SELBERG TYPE INEQUALITIES IN A HILBERT C^*-MODULE AND ITS APPLICATIONS

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Abstract. In this paper, we present a Selberg type inequality in a Hilbert C^*-module, which ia simultaneous extensions of the Cauchy-Schwarz inequality and the Bessel inequality in a Hibert C^*-module. As an application, we give a generalization of the Selberg inequality in a Hilbert C^*-module.

1 Introduction

The theory of Hilbert C^*-modules over non-commutative C^*-algebras firstly appeared in Paschke [18] and Rieffel [19], and it has contributed greatly to the developments of operator algebras. Recently, many researchers have studied geometric properties of Hilbert C^*-modules from a viewpoint of the operator theory. For example, Dragomir, Khosravi and Moslehian [4], and Bounader and Chahbi [3] showed several variants of the Bessel inequality, the Selberg inequality and these generalizations in the framework of a Hilbert C^*-module. We showed in [6] the new Cauchy-Schwarz inequality in a Hilberet C^*-module by means of the operator geometric mean. From the viewpoint, we show a Hilbert C^*-module version of the Selberg inequality which is simultaneous extensions of the Cauchy-Schwarz inequality and the Bessel one in a Hilbert C^*-module.

We briefly review the Selberg inequality and its generalization in a Hilbert space.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. The Selberg inequality [2, 17] states that if $y_1, y_2, \ldots, y_n$ and x are nonzero vectors in H, then

$$\sum_{i=1}^{n} \frac{|\langle y_i, x\rangle|^2}{\sum_{j=1}^{n} |\langle y_j, y_i\rangle|} \leq \| x \|^2 . \tag{1.1}$$

Moreover, Furuta [10] posed conditions enjoying the equality: The equality in (1.1) holds if and only if $x = \sum_{i=1}^{n} a_i y_i$ for some scalars $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that for arbitrary $i \neq j$

$$\langle y_i, y_j \rangle = 0 \quad \text{or} \quad |a_i| = |a_j| \ \text{ with } \ \langle a_i y_i, a_j y_j \rangle \geq 0, \tag{1.2}$$

also see [7]. Note that the Selberg inequality is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality.

Fujii and Nakamoto [9] showed a refinement of the Selberg inequality: If $\langle y, y_i \rangle = 0$ for given nonzero vectors $y_1, \ldots, y_n \in H$, then

$$|\langle x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^{n} |\langle y_j, y_i\rangle|} \| y \|^2 \leq \| x \|^2 \| y \|^2 \tag{1.3}$$

holds for all $x \in H$. Also, Bombieri [1] showed the following generalization of the Bessel inequality: If $x, y_1, \ldots, y_n$ are nonzero vectors in H, then

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \| x \|^2 \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\langle y_j, y_i\rangle|. \tag{1.4}$$

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Moreover, Mitrinović, Pecǎrić and Fink [17, Theorem 5 in pp394] mentioned the following inequality equivalent to Bombieri's type: If $x, y_1, \ldots, y_n$ are nonzero vectors in H and $a_1, \ldots, a_n \in \mathbb{C}$, then

$$|\sum_{i=1}^{n} a_i \langle x, y_i\rangle|^2 \leq \| x \|^2 \sum_{i=1}^{n} |a_i|^2 \sum_{j=1}^{n} |\langle y_j, y_i\rangle|. \tag{1.5}$$

In this paper, from a viewpoint of the operator theory, we propose a Selberg type inequality in a Hilbert C^*-module, which ia simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality in a Hibert C^*-module. As applications, we show Hilbert C^*-module versions of Fujii-Nakamoto type (1.3), Bombieri type (1.4) and Mitrinović, Pecǎrić and Fink type (1.5). Moreover, we give a generalization of the Selberg inequality in a Hilbert C^*-module.

2 Preliminaries Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e. An element $a \in \mathscr{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathscr{A}$, the operator geometric mean of a and b is defined by

$$a \sharp b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

for invertible a. If a and b are non invertible, then $a \sharp b$ belongs to the double commutant $\mathscr{A}''$ in general. In fact, since $a \sharp b$ satisfies the upper semicontinuity, it follows that $a \sharp b = \lim_{\varepsilon \to +0}(a + \varepsilon e) \sharp (b + \varepsilon e)$ in the strong operator topology. If $\mathscr{A}$ is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \sharp b \in \mathscr{A}$, see [13]. The operator geometric mean has the symmetric property: $a \sharp b = b \sharp a$. In the case that a and b commute, we have $a \sharp b = \sqrt{ab}$. For more details on the operator geometric mean, see [12, 8].

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle \quad (x, y, x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,

(ii) $\langle x, ya\rangle = \langle x, y\rangle a \quad (x, y \in \mathscr{X}, a \in \mathscr{A})$,

(iii) $\langle y, x\rangle = \langle x, y\rangle^* \quad (x, y \in \mathscr{X})$,

(iv) $\langle x, x\rangle \geq 0$ $(x \in \mathscr{X})$ and if $\langle x, x\rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathscr{A}$ and $\mathscr{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y\rangle = a^*\langle x, y\rangle$ for all $x, y \in \mathscr{X}, a \in \mathscr{A}$. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module.

In this case, we write $\| x \| := \sqrt{\| \langle x, x\rangle \|}$, where the latter norm denotes the C^*-norm of $\mathscr{A}$. If an inner-product $\mathscr{A}$-module $\mathscr{X}$ is complete with respect to its norm, then $\mathscr{X}$ is called a *Hilbert C^*-module.* In [6], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product C^*-module over a unital C^*-algebra: If $x, y \in \mathscr{X}$ such that the inner product $\langle x, y\rangle$ has a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then

$$|\langle x, y\rangle| \ \leq \ u^*\langle x, x\rangle u \sharp \langle y, y\rangle. \tag{2.1}$$

An element x of a Hilbert C^*-module $\mathscr{X}$ is called nonsingular if the element $\langle x,x\rangle \in \mathscr{A}$ is invertible. The set $\{x_i\} \subset \mathscr{X}$ is called orthonormal if $\langle x_i, x_j\rangle = \delta_{ij}e$. For more details on Hilbert C^*-modules, see [16].

In [4], Dragomir, Khosravi and Moslehian showed a version of the Bessel inequality and some generalizations of this inequality in the framework of Hilbert C^*-modules. Moreover, in [3], Bounader and Chahbi showed a type and refinement of Selberg inequality in Hilbert C^*-modules. We shall show an improvement of the Selberg type inequality due to Bounader and Chahbi.

3 Main theorem

Fiest of all, we show the following Selberg type inequality in a Hilbert C*-module.

Theorem 1. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then*

$$\sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle| \right)^{-1} \langle y_i, x\rangle \leq \langle x, x\rangle. \tag{3.1}$$

The equality in (3.1) *holds if and only if $x = \sum_{i=1}^{n} y_i a_i$ for some $a_i \in \mathscr{A}$ and $i = 1, \ldots, n$ such that for arbitrary $i \neq j$ $\langle y_i, y_j\rangle = 0$ or $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$.*

Theorem 1 is simultaneous extensions of the Bessel inequality [4] and the Cauchy-Schwarz inequality [6] in a Hilbert C^*-module. As a matter of fact, if $\{y_1, \ldots, y_n\}$ is orthonormal in Theorem 1, then we have the Bessel inequality:

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \leq \langle x, x\rangle$$

holds for all $x \in \mathscr{X}$. If $n = 1$ and $y = y_1$ in Theorem 1 and $\langle x, y\rangle$ has a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then we have $u|\langle x, y\rangle| \langle y, y\rangle^{-1} |\langle y, x\rangle| u^* \leq \langle x, x\rangle$ and hence

$$|\langle x, y\rangle| = |\langle x, y\rangle| \langle y, y\rangle^{-1} |\langle y, x\rangle| \sharp \langle y, y\rangle \leq u^* \langle x, x\rangle u \sharp \langle y, y\rangle.$$

This implies the Cauchy-Schwarz inequality (2.1).

To prove Theorem 1, we need the following two lemmas:

Lemma 2. *If $a \in \mathscr{A}$, then the operator matrix on $\mathscr{A} \oplus \mathscr{A}$*

$$A = \begin{pmatrix} |a^*| & -a \\ -a^* & |a| \end{pmatrix}$$

is positive, and $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathrm{N}(A)$ if and only if $|a^*|\xi = a\eta$, where $N(A)$ is the kernel of A.

Proof. Let $a = u|a|$ be the polar decomposition of a, where u is the partial isometry in the double commutant $\mathscr{A}''$. Since it follows that $|a^*| = u|a|u^*$, we have

$$A = \begin{pmatrix} u|a|u^* & -u|a| \\ -|a|u^* & |a| \end{pmatrix} = \begin{pmatrix} u|a|^{1/2} & 0 \\ 0 & |a|^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u|a|^{1/2} & 0 \\ 0 & |a|^{1/2} \end{pmatrix}^* \geq 0.$$

Next, it is obvious that $\begin{pmatrix}\xi\\ \eta\end{pmatrix} \in \mathrm{Ker}(A)$ if and only if $|a|\eta = a^*\xi$ and $|a^*|\xi = a\eta$. Moreover, it follows that $|a|\eta = a^*\xi$ if and only if $|a^*|\xi = a\eta$. In fact, if $|a|\eta = a^*\xi$, then we have $a\eta = u|a|\eta = ua^*\xi = u|a|u^*\xi = |a^*|\xi$. Conversely, if $|a^*|\xi = a\eta$, then we have $a^*\xi = u^*|a^*|\xi = u^*a\eta = u^*u|a|\eta = |a|\eta$. ☐

Lemma 3. *For any* $y_1, y_2, \dots, y_n \in \mathscr{X}$

$$\begin{pmatrix} \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ & \ddots & \\ \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{pmatrix} \le \begin{pmatrix} \sum_{j=1}^n |\langle y_j, y_1\rangle| & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^n |\langle y_j, y_n\rangle| \end{pmatrix}. \tag{3.2}$$

Proof. The difference between both sides of (3.2) is the following form:

$$\sum_{i,j=1}^n \begin{pmatrix} 0 & & & 0 \\ & |\langle y_j, y_i\rangle| & -\langle y_i, y_j\rangle & \\ & & & \\ & -\langle y_i, y_j\rangle & |\langle y_i, y_j\rangle| & \\ 0 & & & 0 \end{pmatrix}$$

and for each pair i, j it is positive by Lemma 2. ☐

Proof of Theorem 1 For each $i = 1, \dots, n$, put $c_i = \sum_{j=1}^n |\langle y_j, y_i\rangle|$. Since y_i is nonsingular, it follows that c_i is invertible in $\mathscr{A}$. It follows from Lemma 3 that

$$\begin{aligned} &\sum_{i=1}^n \langle x, y_i\rangle c_i^{-1}\langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle \\ &= (\langle x, y_1\rangle c_1^{-1} \cdots \langle x, y_n\rangle c_n^{-1}) \begin{pmatrix} \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ & \ddots & \\ \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{pmatrix} \begin{pmatrix} c_1^{-1}\langle y_1, x\rangle \\ \vdots \\ c_n^{-1}\langle y_n, x\rangle \end{pmatrix} \\ &\le (\langle x, y_1\rangle c_1^{-1} \cdots \langle x, y_n\rangle c_n^{-1}) \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \begin{pmatrix} c_1^{-1}\langle y_1, x\rangle \\ \vdots \\ c_n^{-1}\langle y_n, x\rangle \end{pmatrix} \\ &= \sum_{i=1}^n \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle \end{aligned}$$

and this implies

$$\begin{aligned} 0 &\le \langle x - \sum_{i=1}^n y_i c_i^{-1}\langle y_i, x\rangle, x - \sum_{i=1}^n y_i c_i^{-1}\langle y_i, x\rangle\rangle \\ &= \langle x, x\rangle - 2\sum_{i=1}^n \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle + \sum_{i=1}^n \langle x, y_i\rangle c_i^{-1}\langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle \\ &\le \langle x, x\rangle - \sum_{i=1}^n \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle. \end{aligned}$$

Hence we have the desired inequality (3.1).

The equality in (3.1) holds if and only if the following (3.3) and (3.4) are satisfied:

$$x = \sum_{i=1}^{n} y_i c_i^{-1} \langle y_i, x\rangle \tag{3.3}$$

and for arbitrary $i \neq j$

$$\begin{pmatrix} \langle x, y_i\rangle c_i^{-1} & \langle x, y_j\rangle c_j^{-1} \end{pmatrix} \begin{pmatrix} |\langle y_j, y_i\rangle| & -\langle y_i, y_j\rangle \\ -\langle y_j, y_i\rangle & |\langle y_i, y_j\rangle| \end{pmatrix} \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = 0. \tag{3.4}$$

Put $A = \begin{pmatrix} |\langle y_j, y_i\rangle| & -\langle y_i, y_j\rangle \\ -\langle y_j, y_i\rangle & |\langle y_i, y_j\rangle| \end{pmatrix}$ and it follows that the condition (3.4) holds if and only if

$$A^{1/2} \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff A \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence it follows from Lemma 2 that the condition (3.4) is equivalent to the following (3.5) and (3.6): For arbitrary $i \neq j$

$$\langle y_i, y_j\rangle = 0 \tag{3.5}$$

or

$$|\langle y_j, y_i\rangle| c_i^{-1}\langle y_i, x\rangle = \langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle. \tag{3.6}$$

Conversely, suppose that $x = \sum_{i=1}^{n} y_i a_i$ for some $a_i \in \mathscr{A}$ and for $i \neq j$ $\langle y_i, y_j\rangle = 0$ or $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right)^{-1} \langle y_i, x\rangle &= \sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right)^{-1} \sum_{j=1}^{n} \langle y_i, y_j\rangle a_j \\
&= \sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right)^{-1} \sum_{j=1}^{n} |\langle y_j, y_i\rangle| a_i \\
&= \sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right)^{-1} \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right) a_i \\
&= \sum_{i=1}^{n} \langle x, y_i\rangle a_i \\
&= \langle x, x\rangle.
\end{aligned}$$

Whence the proof is complete. □

Remark 4. (1) In the case that $\mathscr{X}$ is a Hilbert space, the equality condition $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$ in Theorem 1 implies the condition (1.2). In fact, for some scalars $a_i, a_j \in \mathbb{C}$, it follows that $\langle a_i y_i, a_j y_j\rangle = a_i^* \langle y_i, y_j\rangle a_j = a_i^* |\langle y_j, y_i\rangle| a_i \geq 0$, and $|\langle y_j, y_i\rangle| = |\langle y_j, y_i\rangle^*|$ implies $|a_i| = |a_j|$.
(2) In the Hilbert space setting, K. Kubo and F. Kubo [15] showed another proof of Selberg's inequality (1.1) using Geršgorin's location of eigenvalues [14, Theorem 6.1.1] and a diagonal domination theorem of positive semidefinite matrix.

4 Applications In this section, by using Theorem 1, we consider several Hilbert C^*-module versions of the Selberg inequality and the Bessel inequality.

Bounader and Chahbi in [3, Theorem 3.1] showed that if $\mathscr{X}$ is an inner product C^*-module and $y_1,\dots,y_n$ are nonzero vectors in $\mathscr{X}$, and $x\in\mathscr{X}$, then

$$\sum_{i=1}^{n}\frac{|\langle y_i,x\rangle|^2}{\sum_{j=1}^{n}\parallel\langle y_j,y_i\rangle\parallel}\le\langle x,x\rangle. \tag{4.1}$$

By Theorem 1, we have the following corollary, which is an improvement of (4.1):

Corollary 5. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. If $x,y_1,\dots,y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1,\dots,y_n$ are nonsingular, then*

$$\sum_{i=1}^{n}\frac{|\langle y_i,x\rangle|^2}{\parallel\sum_{j=1}^{n}|\langle y_j,y_i\rangle|\parallel}\le\langle x,x\rangle.$$

Proof. By assumption it follows that $\sum_{i=1}^{n}|\langle y_j,y_i\rangle|$ is invertible in $\mathscr{A}$ and hence

$$\left(\sum_{i=1}^{n}|\langle y_j,y_i\rangle|\right)^{-1}\ge\parallel\sum_{i=1}^{n}|\langle y_j,y_i\rangle|\parallel^{-1}.$$

Therefore, Theorem 1 implies Corollary 5. □

Moreover, Bounader and Chahbi showed a Hilbert C^*-module version of Fujii-Nakamoto type (1.3), which is a refinement of (4.1): If y and $y_1,\dots,y_n$ are nonzero vectros in $\mathscr{X}$ such that $\langle y,y_i\rangle=0$ for $i=1,\dots,n$, and $x\in\mathscr{X}$, then

$$|\langle y,x\rangle|^2+\sum_{i=1}^{n}\frac{|\langle y_i,x\rangle|^2}{\sum_{j=1}^{n}\parallel\langle y_i,y_j\rangle\parallel}\parallel\langle y,y\rangle\parallel\le\parallel\langle y,y\rangle\parallel\langle x,x\rangle. \tag{4.2}$$

We show a Hilbert C^*-module version of a refinement of the Selberg inequality due to Fujii and Nakamoto, which is another version of (4.2):

Theorem 6. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. If $x,y,y_1,\dots,y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1,\dots,y_n$ are nonsingular, $\langle y,y_i\rangle=0$ for $i=1,\cdots,n$ and $\langle x,y\rangle=u|\langle x,y\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u\in\mathscr{A}$ is a partial isometry, then*

$$|\langle y,x\rangle|\le u^*\langle y,y\rangle u\ \sharp\ \left(\langle x,x\rangle-\sum_{i=1}^{n}\langle x,y_i\rangle\left(\sum_{j=1}^{n}|\langle y_j,y_i\rangle|\right)^{-1}\langle y_i,x\rangle\right) \tag{4.3}$$

$$\Big(\le u^*\langle y,y\rangle u\ \sharp\ \langle x,x\rangle\Big).$$

Proof. Put $z=x-\sum_{i=1}^{n}y_i\left(\sum_{j=1}^{n}|\langle y_j,y_i\rangle|\right)^{-1}\langle y_i,x\rangle$. By the proof of Theorem 1, we have

$$\langle z,z\rangle\le\langle x,x\rangle-\sum_{i=1}^{n}\langle x,y_i\rangle\left(\sum_{j=1}^{n}|\langle y_j,y_i\rangle|\right)^{-1}\langle y_i,x\rangle.$$

Since $\langle y, z\rangle = \langle y, x\rangle$, it follows from the monotonicity of the operator geometric mean that

$$\begin{aligned} |\langle y, x\rangle| = |\langle y, z\rangle| &\le u^*\langle y, y\rangle u \sharp \langle z, z\rangle \quad \text{by the Cauchy-Schwarz inequality (2.1)} \\ &\le u^*\langle y, y\rangle u \sharp \left(\langle x, x\rangle - \sum_{i=1}^n \langle x, y_i\rangle \left(\sum_{j=1}^n |\langle y_j, y_i\rangle| \right)^{-1} \langle y_i, x\rangle \right). \end{aligned}$$

□

In [3, Corollary 3.5], Bounader and Chahbi showed a Hilbert C^*-module version of Bombieri type (1.4): If $y_1, \dots, y_n$ are nonzero vectors in $\mathscr{X}$ and $x \in \mathscr{X}$, then

$$\sum_{i=1}^n |\langle y_i, x\rangle|^2 \le \langle x, x\rangle \max_{1\le i\le n} \sum_{j=1}^n \| \langle y_i, y_j\rangle \| . \tag{4.4}$$

We show a Hilbert C^*-module version of Bombieri type, which is an improvement of (4.4):

Theorem 7. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n |\langle y_i, x\rangle|^2 \le \langle x, x\rangle \max_{1\le i\le n} \| \sum_{j=1}^n |\langle y_j, y_i\rangle| \| .$$

Proof. Since for $i = 1, \dots, n$

$$\sum_{j=1}^n |\langle y_j, y_i\rangle| \le \| \sum_{j=1}^n |\langle y_j, y_i\rangle| \| \le \max_{1\le i\le n} \| \sum_{j=1}^n |\langle y_j, y_i\rangle| \|,$$

we have this theorem by virtue of Theorem 1. □

As a corollary, we have the following Boas-Bellman type inequality [3, Corollary 3.6]:

Corollary 8. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n |\langle y_i, x\rangle|^2 \le \langle x, x\rangle \left(\max_{1\le i\le n} \| \langle y_i, y_i\rangle \| + (n-1) \max_{j\ne i} \| \langle y_j, y_i\rangle \| \right).$$

Finally, we show a Mitrinović-Pečarić-Fink type inequality [17, Theorem 5 in pp394] in Hilbert C^*-modules, which is another version of [4, Theorem 3.8]:

Theorem 9. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathscr{X}$ and $a_1, \cdots, a_n \in \mathscr{A}$ such that $y_1, \dots, y_n$ are nonsingular and $\langle x, \sum_{i=1}^n y_i a_i\rangle = u|\langle x, \sum_{i=1}^n y_i a_i\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u \in \mathscr{A}$ is a partial isometry, then*

$$|\sum_{i=1}^n \langle x, y_i\rangle a_i| \le u^*\langle x, x\rangle u \sharp \left(\sum_{i=1}^n a_i^* \left(\sum_{j=1}^n |\langle y_j, y_i\rangle| \right) a_i \right).$$

Proof. By the Cauchy-Schwarz inequality (2.1), we have

$$\begin{aligned}|\sum_{i=1}^{n}\langle x, y_i\rangle a_i| &= |\langle x, \sum_{i=1}^{n} y_i a_i\rangle| \\ &\le u^*\langle x, x\rangle u \ \sharp \ \left(\langle \sum_{i=1}^{n} y_i a_i, \sum_{i=1}^{n} y_i a_i\rangle\right) \\ &= u^*\langle x, x\rangle u \ \sharp \ \left(\sum_{i,j=1}^{n} a_i^*\langle y_i, y_j\rangle a_j\right) \\ &\le u^*\langle x, x\rangle u \ \sharp \ \left(\sum_{i=1}^{n} a_i^*\left(\sum_{j=1}^{n}|\langle y_j, y_i\rangle|\right) a_i\right) \qquad \text{by Lemma 3.}\end{aligned}$$

□

5 Generalization In this section, we present a generalization of the Selberg inequality in a Hilbert C^*-module.

We review the basic concepts of adjointable operators on a Hilbert C^*-module$\mathscr{X}$ over a unital C^*-algebra $\mathscr{A}$. We define $\mathcal{L}(\mathscr{X})$ to be the set of all maps $T : \mathscr{X} \mapsto \mathscr{X}$ for which there is a map $T^* : \mathscr{X} \mapsto \mathscr{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathscr{X}$. For $T \in \mathcal{L}(\mathscr{X})$, we denote the kernel of T by $N(T)$. A closed submodule $\mathscr{M}$ of $\mathscr{X}$ is said to be complemented if $\mathscr{X} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Suppose that the closures of the ranges of T and T^* are both complemented. Then it follows from [16, Proposition 3.8] that T has a polar decomposition $T = U|T|$ with a partial isometry $U \in \mathcal{L}(\mathscr{X})$ and $N(U) = N(|T|)$, and the following hold:

(i) $N(|T|) = N(T)$.

(ii) $|T^*|^q = U|T|^q U^*$ for any positive number $q > 0$.

(iii) $N(S^q) = N(S)$ for any positive operator $S \in \mathcal{L}(\mathscr{X})$ and $q > 0$,

also see [5, 20].

Theorem 10. *Let T be an operator in $\mathcal{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented. If $y_1, \dots, y_n \notin \mathrm{N}(T^*)$ are nonsingular, then*

$$\sum_{i=1}^{n}\langle Tx, y_i\rangle\left(\sum_{j=1}^{n}|\langle |T^*|^{2\beta)} y_j, y_i\rangle|\right)^{-1}\langle y_i, Tx\rangle \le \langle |T|^{2\alpha}x, x\rangle \tag{5.1}$$

holds for every $x \notin \mathrm{N}(T)$ and for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. In particular,

$$\sum_{i=1}^{n}\langle Tx, y_i\rangle\left(\sum_{j=1}^{n}|\langle TT^* y_j, y_i\rangle|\right)^{-1}\langle y_i, Tx\rangle \le \langle U^*Ux, x\rangle \tag{5.2}$$

and

$$\sum_{i=1}^{n}\langle Tx, y_i\rangle\left(\sum_{j=1}^{n}|\langle UU^* y_j, y_i\rangle|\right)^{-1}\langle y_i, Tx\rangle \le \langle T^*Tx, x\rangle. \tag{5.3}$$

Moreover, the equality in (5.1) *holds if and only if* $Tx = \sum_{i=1}^n |T^*|^{2\beta} y_i a_i$ *for some* $a_1, \dots, a_n \in \mathscr{A}$ *such that for arbitrary* $i \neq j$, $\langle |T^*|^{2\beta} y_i, y_j \rangle = 0$ *or* $|\langle |T^*|^{2\beta} y_j, y_i \rangle| a_i = \langle |T^*|^{2\beta} y_i, y_j \rangle a_j$.

Proof. Let $T = U|T|$ be the polar decomposition of T, where U is the partial isometry. In the case of $\alpha = 0$ or 1, it follows from Theorem 1 that replacing x by U^*Ux (resp. $|T|x$) and y_i by $|T|U^*y_i$ (resp. U^*y_i) for all $i = 1, \dots, n$, it follows that $\langle U^*Ux, |T|U^*y_i \rangle = \langle Ux, U|T|U^*y_i \rangle = \langle x, U^*|T^*|y_i \rangle = \langle x, T^*y_i \rangle = \langle Tx, y_i \rangle$ and we have (5.2) (resp. (5.3)). In the case of $0 < \alpha < 1$, we replace x by $|T|^\alpha x$ and also replace y_i by $|T|^\beta U^* y_i$ for all $i = 1, \dots, n$. Then we have

$$\langle |T|^\beta U^* y_i, |T|^\beta U^* y_j \rangle = \langle U|T|^{2\beta} U^* y_i, y_j \rangle = \langle |T^*|^{2\beta} y_i, y_j \rangle$$

and $y_1, \dots, y_n \notin \mathrm{N}(T^*) = \mathrm{N}(|T^*|) = \mathrm{N}(|T^*|^\beta)$. Thus we have (5.1) by Theorem 1.

Next, we consider the equality condition in (5.1). By (iii), we have

$$|T|^\alpha x = \sum_{i=1}^n |T|^\beta U^* y_i a_i \quad \Longleftrightarrow \quad |T|^{2\alpha} x = \sum_{i=1}^n |T| U^* y_i a_i = \sum_{i=1}^n T^* y_i a_i.$$

Hence we have the following implication:

$$\begin{aligned} |T|^\alpha x = \sum_{i=1}^n |T|^\beta U^* y_i a_i \quad &\Longleftrightarrow \quad |T|x = |T|^{\alpha+\beta} x = \sum_{i=1}^n |T|^{2\beta} U^* y_i a_i \quad \text{by (iii)} \\ &\Longleftrightarrow \quad U|T|x = \sum_{i=1}^n U|T|^{2\beta} U^* y_i a_i \quad \text{by (i) and (iii)} \\ &\Longleftrightarrow \quad Tx = \sum_{i=1}^n |T^*|^{2\beta} y_i a_i. \quad \text{by (ii).} \end{aligned}$$

Whence the proof is complete. ☐

References

[1] E. Bombieri, *A note on the large sieve*, Acta. Arith., **18** (1971), 401–404.

[2] E. Bombieri, *Le Grand Grible dans la Theórie Analytique des Nombres*, Asterisque 18, Societe Mathematique de France, 1974.

[3] N. Bounader and A. Chahbi, *Selberg type inequalities in Hilbert C^*-modules*, Int. J. Analy., **7** (2013), 385–391.

[4] S.S. Dragomir, M. Khosravi and M.S. Moslehian, *Bessel type inequalities in Hilbert C^*-modules*, Linear Multilinear Algebra, **58** (2010), 967–975.

[5] J.I. Fujii, M. Fujii and Y. Seo, *Operator inequalities on Hilbert C^*-modules via the Cauchy-Schwarz inquality*, to appear in Math. Ineq. Appl.

[6] J.I. Fujii, M. Fujii, M.S. Moslehian and Y. Seo, *Cauchy-Schwarz inequality in semi-inner product C*-modules via polar decomposition*, J. Math. Anal. Appl., **394** (2012), 835-840.

[7] M. Fujii, K. Kubo and S. Otani, *A graph theoretical observation on the Selberg inequality*, Math. Japon., **35** (1990), 381–385.

[8] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

[9] M. Fujii and R. Nakamoto, *Simultaneous extensions of Selberg inequality and Heinz-Kato-Furuta inequality*, Nihonkai Math. J., **9** (1998), 219–225.

[10] T. Furuta, *When does the equality of a generalized Selberg inequality hold?*, Nihonkai Math. J., **2** (1991), 25–29.

[11] T. Furuta, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.

[12] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[13] M. Hamana, *Partial $*$-automorphisms, normalizers, and submodules in monotome complete C^*-algebras*, Canad. J.Math., **58** (2006), 1144–1202.

[14] R.A. Horn and C.A. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[15] K. Kubo and F. Kubo, *Diagonal matrix that dominates a positive semidefinite matrix*, Technical note, 1988.

[16] E.C. Lance, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

[17] D.S. Mitrinović, J. Pecǎrić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[18] W.L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443-468.

[19] M.A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Applied Algebra, **5** (1974), 51-96.

[20] Y. Seo, *Hölder type inequalities on Hilbert C^*-modules and its reverses*, Ann. Funct. Anal., **5** (2014), 1-9.

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