$$T_X Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi \quad \text{for all } X, Y \in TM. \tag{5.7}$$

Using (3.3), (3.4), (3.5), Theorem B and Lemma C repeatedly, we can see that the tensor T given by (5.7) is a naturally reductive homogeneous structure on $\widetilde{M}$ (see Theorem 9 in [13]). Thus we get Condition 2_d).

We suppose Condition 2_e). For a unit vector X orthogonal to ξ with $AX = \lambda X$, Then by assumption we have $(2\lambda - \delta)g(A\phi X, \phi X) = (2\lambda - \delta)g(AX, X)$, which together with Lemma A(1), yields $\lambda(2\lambda - \delta) = \delta\lambda + (c/2)$, so that $2\lambda^2 - 2\delta\lambda - (c/2) = 0$. Thus we know that our Hopf hypersurface M has either two constant principal curvatures λ_1, δ, or λ_2, δ or three constant principal curvatures $\lambda_1, \lambda_2, \delta$ with $\lambda_1 + \lambda_2 = \delta$ and $\lambda_1\lambda_2 = -c/4$. This, combined with Lemma A, shows that M satisfies $\phi A = A\phi$ (cf. [10]). Then our real hypersurface M is locally congruent to a hypersurface of type (A) (see Theorem B). Hence we obtain Condition 2_a).

Conversely, we suppose Condition 2_a). It is well-known that for each hypersurface M of type (A) every eigenspace V_λ orthogonal to ξ satisfies $\phi V_\lambda = V_\lambda$. This means that the shape operator A is ϕ-invariant. Therefore we have Condition 2_e).

We suppose Condition 2_f). Then by the discussion in the assumption 2_d) we get Condition 2_a).

Conversely, we suppose Condition 2_a). Then by the above discussion we have Condition 2_d). Hence we get Condition 2_f) (see Section 4).

Therefore we complete the proof of our Theorem.

Remark. (1) In [15, 12], they already proved that in a nonflat complex space form all hypersurfaces of type (A) are the only examples of normal real hypersurfaces.

(2) If we omit the hypothesis that M is a Hopf hypersurface in Condition 2_e), then our Theorem is no longer true. In fact, for each ruled real hypersurface M in $\widetilde{M}_n(c)$ we see $g(A\phi X, \phi Y) = 0 = g(AX, Y)$ for all $X, Y (\perp \xi) \in TM$, so that the shape operator A of M is ϕ-invariant in a trivial sense (cf. [10]).

(3) As a consequence of our Theorem 2_a) and 2_b) we obtain the following:

Fact. *Let M be a connected Sasakian real hypersurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then M is locally congruent to one of the following homogeneous real hypesurfaces of the ambient space $\widetilde{M}_n(c)$:*

i) *A geodesic sphere $G(r)$ of radius r with $\tan(\sqrt{c}\, r/2) = \sqrt{c}/2\, (0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;*

ii) *A horosphere in $\mathbb{C}H^n(-4)$;*

iii) *A geodesic sphere $G(r)$ of radius r with $\tanh(\sqrt{|c|}\, r/2) = \sqrt{|c|}/2\, (0 < r < \infty)$ in $\mathbb{C}H^n(c)\, (-4 < c < 0)$;*

iv) *A tube of radius r around a totally geodesic complex hypersurface $\mathbb{C}H^{n-1}(c)$ with $\tanh(\sqrt{|c|}\, r/2) = 2/\sqrt{|c|}\ (0 < r < \infty)$ in $\mathbb{C}H^n(c)\, (c < -4)$.*

In these cases, M is automatically a Sasakian space form. It has constant ϕ-sectional curvature $c + 1$.

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