ON 0-MINIMAL IDEALS IN A DUAL ORDERED SEMIGROUP WITH ZERO

Thawhat Changphas and Wichayaporn Jantanan

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Abstract. An ordered semigroup S is called a *dual ordered semigroup* if $l(r(L)) = L$ for every left ideal L of S and $r(l(R)) = R$ for every right ideal R of S where $r(A)$ and $l(A)$ denoted the *right annihilator* and the *left annihilator* of a nonempty subset A of S, respectively. The main result of this paper is to show the existence of 0-minimal ideals of a dual ordered semigroup.

1 Preliminaries Dual ring credited to Baer [1] and Kaplansky [8] have been widely studied (see [3], [5], [4], [9]). Using only the multiplication properties of the elements of a ring, Schwarz ([10], [11]) introduced and studied dual semigroups. Let S be a semigroup with zero 0 and let A be a nonempty subset of S. The *left annihilator* of A, denoted by $l(A)$, is defined by $l(A) = \{x \in S \mid xA = \{0\}\}$. Dually, the *right annihilator* of A, denoted by $r(A)$, is defined by $r(A) = \{x \in S \mid Ax = \{0\}\}$. The semigroup S is said to be *dual* if $l(r(L)) = L$ for all left ideals L of S and $r(l(R)) = R$ for all right ideals R of S. In [11], the author proved the existence of 0-minimal ideals of a dual semigroup. The purpose of this paper is to extend the results to ordered semigroups.

A semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy,\ xz \leq yz,$$

is called an *ordered semigroup* ([2], [4]). If A, B are nonempty subsets of S, we let

$$\begin{aligned} AB &= \{xy \in S \mid x \in A, y \in B\}, \\ (A] &= \{x \in S \mid x \leq a \text{ for some } a \in A\}. \end{aligned}$$

If $x \in S$, then we write Ax and xA instead of $A\{x\}$ and $\{x\}A$, respectively.

If A, B are non-empty subsets of an ordered semigroup $(S, \cdot, \leq)$, then it was proved in [6] that the following conditions hold:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A]] = (A]$;

(4) $(A](B] \subseteq (AB]$;

(5) $(A \cup B] = (A] \cup (B]$;

(6) $((A](B]] = (AB]$.

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The concepts of left ideals, right ideals and (two-sided) ideals in an ordered semigroup have been introduced in [6] as follows: let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left ideal* of S if

(i) $SA \subseteq A$;

(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

A nonempty subset A of S is called a *right ideal* of S if $AS \subseteq A$ and (ii) holds. If A is both a left and a right ideal of S, then A is called a (two-sided) *ideal* of S. It is known that, for $x \in S$, $(Sx]$ is a left ideal of S, $(xS]$ is a right ideal of S and $(SxS]$ is an ideal of S.

An element 0 of an ordered semigroup $(S, \cdot, \leq)$ is called a *zero* [2] if

(i) $0x = x0 = 0$ for all $x \in S$;

(ii) $0 \leq x$ for all $x \in S$.

Clearly, $\{0\}$ is an ideal of S which will be denoted by 0. To exclude the trivial case, if an ordered semigroup $(S, \cdot, \leq)$ has a zero 0 then we assume that $S \neq \{0\}$.

Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. A left ideal A of S is said to be 0-*minimal* if $\{0\} \neq A$ and $\{0\}$ is the only left ideal of S properly contained in A. Similarly, we define 0-minimal right ideals and 0-minimal two-sided ideals.

Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. Analogously to [11], if A is a nonempty subset of S, then the *left annihilator* of A, denoted by $l(A)$, is defined by

$$l(A) = \{x \in S \mid xA = 0\}.$$

Dually, the *right annihilator* of A, denoted by $r(A)$, is defined by

$$r(A) = \{x \in S \mid Ax = 0\}.$$

It is easy to see that $l(A)A = 0$ and $Ar(A) = 0$.

Lemma 1.1 *Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0 and A, B nonempty subsets of S. Then the following statements hold:*

(1) *$l(A)$ is a left ideal of S and $r(A)$ is a right ideal of S;*

(2) *$A \subseteq r(l(A))$, $A \subseteq l(r(A))$;*

(3) *if $A \subseteq B$, then $l(B) \subseteq l(A)$ and $r(B) \subseteq r(A)$;*

(4) *if $A_\alpha \subseteq S$, $\alpha \in \Lambda$, then*

$$l(\textstyle\bigcup_\alpha A_\alpha) = \bigcap_\alpha l(A_\alpha),\ r(\bigcup_\alpha A_\alpha) = \bigcap_\alpha r(A_\alpha).$$

Proof. (1) We will show that $l(A)$ is a left ideal of S. Dually, we have $r(A)$ is a right ideal of S. Clearly, $l(A) \neq \emptyset$. If $x \in S, y \in l(A)$, then $(xy)A = x(yA) = 0$, and so $xy \in l(A)$. Let $x \in l(A)$ and $y \in S$ such that $y \leq x$. Then $yA \subseteq (yA] \subseteq (xA] = 0$, and hence $y \in l(A)$.

(2) Since $l(A)A = 0$, so $A \subseteq r(l(A))$. Similarly, $A \subseteq l(r(A))$.

(3) Assume that $A \subseteq B$. Let $x \in l(B)$. Since $A \subseteq B$, we get $xA \subseteq xB = 0$, and so $x \in l(A)$. Thus $l(B) \subseteq l(A)$. Similarly, $r(B) \subseteq r(A)$.

(4) The proof is straightforward.

2 Main Results Analogously to [11], we define a dual ordered semigroup as follows:

Definition 2.1 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is called a dual ordered semigroup if*

(i) $l(r(L)) = L$ *for all left ideals* L *of* S;

(ii) $r(l(R)) = R$ *for all right ideals* R *of* S.

Lemma 2.2 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero* 0.

(1) *If $\{R_\alpha \mid \alpha \in \Lambda\}$ is a family of right ideals of S, then*

$$l(\bigcap_\alpha R_\alpha) = \bigcup_\alpha l(R_\alpha).$$

(2) *If $\{L_\alpha \mid \alpha \in \Lambda\}$ is a family of left ideals of S, then*

$$r(\bigcap_\alpha L_\alpha) = \bigcup_\alpha r(L_\alpha).$$

(3) $l(S) = r(S) = 0$.

(4) *If L is a 0-minimal left ideal of S, then $r(L)$ is a maximal right ideal of S.*

(5) *If A is a 0-minimal ideal of S, then $r(A)$ and $l(A)$ are maximal ideals of S.*

Proof. For (1) and (2), the proofs are straightforward.
(3) We have

$$r(S) = r(S \cup l(0)) = r(S) \cap r(l(0)) = r(S) \cap 0 = 0.$$

Similarly, $l(S) = 0$.

(4) Assume that L is a 0-minimal left ideal of S. Since $L \neq 0$, $r(L) \neq S$. Let R be a proper right ideal of S such that $r(L) \subseteq R$. Then $0 \neq l(R) \subseteq l(r(L)) = L$, and thus $l(R) = L$. Hence $R = r(l(R)) = r(L)$.

(5) Assume that A is a 0-minimal ideal of S. We will show that $r(A)$ is a maximal ideal of S. It is easy to see that $r(A)$ is an ideal of S. Let M be a proper ideal of S such that $r(A) \subseteq M$. Then $0 \neq l(M) \subseteq l(r(A)) = A$, and thus $l(M) = A$. Hence $M = r(l(M)) = r(A)$. Therefore, $r(A)$ is a maximal ideal of S. Similar arguments show that $l(A)$ is a maximal ideal of S.

Lemma 2.3 *If $(S, \cdot, \leq)$ is a dual ordered semigroup with zero 0, then $a \in (Sa]$ and $a \in (aS]$ for every $a \in S$. In particular, $(S^2] = S$.*

Proof. Let $a \in S$. Since $(Sa]$ is a left ideal of S, by assumption, we have $l(r((Sa])) = (Sa]$. If $x \in r((Sa])$, then $(Sa]x = 0$, and hence $(Sax] = 0$. By Lemma 2.2, $ax \in r(S)$, and so $ax = 0$. This proves that $a \in l(r((Sa]))$. Hence $a \in (Sa]$. Dually, $a \in (aS]$.

Lemma 2.4 *Let $(S, \cdot, \leq)$ is a dual ordered semigroup with zero 0 and $a \in S$. If $(aS] = 0$ or $(Sa] = 0$, then $a = 0$.*

Proof. This follows by Lemma 2.3.

Lemma 2.5 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. If $S = (aS]$ for every $a \in S \setminus \{0\}$, then S is itself a 0-minimal right ideal of S.*

Proof. Assume that $S = (aS]$ for every $a \in S \setminus \{0\}$. Let A be a right ideal of S such that $A \neq \{0\}$. Then there exists $a \in A \setminus \{0\}$. By assumption, $S = (aS]$, and thus $S = A$. This shows that S contains only the right ideals S and $\{0\}$. Therefore, the assertion follows.

We now prove the main result analogue to ([11], Theorem 4).

Theorem 2.6 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every nonzero right ideal of S contains a 0-minimal right ideal of S.*

Proof. Let R be a non-zero right ideal of S. There are two cases to consider:

Case 1: $S = (aS]$ for every $a \in S \setminus \{0\}$. By Lemma 2.5, we have S is itself a 0-minimal right ideal of S. Therefore, R contains a 0-minimal right ideal of S.

Case 2: $(aS] \neq S$ for some $a \in S \setminus \{0\}$. We have $a \in (aS] \subseteq S$. Since $a \in (Sa]$, there exists $y \in S$ such that $a \leq ya$. If $y \in l(aS)$, then $yaS = 0$, and so $(yaS] = 0$. Hence $ya = 0$. This is a contradiction. This shows that $y \notin l(aS)$ which implies $y \notin l((aS])$. If $l((aS]) = 0$, then $(aS] = r(l((aS])) = r(0) = S$. This is a contradiction. We have $l((aS]) \neq 0$.

Let L_0 be the union of all left ideals of S which does not contain y. Since

$$l((aS]) \subseteq L_0 \neq S,$$

it follows that

$$r(L_0) \subseteq r(l((aS])) = (aS] \subseteq S$$

and $r(L_0) \neq 0$.

We will show that $r(L_0)$ is a 0-minimal right ideal of S. Let R_1 be a right ideal of S such that $0 \neq R_1 \subset r(L_0)$. Then $L_0 \subset l(R_1) \subset S$, and thus $y \in l(R_1)$. Since $l(R_1)R_1 = 0$, $yR_1 = 0$. Since $l((aS]) \subseteq L_0$, $R_1 \subseteq r(L_0) \subseteq (aS]$. If $x \in R_1 \subseteq (aS]$, then there is $z \in S$ such that $x \leq az \leq yaz = 0$, and thus $R_1 = 0$. This is a contradiction. Hence the proof is completed.

Theorem 2.7 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every nonzero left ideal of S contains a 0-minimal left ideal of S.*

Proof. This can be proved similarly to Theorem 2.6.

Corollary 2.8 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every right ideal R of S such that $R \neq S$ is contained in a maximal right ideal of S.*

Proof. Let R be a right ideal of S such that $R \neq S$. Since $l(R)$ is a left ideal of S, by Theorem 2.6(6), $l(R)$ contains a 0-minimal left ideal L_0 of S. Since $0 \neq L_0 \subseteq l(R)$, we have $R \subseteq r(L_0) \subset S$, By Lemma 2.2, $r(L_0)$ is a maximal right ideal of S.

Theorem 2.9 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every 0-minimal left ideal of S is contained in a 0-minimal ideal of S.*

Proof. Let L_0 be a 0-minimal left ideal of S. By Lemma 2.3, $L_0 \subseteq (L_0S]$. We have $(L_0S]$ is a 0-minimal ideal of S. This proves the assertion.

We will show that $M_0 := (L_0S]$ is a 0-minimal ideal of S. It is easy to see that M_0 is an ideal of S. Setting

$$Z = S \setminus r(L_0) := \{z_\alpha \mid \alpha \in \Lambda\},$$

we have

$$M_0 = (L_0(r(L_0) \cup Z)] = (L_0 Z] = \bigcup_{\alpha\in\Lambda}(L_0 z_\alpha].$$

Note that for $a \in S$, $(L_0 a] = 0$ or $(L_0 a]$ is a 0-minimal left ideal of S. In fact: we assume that $(L_0 a] \neq 0$. Let L be a left ideal of S such that $0 \neq L \subseteq (L_0 a]$. Setting $L_1 = \{x \in L_0 \mid xa \in L\}$. It is easy to see that L is a left ideal of S. By the minimality of L_0, we obtain $L = L_0$. Hence, $L = (L_0 a]$.

Now, since $L_0 \subseteq M_0$, there exists $z_0 \in Z$ such that $L_0 = (L_0 z_0]$.

Let M be an ideal of S such that $0 \neq M \subseteq M_0$. We claim that $L_0 \subseteq M$. Suppose not, then

$$M = \bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha]$$

for some $\Lambda_1 \subseteq \Lambda$ such that $z_0 \notin \{z_\alpha \mid \alpha \in \Lambda_1\}$. Since $MS \subseteq M$, we obtain

$$\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha]S \subseteq \bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha],$$

thus

$$(\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha](S]] = (\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha S] \subseteq (\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha]] \subseteq \bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha].$$

Since

$$(\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha](S]] = \bigcup_{\alpha\in\Lambda_1}((L_0 z_\alpha](S]] = \bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha S],$$

we get $\bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha S] \subseteq \bigcup_{\alpha\in\Lambda_1}(L_0 z_\alpha]$.

Let $\alpha \in \Lambda_1$. Since

$$(L_0 z_\alpha S] = ((L_0](z_\alpha S]] = (L_0(z_\alpha S]]$$

and $(L_0 z_0]$ is not contained in M, we have $z_0 \notin (z_\alpha S]$. Since $r(L_0)$ is a maximal right ideal of S, it follows that $S = (z_\alpha S] \cup r(L_0)$. This is a contradiction sine $z_0 \notin r(L_0)$. So we have the claim.

Now, we get $L_0 \subseteq M \subseteq (L_0 S]$, and thus $(L_0 S] \subseteq (MS] \subseteq (L_0 S]$. Since $M = (MS]$, we have $M = (L_0 S] = M_0$. This completes the proof.

Corollary 2.10 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every ideal of S contains (at least one) 0-minimal ideal of S.*

Proof. This follows by Theorem 2.9.

Corollary 2.11 *Let $(S, \cdot, \leq)$ be a dual ordered semigroup with zero 0. Every maximal left ideal of S contains a maximal ideal of S.*

Proof. Let L be a maximal left ideal of S. By Theorem 2.9, the 0-minimal right ideal $r(L)$ is contained in the 0-minimal ideal $(Sr(L)]$. Since $r(L) \subseteq (Sr(L)] \subseteq S$, we have $0 \subseteq l((Sr(L)]) \subseteq L$. By Lemma 2.2, $l((Sr(L)])$ is a maximal ideal of S.

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Communicated by *Klaus Denecke*

Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen, 40002, Thailand
e-mail: thacha@kku.ac.th, jantanan-2903@hotmail.com