QUANTITATIVE INVESTIGATIONS FOR ODE MODEL DESCRIBING FISH SCHOOLING

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Abstract. This paper is devoted to investigating quantitatively the ODE model for fish schooling which was introduced in the paper [15]. First, we will study how each parameter in the model equations contributes to the geometrical structure of the school created by fish such as school diameter, connectedness, graph, etc. Second, we will concentrate on studying effects of the noise imposed to the model equations. In particular, it will be shown that, if the noise's magnitude is larger than a certain threshold, then fish can no longer form a school.

1 Introduction In the preceding paper [15], we have introduced an ordinary differential equation model:

$$
\begin{cases}
dx_i(t) = & v_i dt + \sigma_i dw_i(t), \qquad i = 1, 2, \dots, N, \\
dv_i(t) = & \Big[-\alpha \displaystyle\sum_{j=1,\, j\neq i}^{N} \left(\frac{r^p}{\|x_i - x_j\|^p} - \frac{r^q}{\|x_i - x_j\|^q} \right)(x_i - x_j) \\
& -\beta \displaystyle\sum_{j=1,\, j\neq i}^{N} \left(\frac{r^p}{\|x_i - x_j\|^p} + \frac{r^q}{\|x_i - x_j\|^q} \right)(v_i - v_j) \\
& +F_i(t, x_i, v_i)\Big] dt, \qquad i = 1, 2, \dots, N,
\end{cases}
\tag{1.1}
$$

for describing the process of schooling of N-fish system. Each fish is regarded as a moving particle in the Euclidean space $\mathbb{R}^d$, where $d = 2$ or 3. The unknown $x_i(t)$ is a stochastic process with values in $\mathbb{R}^d$ denoting a position of the i-th fish of system at time t; meanwhile, $v_i(t)$ is a stochastic process with values in $\mathbb{R}^d$ denoting a velocity of the i-th fish at time t. The fish are allowed to swim in the unbounded, continuous and homogeneous space $\mathbb{R}^d$.

The first equations of (1.1) are stochastic equations concerning x_i, where $\sigma_i dw_i$ denote noise resulting from the imperfectness of information-gathering and action of the i-th fish. In fact, $\{w_i(t),\, t \geqslant 0\}$ $(i = 1, 2, ..., N)$ are independent d-dimensional Brownian motions defined on a complete probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ satisfying the usual conditions. The second equations are deterministic equations on v_i, where $1 < p < q < \infty$ are fixed exponents, α, β are positive coefficients for interaction between fish and velocity matching, respectively, and $r > 0$ is a fixed distance. Since $1 < p < q < \infty$, if $\|x_i - x_j\| > r$ then the i-th fish moves toward the j-th; to the contrary, if $\|x_i - x_j\| < r$, then the i-th fish acts in order to avoid collision with the j-th fish. The number $r > 0$ therefore denotes a critical distance. Finally, the functions $F_i(t, x_i, v_i)$ denote external forces at time t which are given functions defined for (x_i, v_i) with values in $\mathbb{R}^d$. It is assumed that $F_i(t, x_i, v_i)$ $(i = 1, 2, \dots, N)$ are locally Lipschitz continuous. In building up such a differential equation model we have referred to the fish's behavioral rules:

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1. The school has no leaders and each fish follows the same behavioral rules.
2. To decide where to move, each fish uses some form of weighted average of the position and orientation of its nearest neighbors.
3. There is a degree of uncertainty in the individual's behavior that reflects both the imperfect information-gathering ability of a fish and the imperfect execution of the fish's actions.

introduced by Camazine-Deneubourg-Franks-Sneyd-Theraulaz-Bonabeau [4, Chapter 11]. We have also referred to the idea due to Reynolds [14]. For the details, however, consult the paper [15].

The objective of the present paper is to investigate geometrical structures of the fish school when the fish move by obeying the kinematic equations (1.1) and create a swarm. For this purpose, we intend to introduce several quantitative notions: Distance to School Mates, Minimum Distance, Mean Distance to School Mates, Diameter of School, Variance of Velocity, and ε-Graph, to measure the geometrical structure of school. We in addition introduce a notion of ε-schooling where ε is fixed almost equally to r. We then perform many numerical computations to clarify effects of each parameter or exponent of the equations in determining geometry of structures of school. These will be presented in Section 2 with absence of noise. Next, in Section 3, we focus on studying effects of the noise which is an indispensable factor in the real world.

Empirical study on fish schooling has been done in [1, 3, 5, 8, 13]. As for the theoretical approach we want to quote [7, 10, 11, 16]. Vicsek et al. [16] introduced a simple difference model, assuming that each particle is driven with a constant absolute velocity and the average direction of motion of the particles in its neighborhood together with some random perturbation. Oboshi et al. [10] presented another difference model in which an individual selects one basic behavioral pattern from four based on the distance between it and its nearest neighbor. Olfati-Saber [11] and D´Orsogna et al. [7] constructed deterministic differential models using a generalized Morse and attractive/repulsive potential functions, respectively. We use the ODE model mentioned above. Such a model can describe the fish's behavior precisely. Moreover, an ODE model is tractable for making numerical simulations. In this paper, we will use the Euler scheme for stochastic differential equations which has been introduced by Kloeden and Platen [6].

2 Various Measures for Geometrical Structures In this section we want to introduce various measures to study the geometrical structures of school. Using these measures we will also clarify contributions of exponents and parameters included in (1.1) to the geometrical structure of school by examining many numerical examples.

For simplicity, we consider throughout this section the deterministic case, i.e., $\sigma_i = 0$ for all i. Therefore, $(x_i(t), v_i(t))$ denotes a trajectory of the i-th fish in the phase space $\mathbb{R}^d \times \mathbb{R}^d$.

2.1 Distance to School Mates For each fish i, put

$$\mathrm{DS}_i(t) = \min_{1\leqslant j\leqslant N,\, j\neq i} \|x_j(t) - x_i(t)\|, \qquad 0 < t < \infty,\ i = 1, 2, \dots, N.$$

By definition, $\mathrm{DS}_i(t)$ denotes the distance between the i-th individual to its nearest mates at time t. We call $\mathrm{DS}_i(t)$ the distance of i to the school mates. It is observed that $\mathrm{DS}_i(t)$ depends on the position $x_i(t)$ considerably. If $x_i(t)$ is near the center of school, i.e., $\bar{x}(t) =$

$\frac{1}{N}\sum_{j=1}^{N} x_i(t)$, then $\mathrm{DS}_i(t)$ is much smaller than r; on the contrary, if $x_i(t)$ is in the periphery of school, then $\mathrm{DS}_i(t)$ can be almost equal to the maximum value r.

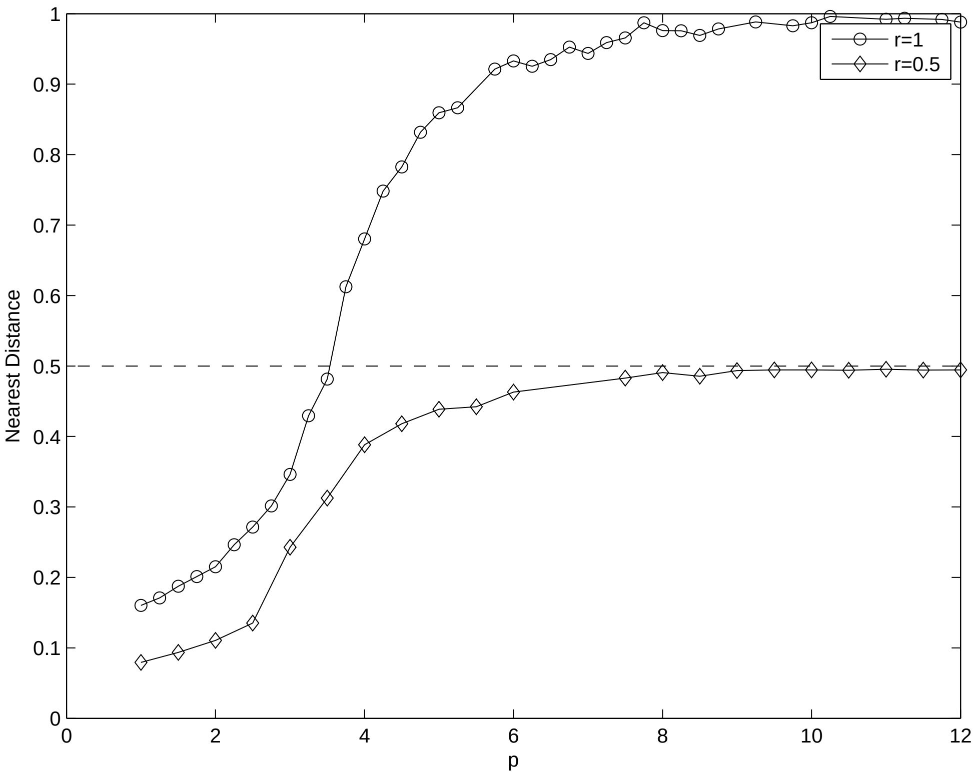

Figure 1: Dependence of MiDS on the exponent p

2.2 Minimum Distance We define

$$\mathrm{MiDS}(t) = \min_{1\leqslant i\leqslant N} \mathrm{DS}_i(t), \qquad 0 < t < \infty,$$

and call this value the minimum distance of school. This is the nearest distance between two fish in a group of N individuals at time t. Basically, $\mathrm{MiDS}(t)$ is dependent on r. But, it is seen that $\mathrm{MiDS}(t)$ depends on the exponents p and q, too. For example, we have

$$\lim_{p\to\infty} \mathrm{MiDS}(T) = r,$$

provided that T is a sufficiently large time. That is the nearest distance tends to the critical distance as power p tends to infinity for sufficiently large time T. By simulations, we shall find such a relationship between r and $\mathrm{MiDS}(T)$.

We consider a 100-fish system in the 2-dimensional space with $F_i = -5.0v_i$, which is often used to present the resistance against the moving particles. We fix two initial positions for two examples of 100-individual system (the initial positions $x_i(0)$, $1 \leqslant i \leqslant 100$, are randomly distributed in the square domain $[0, 10]^2 \subset \mathbb{R}^2$) with all null initial velocities

$v_i = (0,0)$, $(1 \leqslant i \leqslant 100)$. Taking the critical distance $r = 1$ for the first example and $r = 0.5$ for the second, we tune the exponent p from 1 to 12 and always keep the relation $q = p+1$. Other parameters are chosen as follows: $\alpha = 1$, $\beta = 0.5$, step size $\delta = 0.001$. The result is got after 30.000 running steps, that is at time $T = 30$. Figure 1 illustrates dependence of MiDS(T) on the exponent p.

Remark 2.1. The model we consider contains many parameters, but we can find that the powers p and q are especially meaningful. p and q are concerned with a range of interactions among fish. As p and q increase, the range shortens and approaches sharply to the critical length r, namely, if $\|x_i - x_j\| > r$ the attraction between i and j is weak and if $\|x_i - x_j\| < r$ the repulsive is very strong. □

In order to simplify our arguments, in what follows, we will always take q so that $q = p+1$. This assures the condition $q > p$ in modeling and the difference is similar to that of the Van der Waals and the Newton's law, where $p = 3$ and $q = 4$.

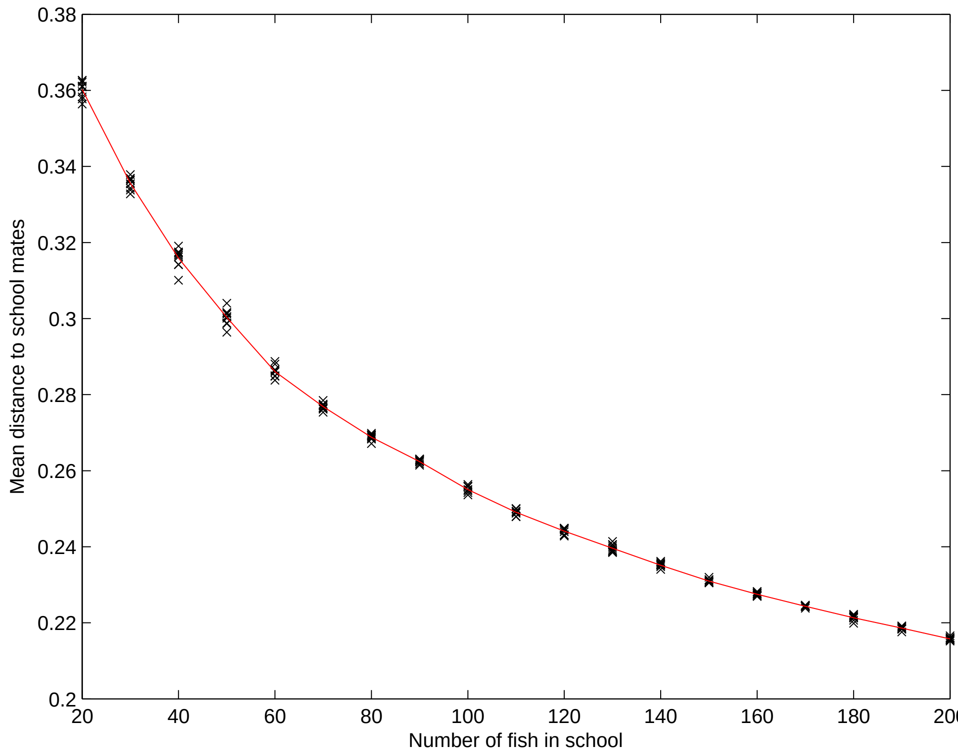

Figure 2: Dependence of MDS on the total number of fish

2.3 Mean Distance to School Mates

We consider the mean of $\mathrm{DS}_i(t)$, i.e.,

$$\mathrm{MDS}(t) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{DS}_i(t), \qquad 0 < t < \infty.$$

This quantity is called the mean distance of school mates and is one of quantitative measures which are used to study the internal structure of the fish school.

It may be a very interesting question to know how MDS(t) depends on the total number of fish. In order to examine this, we consider an N-fish system in the 3-dimensional space with $F_i = -5.0v_i,\ 1 \leqslant i \leqslant N$. Let $\alpha = 5, \beta = 1, p = 3, q = 4$ and $r = 0.5$. We take various values N between 20 and 200. Initial positions $x_i(0)$, $1 \leqslant i \leqslant N$, are randomly distributed in the cubic domain $[0, 20]^3$ with all null initial velocities $v_i(0) = (0, 0, 0)$. The time T is fixed as $T = 120$ throughout the simulations. Figure 2 then shows dependence of MDS(T) on the total number N. In order to reduce the effect of the random initial positions to the result, for each value of N, we run 10 simulations each with different random initial positions in $[0, 20]^3 \subset \mathbb{R}^3$. The mean distance for each N is drawn by a cross $\times$. After that we take the mean value of these and then interpolate these values by a smooth curve.

As seen, MDS(T) decreases monotonically as N increases. This means that the school becomes "more condensed" as N is larger. This agrees with the results stated in a number of works, such as [2, 8, 9, 12] in which the authors show that the mean distance to school mates decreases as a function of the number of fish. From Figure 2, we also see that the range of the simulation results for MDS decreases as N increases.

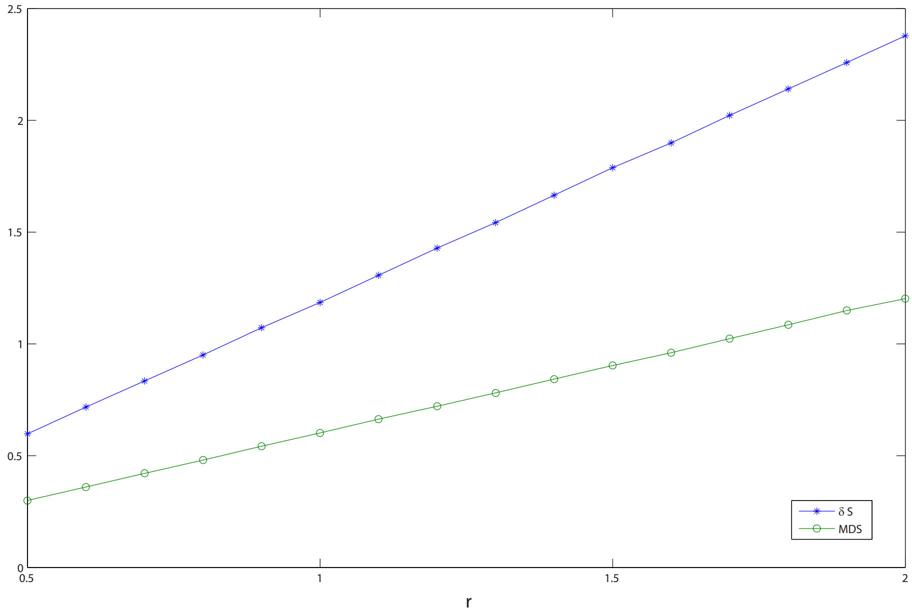

Figure 3: Dependence of MDS and δS on the critical distance r

2.4 Diameter of School The diameter of school is defined by

$$\delta S(t) = \sup_{1 \leqslant i \leqslant N} \|x_i(t) - \bar{x}(t)\|, \qquad 0 < t < \infty,$$

where $\bar{x}(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t)$ is the center of the group at time t.

The diameter of school is, by definition, the radius of the minimal ball centered at $\bar{x}(t)$ and containing all the individuals at time t.

The following numerical example shows that MDS(T) and $\delta S(T)$ are linearly dependent on r for sufficiently large time T. We consider a 50-fish system in the 3-dimensional space with $F_i = -5v_i$. Let $\alpha = 5$, $\beta = 1$, $p = 3$ with $q = p + 1$. Now, r is a tuning parameter which varies from 0.5 to 2. Initial positions $x_i(0)$, $1 \leqslant i \leqslant 50$, are randomly distributed in the cubic domain $[0, 20]^3$ with null initial velocities $v_i(0) = (0, 0, 0)$. The time T is fixed as $T = 150$. Figure 3 then illustrated the dependence of MDS(T) and δS(T) on the critical distance r. The plots of these values are approximately on linear lines $\delta S(T) = ar$ and MDS(T) $= br$, respectively. In this parameter setting we observe that $a = 1.18984$ and $b = 0.60158$.

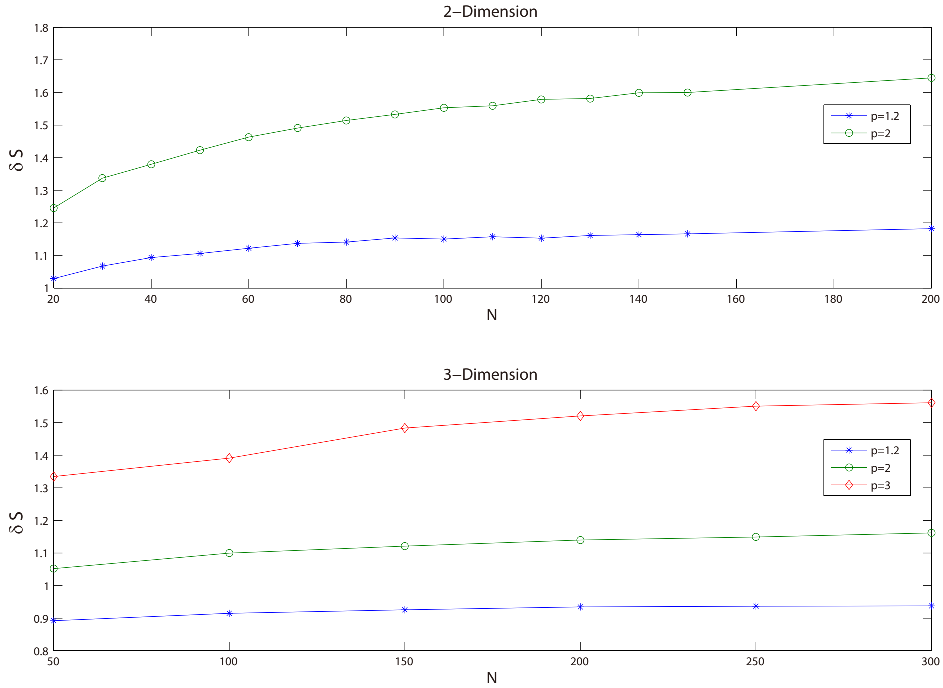

Figure 4: Dependence of δS on the total number N

How does $\delta S(T)$ respond when the total number N increases? To examine this question, we consider an N-fish system in the 2 or 3-dimensional space with $F_i = -5.0v_i$, and set $\alpha = 1$, $\beta = 0.5$, $p = 3$, $q = p + 1$, $r = 1$ and $T = 20$. As stated before, in order to simplify the arguments, each value shown in the figure is calculated by taking the mean value of the corresponding values for 10 simulations with different initial positions. Figure 4 shows that the diameter of school typically increases with the fish number. This is generally true in animal flocks, cf. also [7].

By observing the figure we find that the slope of the school radius as function of N is larger when p becomes larger.

2.5 Variance of Velocity In order to measure matching of velocity each other, we will use the ordinary variance

$$\sigma\mathrm{VS}(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\|v_i(t) - \bar{v}(t)\|^2}, \qquad 0 < t < \infty,$$

where $\bar{v}(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t)$ is the average of all velocities of fish at time t.

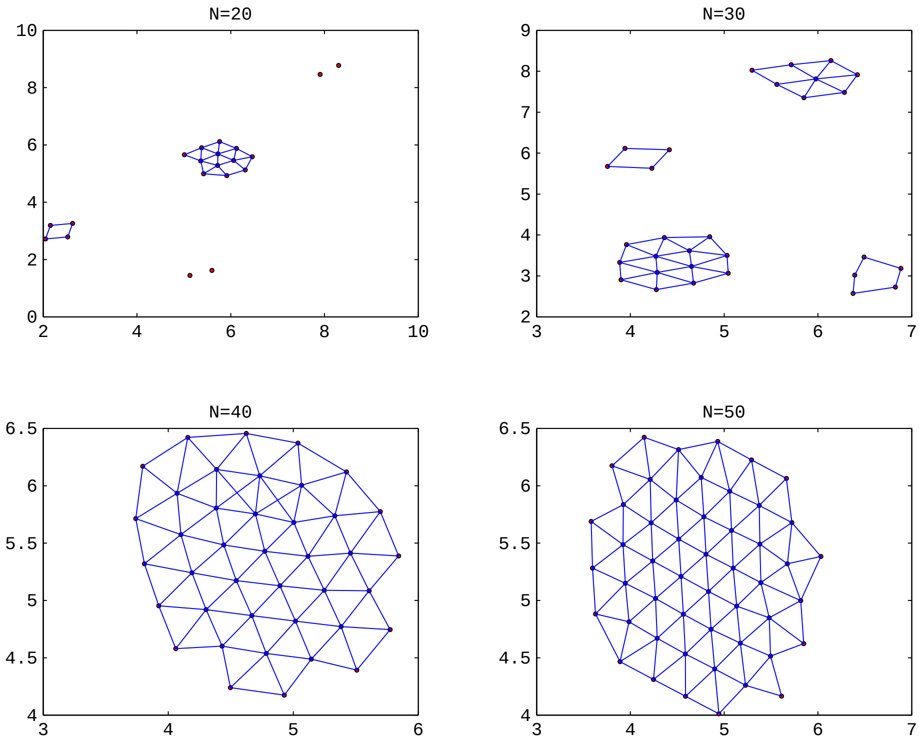

Figure 5: Effect of the total number N for N_ε

2.6 ε-Graph of School We finally introduce the ε-graph. Let $\varepsilon > 0$ be a fixed length. The vertices of graph at time t are all the positions of particles, $x_i(t)$, $1 \leqslant i \leqslant N$. Two vertices $x_i(t)$ and $x_j(t)$ are connected by the edge of graph if and only if $\|x_i(t) - x_j(t)\| \leqslant \varepsilon$. This graph is called the ε-graph of school at time t and is denoted by $\mathrm{GS}_\varepsilon(t)$. We also denote by $N_\varepsilon(t)$ the number of connected components of $\mathrm{GS}_\varepsilon(t)$. When $N_\varepsilon(t) = 1$, we consider that the fish have created a school with $\max_{1\leqslant i\leqslant N} \mathrm{DS}_i(t) \leqslant \varepsilon$. If $N_\varepsilon(t) \geqslant 2$, $N_\varepsilon(t)$ denotes the number of sub-schools.

Let us now examine effects of the total population N on $N_\varepsilon(t)$ for sufficiently large time t. To create a single school, N must be sufficiently large. To see this fact, consider an

N-fish system in the 2-dimensional space with $F_i = -5.0v_i$. Let $\alpha = 1$, $\beta = 0.5$, $p = 4$, $q = p+1$, $r = \varepsilon = 0.5$. Initial positions $x_i(0)$, $1 \leqslant i \leqslant N$, are randomly distributed in $[0,10]^2$ with null initial velocities $v_i(0) = (0,0)$. The population number N changes from 20 to 50. Figure 5 illustrates the graph $\mathrm{GS}_{0.5}(400)$ for each N. Up to $N = 39$, $N_{0.5}(400) \geqslant 2$ and so the fish are divided into a few sub-schools. But after a threshold number $N = 40$, they can create a single school.

3 Robustness of ε, θ-Schooling against Noise In this section, we consider the stochastic model (1.1). Under $\sigma_i > 0$, we want to study how the terms $\sigma_i dw_i(t)$ affect the geometrical structure of school. Can the fish system still create a school?

Let us here give a mathematical definition of school.

Definition 3.1 (ε, θ-Schooling)**.** For a given length $\varepsilon > 0$ and a tolerance $\theta > 0$, we say that the fish system is in ε, θ-*schooling* if there exists a time $T > 0$ such that $N_\varepsilon(t) = 1$ and $\sigma VS(t) < \theta$ for every $t \geqslant T$.

According to the above definition, a system forms a school only if velocities of all the fish tend to their average with the error less than tolerance θ. Therefore, the distance $\|x_i(t) - x_j(t)\|$ between any pair (i, j) will almost remain unchanged for $t \geqslant T$. So, the school structure remains unchanged, too. The second condition ensures that all the fish keep the relation $\mathrm{DS}_i(t) \leqslant \varepsilon$ for $t \geqslant T$. As a consequence, $N_\varepsilon(t) = 1$ remains to hold for $t \geqslant T$.

Assume that a system is in ε, θ-schooling for $t \geqslant T$. According to Remark 2.1 (cf. also Figure 1), if $\|x_i(t) - x_j(t)\| > \varepsilon$, then i and j keep their distance far away and consquently

$$\left(\frac{r^p}{\|x_i(t) - x_j(t)\|^p} - \frac{r^q}{\|x_i(t) - x_j(t)\|^q} \right) (x_i(t) - x_j(t)) \tag{3.1}$$

is sufficiently small. In the meantime, if $\|x_i(t) - x_j(t)\| \approx \varepsilon$, then their distance is $\|x_i(t) - x_j(t)\| \approx r$ and consequently (3.1) is again sufficiently small. In addition, it is clear that

$$\left(\frac{r^p}{\|x_i(t) - x_j(t)\|^p} + \frac{r^q}{\|x_i(t) - x_j(t)\|^q} \right) (v_i(t) - v_j(t))$$

is sufficiently small because of $\|v_i(t) - v_j(t)\| \approx 0$. We thus verify that

$$\sum_{i=1}^{n} dv_i \approx \sum_{i=1}^{N} F_i(t, x_i, v_i)dt.$$

In particular, if we take $F_i(t, x_i, v_i) = -cv_i$ $(1 \leqslant i \leqslant N)$, then

$$\sum_{i=1}^{N} dv_i \approx -c \left(\sum_{i=1}^{N} v_i \right) dt.$$

Consequently, $\sum_{i=1}^{N} v_i(t)$ decays exponentially as $t \to \infty$ and the system converges to a steady state.

Figure 6 shows an example of ε, θ-schooling generated by (1.1). 100 fish are situated at random positions in $[0,10]^2 \subset \mathbb{R}^2$ with null velocities at time $t = 0$. Then they interact

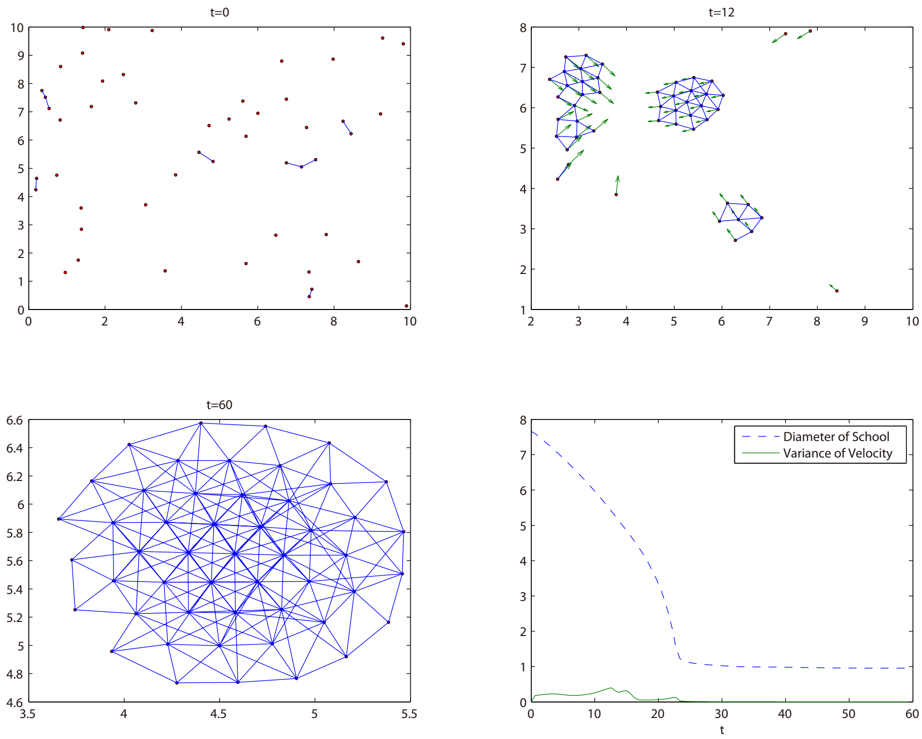

Figure 6: Example of ε, θ-schooling

with each other with $\alpha = 5$, $\beta = 1$, $p = 3$, $q = 4$, $r = 0.5$, $\sigma = 0$, $F_i = -5v_i$, $(1 \leqslant i \leqslant 100)$, we set $\varepsilon = 0.5 = r$ and $\theta = 10^{-6}$.

In the first three subfigures, we show ε-graphs of the system at different instants t. Each of these figure shows the positions of fish by points, their velocities by vectors and ε-graph edges by lines. The last subfigure draws the variance of velocity and the radius of school as functions of t.

Of course whether a system creates a school or not depends strongly on initial positions. It is also observed that 3-dimensional systems can create schools much easier than 2-dimensional ones.

Let us next study effects of the noise. We set $\sigma_i(t) = \sigma$, for $i = 1, 2, \ldots, N$. Simulations are implemented in the 3-dimensional space. We fix initial positions taking randomly in $[0, 5]^3 \subset \mathbb{R}^3$ with 50 fish, run 10 simulations with different realizations of the Wiener process for each value of σ. We observe the end point of each trajectories of $\sigma\mathrm{VS}(T)$ and $\delta\mathrm{S}(T)$ at $T = 50$. Other parameters are set as $p = 3$, $q = p + 1$, $\alpha = 5$, $\beta = 1$, $r = 0.5$, $F_i = -5.0v_i$, step size $\delta = 0.001$. Figure 7 shows that the fish can keep schooling against the noises when their magnitude σ is small enough. To the contrary, when it is large, the noises prevent the fish from creating a single school. It might be allowed, however, to insist that the swarming behavior described by our model (1.1) possesses the robustness for schooling. Figure 8 shows the expectation of school diameter as a function of σ. From this figure, too, we can find a similar tendency.

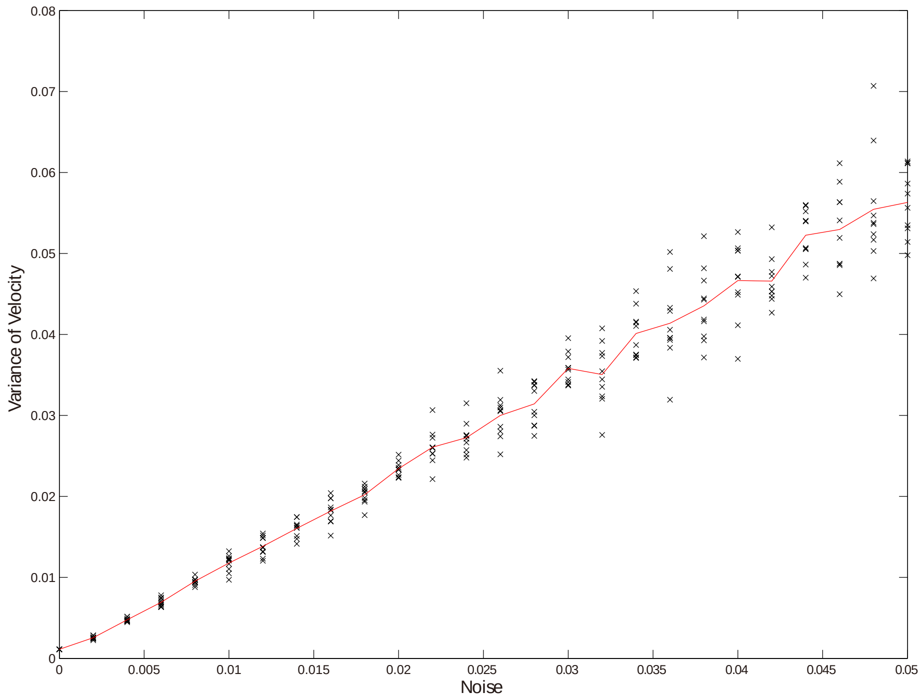

Figure 7: Influence of the noise magnitude σ for schooling

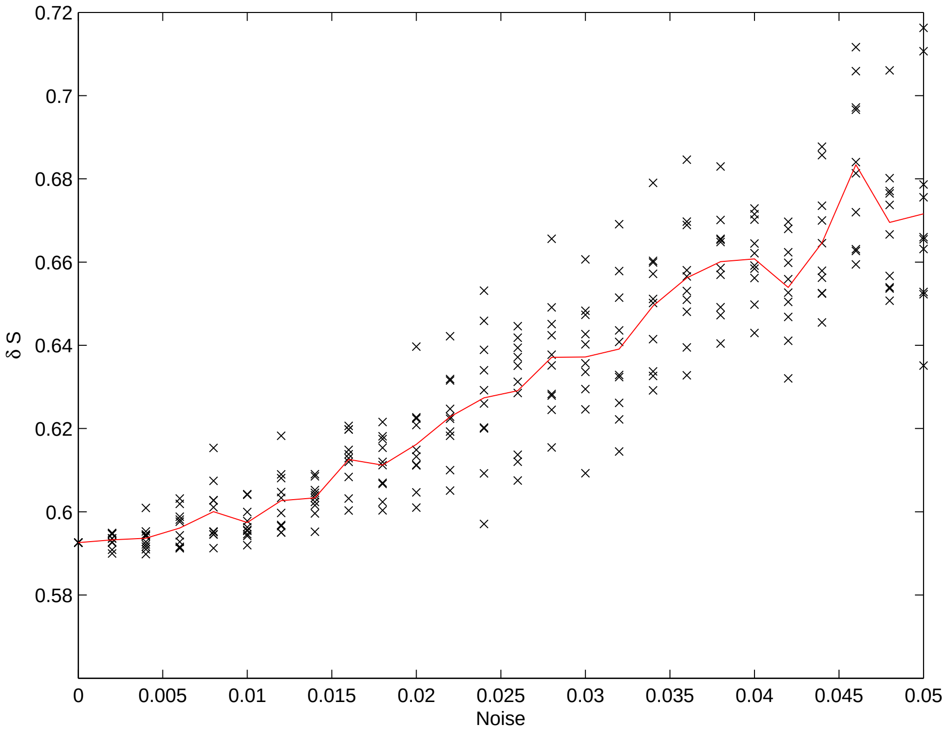

Figure 8: Influence of the noise magnitude σ to δS

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