LOWER DECAY ESTIMATES FOR NON-DEGENERATE DISSIPATIVE WAVE EQUATIONS OF KIRCHHOFF TYPE

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Abstract. Consider the initial-boundary value problem for non-degenerate dissipative wave equations of Kirchhoff type. Using the energy method, we see that the energies have exponential decay rates. Also, we show that the decay rates from below of the solutions are exponentially.

1 Introduction In this paper, we study on the asymptotic behavior of solutions to the initial boundary value problem for the following non-degenerate dissipative wave equations of Kirchhoff type :

$$(1.1)\qquad \begin{cases} \rho u'' + \left(1 + \|A^{1/2}u(t)\|^{2\gamma}\right) Au + u' = 0 \quad \text{in} \quad \Omega\times(0,\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty)\,, \end{cases}$$

where $u = u(x,t)$ is an unknown real value function, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega)\cap H_0^1(\Omega)$, $\|\cdot\|$ is the usual norm of $L^2 = L^2(\Omega)$, and $0 < \rho \le 1$ and $\gamma > 0$ are constants.

In the case of $N = 1$, Equation (1.1) describes a small amplitude vibration of an elastic string (see Kichhoff [7] for the original equation ; also see [4], [5], [10]).

Many authors have shown the local in time solvability for initial data in suitable Sobolev spaces (see [1], [2], [6], [18], [19]).

By help of dissipation we can show the global in time solvability for initial data in certain Sobolev spaces (see [3], [17] for small data and $\gamma \ge 1$), and we can derive some exponential decay estimates for energies.

In previous paper [13], when $\gamma \ge 1$, we have derive some exponential decay estimates, that is,

$$\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le Ce^{-\theta t}$$

with some constant $\theta > 0$ under the small data condition (see Theorem 5.1).

Ghisi and Gobbino [9] have given some decay estimates of the solutions of (1.1) :

$$\begin{aligned} &C'e^{-\theta_2 t} \le \|A^{1/2}u(t)\|^2 \le Ce^{-\theta_1 t}\,, \\ &C'e^{-\theta_2 t} \le \|Au(t)\|^2 \le Ce^{-\theta_1 t}\,, \\ &\|u'(t)\|^2 \le Ce^{-\theta t} \quad \text{for} \quad t \ge 0\,. \end{aligned}$$

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under the smallness condition for the coefficient $\rho > 0$. However, from their results we can not know the lower decay estimate of the norm $\|u(t)\|^2$ (cf. [8], [9], [11], [14] and the references cited therein for mildly degenerate cases).

The purpose of this paper is to give the condition for the global solvability of (1.1) for any $\gamma > 0$ (see Theorem 3.1), and to derive a lower decay estimate of the L^2 norm of the solution $u(t)$ (see Theorem 4.6).

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2 = L^2(\Omega)$ or sometimes duality between the space X and its dual X'. Positive constants will be denoted by C and will change from line to line.

2 A-priori Estimate

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem. The proof is standard and we omit it here (see [1], [2], [15], [16]).

Proposition 2.1 *If the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, then the problem* (1.1) *admits a unique local solution $u(t)$ in the lass $C^0([0,T);\mathcal{D}(A)) \cap C^1([0,T);\mathcal{D}(A^{1/2})) \cap C^0([0,T);L^2(\Omega))$ for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$. Moreover, $\|Au(t)\| + \|A^{1/2}u(t)\| < \infty$ for $t \geq 0$, then we can take $T = \infty$.*

In what follows in this section, let $u(t)$ be a solution of (1.1) and we assume that

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{\gamma+1}\,. \tag{2.1}$$

By fundamental calculation, we have the energy identity

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \quad \text{or} \quad E(t) + 2\int_0^t \|u'(s)\|^2\,ds = E(0)\,, \tag{2.2}$$

where $E(t)$ is defined by

$$E(t) \equiv \rho\|u'(t)\|^2 + \left(1 + \frac{1}{\gamma+1}M(t)^\gamma\right)M(t) \quad \text{with} \quad M(t) \equiv \|A^{1/2}u(t)\|^2\,. \tag{2.3}$$

Proposition 2.2 *Under the assumption* (2.1)*, it holds that*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0) \tag{2.4}$$

where

$$G(t) \equiv \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\,, \tag{2.5}$$

$$Q(t) \equiv \frac{1}{(1+M(t)^\gamma)M(t)^2}\left(\|A^{1/2}u'(t)\|^2 M(t) - \frac{1}{4}|M'(t)|^2\right). \tag{2.6}$$

Proof. From Equation (1.1) we observe

$$\begin{aligned}
&\frac{d}{dt}\frac{\|Au(t)\|^2}{M(t)}\\
&= \frac{1}{(1+M(t)^\gamma)M(t)^2}\left(2\left((1+M(t)^\gamma)Au, Au'\right)M(t) - \left((1+M(t)^\gamma)Au, Au\right)M'(t)\right)\\
&= \frac{-1}{(1+M(t)^\gamma)M(t)^2}\Bigg(2\left(\|A^{1/2}u'(t)\|^2 + \rho(A^{1/2}u'', A^{1/2}u')\right)M(t)\\
&\quad - \left(\frac{1}{2}|M'(t)|^2 + \rho\left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)\right)\Bigg)\\
&= -2Q(t) + \rho R(t) \qquad (2.7)
\end{aligned}$$

where

$$R(t) \equiv \frac{1}{(1+M(t)^\gamma)M(t)^2}\left(2(A^{1/2}u'', A^{1/2}u')M(t) + \left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)\right).$$

On the other hand, by simple calculation we have

$$\frac{d}{dt}Q(t) = -\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma}Q(t) - R(t)\,. \qquad (2.8)$$

Thus, from (2.7) and (2.8) we obtain

$$\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\right) + 2\left(1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma}\right)Q(t) = 0\,.$$

Since it follows from (2.1) and (2.5) that

$$1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma} \geq 0 \quad \text{and} \quad Q(t) \geq 0\,,$$

we conclude the desired estimate (2.5). □

Proposition 2.3 *Under the assumption* (2.1)*, it holds that*

$$\frac{\|u'(t)\|^2}{M(t)} \leq B(0) \qquad (2.9)$$

where

$$B(0) = \max\left\{\frac{\|u_1\|^2}{M(0)}\,,\ \frac{\gamma+1}{\gamma}G(0)(1+E(0)^\gamma)^2\right\}\,. \qquad (2.10)$$

Proof. Multiplying (1.1) by $2M(t)^{-1}u'$ and integrating it over Ω, we have

$$\begin{aligned}
\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)} + 2\left(1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)} &= -\frac{1+M(t)^\gamma}{M(t)}M'(t)\\
&\leq 2\frac{\|u'(t)\|}{M(t)^{\frac{1}{2}}}\frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}}(1+M(t)^\gamma)\,.
\end{aligned}$$

Since it follows from (2.1) that

$$1 + \frac{\rho}{2}\frac{M'(t)}{M(t)} \geq \frac{2\gamma+1}{2(\gamma+1)}\,, \tag{2.11}$$

the Young inequality yields

$$\begin{aligned}\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)} + \frac{\gamma}{\gamma+1}\frac{\|u'(t)\|^2}{M(t)} &\leq \frac{\|Au(t)\|^2}{M(t)}(1+M(t)^\gamma)^2 \\ &\leq G(0)(1+E(0)^\gamma)^2\end{aligned}$$

where we used the estimates (2.2) and (2.4) at the last inequality. Thus, by standard calculation for ODE, we obtain the desired estimate (2.9). □

3 Global Solvability for $\gamma > 0$

Theorem 3.1 *Let the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$*. Suppose that the coefficient* $\rho > 0$ *and the initial data* $[u_0, u_1]$ *satisfy*

$$2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}} < \frac{1}{\gamma+1} \tag{3.1}$$

where $G(0)$ *and* $B(0)$ *are given by* (2.5) *and* (2.10)*, respectively. Then, the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty);\mathcal{D}(A)) \cap C^1([0,\infty);\mathcal{D}(A^{1/2})) \cap C^0([0,\infty);L^2(\Omega))$*, and moreover, the solution* $u(t)$ *satisfies*

$$\rho\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \textit{and} \quad M(t) \leq E(t) \leq E(0)\,, \tag{3.2}$$

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(0) \quad \textit{and} \quad \frac{\|u'(t)\|^2}{M(t)} \leq B(0) \tag{3.3}$$

for $t \geq 0$.

Proof. Let $u(t)$ be a solution of (1.1) on $[0, T]$. Since it follows from (2.5), (2.10), and (3.1) that

$$\rho\frac{|M'(0)|}{M(0)} \leq 2\rho\frac{\|u_1\|}{M(0)^{\frac{1}{2}}}\frac{\|Au_0\|}{M(0)^{\frac{1}{2}}} \leq 2\rho B(0)^{\frac{1}{2}}G(0)^{\frac{1}{2}} < \frac{1}{\gamma+1}\,,$$

putting

$$T_1 \equiv \sup\left\{\, t \in [0,\infty) \;\middle|\; \rho\frac{|M'(s)|}{M(s)} < \frac{1}{\gamma+1} \text{ for } 0 \leq s < t \,\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$\rho\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \text{for} \quad 0 \leq t < T_1 \quad \text{and} \quad \rho\frac{|M'(T_1)|}{M(T_1)} = \frac{1}{\gamma+1}\,. \tag{3.4}$$

On the other hand, from Proposition 2.2 and Proposition 2.3, we observe

$$\rho\frac{|M'(t)|}{M(t)} \leq 2\rho\frac{\|u'(t)\|}{M(t)^{\frac{1}{2}}}\frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \leq 2\rho B(0)^{\frac{1}{2}}G(0)^{\frac{1}{2}} < \frac{1}{\gamma+1} \quad \text{for} \quad 0 \leq t \leq T_1 \tag{3.5}$$

which is a contradiction to (3.4), and hence, we have that $T_1 \geq T$.

Moreover, for $0 \leq t \leq T$, multiplying (1.1) by $2(1+M(t)^\gamma)^{-1}Au'$ and integrating it over Ω, we have

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}+\|Au(t)\|^2\right)+2\left(1+\frac{\gamma}{2}\rho\frac{M(t)^\gamma}{1+M(t)^\gamma}\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}=0\,.$$

Since it follows from (3.5) that

$$1+\frac{\gamma}{2}\rho\frac{M(t)^\gamma}{1+M(t)^\gamma}\frac{M'(t)}{M(t)}\geq 1-\frac{\gamma}{2}\rho\frac{|M'(t)|}{M(t)}\geq\frac{\gamma+2}{2(\gamma+1)}\,,$$

we observe

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}+\|Au(t)\|^2\right)\leq 0\,,$$

and hence, we see that $\|Au(t)\|+\|A^{1/2}u'(t)\|\leq C$ for $0\leq t\leq T$. Therefore, by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from Proposition 2.2 and Proposition 2.3, we obtain the desired estimate (3.3). □

4 Decay

Proposition 4.1 *Under the assumption of Theorem 3.1, it holds that,*

$$M(t)\leq E(t)\leq\frac{2\alpha}{\rho}E(0)e^{-k_1t} \tag{4.1}$$

with

$$\alpha=\max\left\{\frac{3}{2}\rho\,,\,\rho+c_*^2\right\}\quad\text{and}\quad k_1=\alpha^{-1}=\min\left\{\frac{2}{3\rho}\,,\,\frac{1}{\rho+c_*^2}\right\}\,, \tag{4.2}$$

where c_* is the Sobolev-Poincaré constant such that $\|\phi\|\leq c_*\|A^{1/2}\phi\|$.

Proof. We define $E_1(t)$ by

$$E_1(t)\equiv E(t)+\frac{1}{2\rho}\|u(t)\|^2+(u'(t),u(t))$$

with $E(t)$ given by (2.3). Since $|(u',u)|\leq(\rho/2)\|u'\|^2+(1/2\rho)\|u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2}E(t)\leq E_1(t)\leq\frac{\alpha}{\rho}E(t)\quad\text{with}\quad\alpha=\max\left\{\frac{3}{2}\rho\,,\,\rho+c_*^2\right\}\,. \tag{4.3}$$

Multiplying (1.1) by $2u'+\rho^{-1}u$ and integrating it over Ω, we have

$$\frac{d}{dt}E_1(t)+\|u'(t)\|^2+\frac{1}{\rho}(1+M(t)^\gamma)M(t)=0\,,$$

and moreover, it follows from (4.3) that

$$\frac{d}{dt}E_1(t)+k_1E_1(t)\leq 0\quad\text{with}\quad k_1=\alpha^{-1}\,.$$

Thus, we obtain that $E_1(t)\leq E_1(0)e^{-k_1t}$, and hence, from (4.3) we arrive at the desired estimate. □

Proposition 4.2 *Under the assumption of Theorem 3.1, it holds that*

$$H(t) \equiv \rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \le \frac{m_1}{\rho^2} \tag{4.4}$$

with $m_1 = 2\alpha \max\{\rho H(0)\,,\, \gamma^{-1}(\rho(\gamma+1)E(0)^\gamma G(0)+1)\}$.

Proof. We define $H_1(t)$ by

$$H_1(t) \equiv H(t) + \frac{1}{2\rho} + \frac{(A^{1/2}u'(t), A^{1/2}u(t))}{M(t)}\,.$$

Since $|(A^{1/2}u', A^{1/2}u)| \le (\rho/2)\|A^{1/2}u'\|^2 + (1/2\rho)\|A^{1/2}u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2}H(t) \le H_1(t) \le \frac{\alpha}{\rho}H(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}. \tag{4.5}$$

Multiplying (1.1) by $M(t)^{-1}(2Au' + \rho^{-1}Au)$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \left(1 + \rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \\
&\quad = -\left(1 - (\gamma-1)M(t)^\gamma\right)\frac{M'(t)}{M(t)}\frac{\|Au(t)\|^2}{M(t)} - \frac{1}{2\rho}\frac{M'(t)}{M(t)} - \frac{1}{2}\frac{|M'(t)|^2}{M(t)^2}\,.
\end{aligned}$$

Since it follows from (3.2) that

$$1 + \rho\frac{M'(t)}{M(t)} \ge \frac{\gamma}{\gamma+1}\,, \tag{4.6}$$

we have from (3.2) and (3.3) that

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \frac{\gamma}{\gamma+1}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \\
&\quad \le \frac{|M'(t)|}{M(t)}\left((\gamma+1)M(t)^\gamma\frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2\rho} + \frac{1}{2}\frac{|M'(t)|}{M(t)}\right) \\
&\quad \le \frac{1}{\rho(\gamma+1)}\left((\gamma+1)E(0)^\gamma G(0) + \frac{1}{\rho}\right),
\end{aligned}$$

and moreover, we observe from (4.5) that

$$\frac{d}{dt}H_1(t) + \frac{\gamma}{(\gamma+1)\alpha}H_1(t) \le \frac{\gamma}{\rho^2(\gamma+1)}I(0)$$

with $I(0) \equiv \gamma^{-1}\left(\rho(\gamma+1)E(0)^\gamma G(0)+1\right)$. Thus, we obtain

$$H_1(t) \le \max\left\{H_1(0)\,,\, \frac{\alpha}{\rho^2}I(0)\right\}$$

and from (4.5) we conclude the desired estimate (4.4). □

Proposition 4.3 *Under the assumption of Theorem 3.1, it holds that*

$$P(t) \equiv \rho \frac{\|u''(t)\|^2}{M(t)} + \frac{1+M(t)^\gamma}{M(t)} \|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2} M(t)^\gamma \frac{|M'(t)|^2}{M(t)^2} \le \frac{m_2}{\rho^3} \tag{4.7}$$

with $m_2 = 2\alpha \max\{\rho^2 P(0)\,,\, \gamma^{-1}(6(\gamma+1)^2 E(0)^\alpha m_1 + \rho(\gamma+1)\gamma^{-1}B(0))\}$.

Proof. We define $P_1(t)$ by

$$P_1(t) \equiv P(t) + \frac{1}{2\rho}\frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)}\,.$$

Since $|(u'', u')| \le (\rho/2)\|u''\|^2 + (1/2\rho)\|u'\|^2$, we observe from the Sobolev-Poincaré inequality

$$\frac{1}{2}P(t) \le P_1(t) \le \frac{\alpha}{\rho}P(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}. \tag{4.8}$$

Multiplying (1.1) differentiated with respect to t by $M(t)^{-1}(2u'' + \rho^{-1}u')$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}P_1(t) + \left(1 + \rho\frac{M'(t)}{M(t)}\right)\frac{\|u''(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2\rho}M(t)^\gamma\frac{|M'(t)|^2}{M(t)^2} \\
&\quad = -(1-(3\gamma-1)M(t)^\gamma)\frac{M'(t)}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{\gamma(\gamma-2)}{2}M(t)^\gamma\frac{(M'(t))^3}{M(t)^3} \\
&\qquad - \frac{1}{2\rho}\frac{M'(t)}{M(t)}\frac{\|u'(t)\|^2}{M(t)} - \frac{M'(t)}{M(t)}\frac{(u''(t), u'(t))}{M(t)}\,.
\end{aligned}$$

From the Young inequality and (4.6) (or (3.2)) we observe

$$\begin{aligned}
&\frac{d}{dt}P_1(t) + \frac{\gamma}{2(\gamma+1)}\frac{\|u''(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2\rho}M(t)^\gamma\frac{|M'(t)|^2}{M(t)^2} \\
&\quad \le 3(\gamma+1)^2 M(t)^\gamma \frac{|M'(t)|}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\
&\qquad + \frac{1}{2\rho}\frac{|M'(t)|}{M(t)}\frac{\|u'(t)\|^2}{M(t)} + \frac{\gamma+1}{2\gamma}\frac{|M'(t)|^2}{M(t)^2}\frac{\|u'(t)\|^2}{M(t)} \\
&\quad \le \frac{1}{\rho(\gamma+1)}\left(3(\gamma+1)^2 E(0)^\gamma \frac{m_1}{\rho^2} + \frac{\gamma+1}{2\rho\gamma}B(0)\right)
\end{aligned}$$

where we used the estimates (3.2) and (3.3), and moreover, we have from (4.8) that

$$\frac{d}{dt}P_1(t) + \frac{\gamma}{2(\gamma+1)\alpha}P_1(t) \le \frac{\gamma}{2\rho^3(\gamma+1)}J(0)$$

with $J(0) \equiv \gamma^{-1}(6(\gamma+1)^2 E(0)^\gamma m_1 + \rho(\gamma+1)\gamma^{-1}B(0))$. Thus, we obtain

$$P_1(t) \le \max\left\{P_1(0)\,,\, \frac{\alpha}{\rho^3}J(0)\right\}$$

and from (4.8) we conclude the desired estimate (4.7). □

Proposition 4.4 *Under the assumption of Theorem 3.1, it holds that if* $u_0 \ne 0$,

$$M(t) \ge C' e^{-k_2 t} \quad \textit{with} \quad k_2 = \rho^{-1}\max\{2\,,\, \gamma-2\}(1+E(0)^\gamma)^{\frac{1}{2}} G(0)^{\frac{1}{2}}\,, \tag{4.9}$$

where C' *is some positive constant.*

Proof. Multiplying by $2M(t)^{-2}u'$ and integrating it over Ω, we have

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\right)+2\left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^2}=-\frac{2-(\gamma-2)M(t)^\gamma}{M(t)^2}M'(t)\,,$$

and from (3.2), (3.3), and the Young inequality we observe

$$\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\right)\\
&\quad\le 2\max\{2,\gamma-2\}(1+M(t)^\gamma)^{\frac12}\frac{\|Au(t)\|}{M(t)^{\frac12}}\left(\frac{1+M(t)^\gamma}{M(t)}\right)^{\frac12}\frac{\|u'(t)\|}{M(t)}\\
&\quad\le \rho^{-1}\max\{2,\gamma-2\}(1+E(0)^\gamma)^{\frac12}G(0)^{\frac12}\left(\frac{1+M(t)^\gamma}{M(t)}+\rho\frac{\|u'(t)\|^2}{M(t)^2}\right).
\end{aligned}$$

Thus, we obtain

$$\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\le Ce^{k_2t}\quad\text{with}\quad k_2=\rho^{-1}\max\{2\,,\,\gamma-2\}(1+E(0)^\gamma)^{\frac12}G(0)^{\frac12}$$

which gives the desired estimate (4.9). □

Proposition 4.5 *Under the assumption of Theorem 3.1, it holds that if* $u_0\neq 0$*,*

$$\|u(t)\|^2\ge C'e^{-k_3t}\quad\textit{with}\quad k_3=k_2+m_2/\rho^2\,,\tag{4.10}$$

where C' *is some positive constant.*

Proof. From Equation (1.1), we observe

$$\begin{aligned}
\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}&=\frac{-2\rho}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),u''(t)\right)\\
&\quad-\frac{2(1+M(t)^\gamma)}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),Au(t)\right)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}+\frac{2(1+M(t)^\gamma)}{\|u(t)\|^2}\|Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t)\|^2\\
&\quad=\frac{-2\rho}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),u''(t)\right)\\
&\quad\le 2\rho\frac{1}{\|u(t)\|}\|Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t)\|\frac{\|u''(t)\|}{\|u(t)\|}\,.
\end{aligned}$$

The Young inequality yields

$$\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}\le\rho^2\frac{\|u''(t)\|^2}{\|u(t)\|^2}=\rho^2\frac{\|u''(t)\|^2}{M(t)}\frac{M(t)}{\|u(t)\|^2}\le\frac{m_2}{\rho^2}\frac{M(t)}{\|u(t)\|^2}$$

where we used the estimate (4.7). Thus, we have

$$\frac{M(t)}{\|u(t)\|^2}\le Ce^{\frac{m_2}{\rho^2}t}\,,$$

and hence, from (4.9) we obtain the desired estimate (4.10). □

From Propositions 4.1–4.5, we arrive at the following theorem.

Theorem 4.6 *Under the assumption of Theorem 3.1, the solution $u(t)$ of (1.1) satisfies that if $u_0 \neq 0$,*

$$C'e^{-k_3 t} \leq \|u(t)\|^2 \leq Ce^{-k_1 t}\,, \tag{4.11}$$

$$C'e^{-k_2 t} \leq \|A^{1/2}u(t)\|^2 \leq Ce^{-k_1 t}\,, \tag{4.12}$$

$$C'e^{-k_2 t} \leq \|Au(t)\|^2 \leq Ce^{-k_1 t}\,, \tag{4.13}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq Ce^{-k_1 t} \quad \textit{for} \quad t \geq 0 \tag{4.14}$$

with constants k_1, k_2, k_3 given by (4.1), (4.9), (4.10), where C and C' are some positive constants.

Proof. (4.12) follows from (4.1) and (4.9). (4.11) follows from (4.12) and (4.10). (4.13) follows from (4.12) and (3.3). (4.14) follows from (4.12) and (4.7). □

5 Appendix : Global Sovability for $\gamma \geq 1$ When $\gamma \geq 1$, if the initial energy $E(0)$ is small, then there exists a unique global solution and the solution decays exponentially. We intoroduce the function $F(t)$ as

$$F(t) \equiv \rho\|A^{1/2}u'(t)\|^2 + (1 + M(t)^\gamma)\|Au(t)\|^2\,. \tag{5.1}$$

Theorem 5.1 *Let the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. Suppose that the initial energy $E(0)$ is small such that*

$$2^5\gamma^2\alpha E(0)^{2\gamma-1}F(0) < 1 \tag{5.2}$$

with $\alpha = \max\{3\rho/2\,,\, \rho + c_*\}$. Then, the problem (1.1) admits a unique global solution $u(t)$ in the class $C^0([0,\infty);\mathcal{D}(A)) \cap C^1([0,\infty);\mathcal{D}(A^{1/2})) \cap C^0([0,\infty);L^2(\Omega))$, and moreover, the solution $u(t)$ satisfies

$$\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq Ce^{-\theta t} \quad \textit{for} \quad t \geq 0 \tag{5.3}$$

with $\theta = (4\rho)^{-1}$, where C is some positive constant.

Proof. Let $u(t)$ be a solution of (1.1) on $[0,T]$. We define $F_1(t)$ by

$$F_1(t) \equiv F(t) + \frac{1}{2\rho}\|A^{1/2}u(t)\|^2 + (A^{1/2}u'(t), A^{1/2}u(t))\,,$$

Since $|(A^{1/2}u', A^{1/2}u)| \leq (\rho/2)\|A^{1/2}u'\|^2 + (1/2\rho)\|A^{1/2}u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2}F(t) \leq F_1(t) \leq \frac{\alpha}{\rho}F(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}. \tag{5.4}$$

Multiplying (1.1) by $2Au' + \rho^{-1}Au$ and integrating it over Ω, we have

$$\frac{d}{dt}F_1(t) + \|A^{1/2}u'(t)\|^2 + \frac{1}{\rho}(1 + M(t)^\gamma)\|Au(t)\|^2 = \gamma M'(t)M(t)^{\gamma-1}\|Au(t)\|^2\,. \tag{5.5}$$

We observe from (2.2) and (5.1) that

$$\gamma M'(t)M(t)^{\gamma-1} \leq 2\gamma M(t)^{\gamma-1}\|Au(t)\| \leq 2\gamma\rho^{-\frac{1}{2}}E(0)^{\gamma-\frac{1}{2}}F(t)^{\frac{1}{2}}\,. \tag{5.6}$$

Since $2^4\gamma^2\rho E(0)^{2\gamma-1}F(0) < 1$ (by (5.2)), putting

$$T_1 \equiv \sup\left\{t \in [0,\infty) \;\middle|\; \mu(s) \equiv 2^4\gamma^2\rho E(0)^{2\gamma-1}F(s) < 1 \text{ for } 0 \le s < t\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$\mu(t) < 1 \quad \text{for} \quad 0 \le t < T_1 \quad \text{and} \quad \mu(T_1) = 1 \tag{5.7}$$

or

$$\gamma M'(t)M(t)^{\gamma-1}\|Au(t)\|^2 \le \frac{1}{2\rho}\|Au(t)\|^2 . \tag{5.8}$$

Thus, for $0 \le t \le T_1$, it follows from (5.4), (5.5), and (5.8) that

$$\frac{d}{dt}F_1(t) + \theta F_1(t) \le 0 \quad \text{with} \quad \theta = (4\rho)^{-1},$$

and hence,

$$F_1(t) \le F_1(0)e^{-\theta t} \quad \text{or} \quad F(t) \le \frac{2\alpha}{\rho}F(0)e^{-\theta t} . \tag{5.9}$$

Then, we observe

$$\|u''(t)\| \le \|\rho^{-1}(1 + M(t)^\gamma)Au(t) + \rho^{-1}u'(t)\|^2 \le CF(t) \le Ce^{-\theta t} \tag{5.10}$$

and

$$\mu(t) \equiv 2^4\gamma^2\rho E(0)^{2\gamma-1}F(t) \le 2^5\gamma^2\alpha E(0)^{2\gamma-1}F(0) < 1 \quad \text{for} \quad 0 \le t \le T_1$$

which is a contradiction to (5.7), and hence, we have that $T_1 \ge T$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| \le C$ for $0 \le t \le T$. Therefore, by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from (5.9) and (5.10) we obtain the desired estimate (5.3). □

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