REMARKS ON ω-CLOSED SETS IN SUNDARAM-SHEIK JOHN'S SENSE OF DIGITAL N-SPACES

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Abstract. The aim of this paper is to study some topological properties, especially, ω-closed sets (in Sundaram-Sheik John's sense) of digital lines and digital n-spaces ($n \geq 2$).

1 Introduction

In 2000, the concept of *ω-closed sets* (in Sundaram-Sheik John's sense) of topological spaces was introduced and investigated by P. Sundaram and M. Sheik John [35] [36] [37] and some results on bitopological version were investigated by [12]. We note that, in 1982, Hdeibe [14] had defined the same named concept: *ω-closed sets* (e.g., [14]); but their definitions are different. Throughout the present paper, we call the ω-closed sets [35] *the ω-closed sets in Sundaram-Sheik John's sense* (cf. Definition 2.1). The concept of Λ_s-sets was introduced and investigated by [4]. In the present paper, for the digital n-space $(\mathbb{Z}^n, \kappa^n)(n \geq 1)$, we try to investigate properties on ω-closed sets in Sundaram-Sheik John's sense and Λ_s-sets. The concept of the digital line $(\mathbb{Z}, \kappa)$ is initiated by Khalimsky [15], [16] and sometimes it is called the *Khalimsky line* (cf. [17] and references there, [33], [19, p.905], [20, p.175]; e.g., [11], [18]). We reference the naming of the *digital n-space* $(\mathbb{Z}^n, \kappa^n)$ in [20, Definition 4]; $(\mathbb{Z}^n, \kappa^n)$ is the topological product of n copies of the digital line $(\mathbb{Z}, \kappa)$ (cf. Section 3).

The purpose of the present paper is to characterlize the ω-closedness in Sundaram-Sheik John's sense in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6). Namely, a subset A is an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ if and only if A is closed in $(\mathbb{Z}^n, \kappa^n)$ (Theorem 4.6). In order to prove the result, we investigate the concept of semi-kernels of subsets in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.5) after checking on some examples in $(\mathbb{Z}^n, \kappa^n)$ (cf. Example 4.2). In Section 2 we recall some definitions and properties on topological spaces which are used in the present paper; moreover in Section 3 we recall the definitions of the digital lines and digital n-spaces ($n \geq 2$) and we give a short survey of important properties which are used in the present paper. In Section 4 we give some examples and we prove a characterization of ω-closed sets in Sundaram-Sheik John's sense for $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6). In order to prove Theorem 4.6, we need the construction of semi-open sets containing a point of $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.4). In the end of Section 4, using Theorem 4.4 and Theorem 4.9, we give an alternative and direct proof of [30, Theorem 4.2] which shows $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2.

Throughout the present paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned.

2 Preliminaries

We recall some concepts and properties on topological spaces.

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Definition 2.1 (i) ([22, Definition 2.1]) A subset A of a topological space (X,τ) is called *generalized closed* (shortly, g-closed) in (X,τ) if $\mathrm{Cl}(A)\subset U$ whenever $A\subset U$ and U is open in (X,τ).

(ii) ([35], [36]) A subset A of a topological space (X,τ) is called ω*-closed in Sundaram-Sheik John's sense* in (X,τ) if $\mathrm{Cl}(A)\subset V$ whenever $A\subset V$ and V is semi-open in (X,τ). The complement of an ω-closed set is called an ω*-open set*.

A subset B of (X,τ) is said to be *semi-open* [21, Definition 1] in (X,τ), if there exists an open set U such that $U\subset B\subset\mathrm{Cl}(U)$. It is shown that [21, Theorem 1] a subset B is semi-open if and only if $B\subset\mathrm{Cl}(\mathrm{Int}(B))$ in (X,τ). A subset E of (X,τ) is said to be *preopen* [25] in (X,τ), if $E\subset\mathrm{Int}(\mathrm{Cl}(E))$ holds in (X,τ). Every open set is semi-open and preopen in (X,τ). The complement of a semi-open set (resp. preopen set) is said to be *semi-closed* (resp. *preclosed*). In the present paper, the famly of all semi-open sets (resp. preopen sets) of (X,τ) is denoted by $\mathrm{SO}(X,\tau)$ (resp. $\mathrm{PO}(X,\tau)$). Namely, for a topological space (X,τ), as notation,

• $SO(X,\tau):=\{B|B\subset\mathrm{Cl}(\mathrm{Int}(B)),B\subset X\}$, $PO(X,\tau):=\{E|E\subset\mathrm{Int}(\mathrm{Cl}(E)),\ E\subset X\}$; and $\tau\subset SO(X,\tau)$ and $\tau\subset PO(X,\tau)$ hold for any topological space (X,τ).

The following concept of *semi-kernels* is due to [4] and the concept of *kernels* is well known (e.g., [28]).

Definition 2.2 Let E be a subset of a topological space (X,τ).

(i) ([4, Definition 1]) The following set τ-sKer(E) (or shortly sKer(E)) is called a *semi-kernel* of E in (X,τ) (in [4], it is denoted by E^{Λ_s}):

• $\tau\text{-sKer}(E)=E^{\Lambda_s}:=\bigcap\{V|E\subset V$ and V is semi-open in $(X,\tau)\}$.

Note that, in the present paper, we use the symbol τ-sKer(E) or sKer(E).

(ii) (e.g., [28]) The following set τ-Ker(E) (or shortly Ker(E)) is called a *kernel of* E in (X,τ):

• $\tau\text{-Ker}(E):=\bigcap\{V|E\subset V$ and V is open in $(X,\tau)\}$.

Note that, in [28] (resp. [24]), the set τ-Ker(E) above is denoted by $\mathrm{Ker}_\tau(E)$ (resp. $E^{\wedge}$).

Definition 2.3 ([4, Definition 2]) In a topological space (X,τ), a subset E is a Λ_s*-set* of (X,τ) if $E=E^{\Lambda_s}$ (i.e., $E=$sKer(E)).

We recall the following property on semi-kernels.

Proposition 2.4 *For a family* $\{E_i|i\in\Omega\}$ *of subsets of a topological space* (X,τ), *where* Ω *is an index set,*

(i) ([4, Proposition 3.1]) $\mathrm{sKer}(\bigcup\{E_i|i\in\Omega\})=\bigcup\{\mathrm{sKer}(E_i)|i\in\Omega\}$ *holds; and*

(ii) (e.g., [24, (2.5)]) $\mathrm{Ker}(\bigcup\{E_i|i\in\Omega\})=\bigcup\{\mathrm{Ker}(E_i)|i\in\Omega\}$ *holds.*

Theorem 2.5 *t60* ([35], [36]) *A subset* A *is* ω*-closed* (*in Sundaram-Sheik John's sense*) *in a topological space* (X,τ) *if and only if* $\mathrm{Cl}(A)\subset\mathrm{sKer}(A)$.

Proposition 2.6 (i) ([4, Proposition 3.7]) *A topological space* (X,τ) *is semi-*T_1 *if and only if every subset is a* Λ_s*-set.*

(ii) ([4, Corollary 3.8]) *Every semi-*T_1*-space is a semi-*R_0*-space.*

We need the following notation.

Definition 2.7 (e.g., [10, p.166]; [39, Definition 2.1] [38, p.47] for the case where $E:=\mathbb{Z}^n$) For a subset E of (X,τ), we define the following subsets E_τ and $E_{\mathcal{F}}$:

$E_\tau:=\{x\in E\mid\{x\}$ is open in (X,τ), i.e., $\{x\}\in\tau\}$;

$E_{\mathcal{F}}:=\{x\in E\mid\{x\}$ is closed in $(X,\tau)\}$.

3 Preliminaries-2 In the present section, we recall some foundamental definitions and topological properties on digital lines and digital n-spaces ($n \geq 2$); this includes a survey on digital lines and digital n-spaces ($n \geq 2$) on our topics. And the notation of Definition 3.11 and ($*$ 20) in (II) below are used in the proofs of results in Section 4.

(I) (digital lines):

• Let us recall some definitions and topological properties on digital lines (cf. ($*$1) - ($*$11) below).

Definition 3.1 (cf. [20, p.175], [19, p.905, p.908], [26, Section 2], [27, Example 4 in Section 2]; e.g., [11, Section 1], [33, Section 6 in p.9]) *The digital line* or so called *the Khalimsky line* $(\mathbb{Z}, \kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase.

Remark 3.2 We put $\mathcal{G} := \{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ in Definition 3.1.

(i) By the definition of κ, a subset U of $\mathbb{Z}$ is open in $(\mathbb{Z}, \kappa)$ (i.e., $U \in \kappa$) if and only if there exists a family of subsets of $(\mathbb{Z}, \kappa)$, say $\{B_i^{(U)} | \ i \in I^{(U)}\}$, where $I^{(U)}$ is an index set, such that $U = \bigcup\{B_i^{(U)} | \ i \in I^{(U)}\}$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} | \ j \in \{1, 2, ..., m\}\}$ for some positive integer m and some subsets $V_j^{(i)} \in \mathcal{G}(1 \leq j \leq m)$, here we assume that $V_j^{(i)} \neq V_{j_1}^{(i)}$ if $j \neq j_1$, where $j, j_1 \in \{1, 2, ..., m\}$).

(ii) For the set $B_i^{(U)} = \bigcap\{V_j^{(i)} | \ j \in \{1, 2, ..., m\}\}$ above, we note that:

$(*)_1$ if $m = 1$ (resp. $m = 2$), then $B_i^{(U)} = \{2t-1, 2t, 2t+1\}$ (resp. $=\{2u+1\}$ or $\emptyset$) for some $t \in \mathbb{Z}$ (resp. for some $u \in \mathbb{Z}$);

$(*)_2$ if $m \geq 3$, then $B_i^{(U)} = \bigcap\{V_j^{(i)} | j \in \{1, 2, ..., m\}\} = \emptyset$.

• For examples, we first have some properties on singletons and two-pointed sets of $(\mathbb{Z}, \kappa)$ (cf. ($*$1) - ($*$3) below): for an integer s,

· ($*$1) *a singleton* $\{2s+1\}$ *is open in* $(\mathbb{Z}, \kappa)$; $\{2s+1\}$ *is not closed in* $(\mathbb{Z}, \kappa)$.

· ($*$2) *a singleton* $\{2s\}$ *is not open in* $(\mathbb{Z}, \kappa)$; *but* $\{2s\}$ *is closed in* $(\mathbb{Z}, \kappa)$.

· ($*$3) *subsets* $\{2s, 2s+1\}$ *and* $\{2s-1, 2s\}$ *are not open in* $(\mathbb{Z}, \kappa)$, *where* $s \in \mathbb{Z}$ (cf. ($*$8)(iii) below).

(*Proof of* ($*$1)). (*Proof of the openness*) It is shown that $\{2s+1\} = V_1 \cap V_2$, where $V_1 := \{2s-1, 2s, 2s+1\} \in \mathcal{G}$ and $V_2 := \{2s+1, 2s+2, 2s+3\} \in \mathcal{G}$. Thus, $\{2s+1\}$ is open in $(\mathbb{Z}, \kappa)$.

(*Proof of the non-closedness*) Suppose that $\{2s+1\}$ is closed. Put $U := \mathbb{Z} \setminus \{2s+1\}$. Then, $U \in \kappa$ and so there exists a family of subsets: $\{B_i^{(U)} | \ i \in I^{(U)}\}$, where $I^{(U)}$ is an index set, such that $U = \bigcup\{B_i^{(U)} | \ i \in I^{(U)}\}$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} | \ j \in \{1, 2, ..., m\}\}$ for some positive integer m and some subsets $V_j^{(i)} \in \mathcal{G}(1 \leq j \leq m)$ (cf. Definition 3.1,Remark 3.2(i)). Pick a point $2s \in U$, where $s \in \mathbb{Z}$. Then, we have

$(*)_a$ $2s \in B_{i'}^{(U)} = \bigcap\{V_j^{(i')} | j \in \{1, 2, ..., m'\}\}$ and $B_{i'}^{(U)} \subset U$ for some $i' \in I^{(U)}$ and positive integer m'.

By Remark 3.2(ii), it is shown that $m' = 1$ and $B_{i'}^{(U)} = \bigcap\{V_j^{(i')} | j \in \{1, 2, ..., m'\}\}$ $=\{2s-1, 2s, 2s+1\}$. Thus, using $(*)_a$, we have $2s+1 \in U$; but this contradicts the definition of U in the first setting. Therefore, the singleton $\{2s+1\}$ is not closed in $(\mathbb{Z}, \kappa)$. ($\circ$)

(*Proof of* ($*$2)). (*Proof of the non-openness*). Suppose that $\{2s\} \in \kappa$. We put $U := \{2s\}$. By the definition of κ (cf. Remark 3.2(i)), there exists subsets $B_i^{(U)} (i \in I^{(U)})$, where $I^{(U)}$ is an index set, such that $2s \in B_i^{(U)} = \bigcap\{V_j^{(i)} | \ j \in \{1, 2, ..., m\}\}$ and $B_i^{(U)} \subset U$ for some positive integer m and $V_j^{(i)} \in \mathcal{G}(1 \leq j \leq m)$. By using Remark 3.2(ii), it is shown that $m = 1$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} | \ j \in \{1, 2, ..., m\}\} = \{2s-1, 2s, 2s+1\} \subset U$; and so $2s+1 \in U$.

This contradicts the definition of $U := \{2s\}$. Therefore, any singleton $\{2s\}$ is not open in $(\mathbb{Z}, \kappa)$.
(*Proof of the closedness*). It is shown that $\{2s\} = \mathbb{Z} \setminus E$, where $E := \bigcup\{\{2s-2j-1, 2s-2j, 2s-2j+1\} | j \in \mathbb{Z}$ and $j \neq 0\}$. Since $E \in \kappa$, $\mathbb{Z} \setminus E$ is closed; and so $\{2s\}$ is closed in $(\mathbb{Z}, \kappa)$. $(\circ)$

(*Proof of* $(*3)$) Suppose that $\{2s-1, 2s\} \in \kappa$. Then, we have a contradiction. Put $U := \{2s-1, 2s\}$. By Definition 3.1 (cf. Remark 3.2 (i)), there exists an index set $I^{(U)}$ and some subsets $B_i^{(U)}$ such that $U = \bigcup\{B_i^{(U)} |\ i \in I^{(U)}\}$, where $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ for some positive integer m and $V_j^{(i)} \in \mathcal{G}(1 \leq j \leq m)$ (cf. Remark 3.2). It is noted that $B_k^{(U)} \subset U$ for any $k \in I^{(U)}$. Then, we have:
$(*)^a$ $2s \in B_a^{(U)}$ for some $a \in I^{(U)}$; $(*)^b$ $2s-1 \in B_b^{(U)}$ for some $b \in I^{(U)}$;
$(*)^c$ $B_a^{(U)} \cup B_b^{(U)} \subset U$, where $U := \{2s-1, 2s\}$.
Using $(*)^a$, $(*)^b$ and $(*)^c$, we have: $(*)^d$ $U = B_a^{(U)} \cup B_b^{(U)}$.
Using Remark 3.2(ii), $(*)^a$ and $(*)^b$ above, we have $B_a^{(U)} = \{2s-1, 2s, 2s+1\}$ and $B_b^{(U)} = \{2s-1\}, \{2s-1, 2s, 2s+1\}$ or $\{2s-3, 2s-2, 2s-1\}$. Thus, using $(*)^d$ above, we have $U = \{2s-1, 2s, 2s+1\}$ or $U = \{2s-3, 2s-2, 2s-1, 2s, 2s+1\}$. These properties above contradict the defininion of $U = \{2s-1, 2s\}$. Therefore, $\{2s-1, 2s\}$ is not open in $(\mathbb{Z}, \kappa)$. Similarly, it is proved that $\{2s+1, 2s\}$ is not open in $(\mathbb{Z}, \kappa)$. In $(*8)$(iii) below, we note that they are semi-open in $(\mathbb{Z}, \kappa)$. $(\circ)$
• For the digital line $(\mathbb{Z}, \kappa)$, the concept of *the smallest open set, say* $U(x)$*, containing a point* x *of* $(\mathbb{Z}, \kappa)$ is very important; throughout the present paper, we put:
$\cdot$ $U(2s) := \{2s-1, 2s, 2s+1\}$; $\cdot$ $U(2s+1) := \{2s+1\}$, where $s \in \mathbb{Z}$.
We first recall the definition of *the smallest open set containing a point* x for a topological space (X, τ).

Definition 3.3 (e.g., [29, Definition 2.4]) Let (X, τ) be a topological space and a point $x \in X$. A subset E is called *the smallest open set containing* x if $x \in E, E \in \tau$ and $A = E$ holds for any open set A such that $x \in A$ and $A \subset E$.

For an open set E and $x \in E, E$ is the smallest open set containing x if and only if $E \subset G$ holds for every open set G containing the point x (e.g., [29, Remark 2.5 (ii)]).
• For the digital line $(\mathbb{Z}, \kappa)$, we recall the concept of *the smallest open set, say* $U(x)$*, containing a point* x *of* $(\mathbb{Z}, \kappa)$. Obviously, every subset belonging to $\mathcal{G} =: \{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ is open in $(\mathbb{Z}, \kappa)$. Then, we have the following important property on $U(x)$, where $x \in \mathbb{Z}$:
$\cdot$**$(*4)$ (i)** *$U(2s) := \{2s-1, 2s, 2s+1\}$ is the smallest open set containing $2s$. Namely, $U(2s)$ is an open set containing the point $2s$ and if A is an any open set such that $2s \in A$ and $A \subset U(2s)$, then $A = U(2s)$. And, if G is any open set containing $2s$ in $(\mathbb{Z}, \kappa)$, then $U(2s) \subset G$.*

(ii) *$U(2s+1) := \{2s+1\}$ is the smallest open set containing $2s+1$.*

(iii) *For each point x of $(\mathbb{Z}, \kappa)$, there exists the smallest open set $U(x)$ containing the point x* (cf. [20, p.175]). *Namely, for the point $x \in \mathbb{Z}$, $U(x)$ is an open set containing the point x and if A is an any open set such that $x \in A$ and $A \subset U(x)$, then $A = U(x)$. And, if G is any open set containing x in $(\mathbb{Z}, \kappa)$, then $U(x) \subset G$.*

(*Proof of* $(*4)$). **(i)** By $(*2)$ and $(*3)$ above, it is shown that:
$(*^e)$ $U(2s)$ is open in $(\mathbb{Z}, \kappa)$ and $2s \in U(2s)$ (because of $U(2s) \in \mathcal{G}$); and
if A is any open subset of $U(2s)$ such that $2s \in A$, then $A = U(2s)$.
Indeed, if $A_1 \subset U(2s)$ such that $2s \in A_1$ and $A_1 \neq U(2s)$, then $A_1 = \{2s\}, \{2s-1, 2s\}$ or $\{2s, 2s+1\}$ and the subset A_1 is not open in $(\mathbb{Z}, \kappa)$ (cf. $(*\ 2)$, $(*\ 3)$ above). Thus, we have

$A = U(2s)$ for any open subset A such that $2s \in A$ and $A \subset U(2s)$. Moreover, we show: $(*^f)$ $U(2s) \subset G$ holds for any open set G containing the point $2s$ and $2s \in U(2s)$. (Indeed, let G be any open set containing the point $2s$. Then, we have $2s \in U(2s) \cap G$ and $U(2s) \cap G$ is an open set such that $U(2s) \cap G \subset U(2s)$; thus we have $U(2s) \cap G = U(2s)$ (cf. $(*^e)$ above). Namely, we have $U(2s) \subset G$.)

Therefore, by $(*^e)$ or $(*^f)$, it is shown that $U(2s)$ is the smallest open set containing $2s$ (cf. Definition 3.3).

(ii) For an odd integer $2s + 1$, where $s \in \mathbb{Z}, U(2s + 1) = \{2s + 1\}$ is the smallest open set containing the point $2s + 1$ (cf. $(*1)$). **(iii)** Using (i) and (ii) above, the set $U(x)$ is the smallest open set containing the point x. $(\circ)$

• We have the form of the *κ-closure* of $\{x\}$, the *κ-interior* of $\{x\}$ and the *κ-kernel* of $\{x\}$, respectively, (cf. $(*5)$, $(*6)$ below): for an integer s,

· **$(*5)$ (i)** $\kappa\text{-Cl}(\{2s + 1\}) = \{2s, 2s + 1, 2s + 2\}$, $\kappa\text{-Cl}(\{2s\}) = \{2s\}$;

(ii) $\kappa\text{-Int}(\{2s + 1\}) = \{2s + 1\}; \kappa\text{-Int}(\{2s\}) = \emptyset$;

(iii) $\kappa\text{-Ker}(\{2s + 1\}) = \{2s + 1\}; \kappa\text{-Ker}(\{2s\}) = \{2s - 1, 2s, 2s + 1\} = U(2s)$.

(*Proof of* $(*5)$). **(i)** They are shown by $(*4)$(i), $(*1)$ and $(*2)$ above, respectively. **(ii)** They are shown by $(*1)$ and $(*2)$ above, respectively. **(iii)** They are shown by $(*1)$ and $(*4)$(i) above. $(\circ)$

· **$(*6)$(i)** *In the digital line* $(\mathbb{Z}, \kappa)$*, a singleton* $\{x\}$ *is open if and only if the integer* x *is odd in* $\mathbb{Z}$.

(ii) *A singleton* $\{x\}$ *is closed in* $(\mathbb{Z}, \kappa)$ *if and only if the integer* x *is even in* $\mathbb{Z}$.

(*Proof of* $(*6)$) **(i)**. It is shown by $(*5)$(ii) above. **(ii)** By the closure form in $(*5)$(i) above, (ii) is shown. $(\circ)$

By $(*6)$ above, it is shown that:

· $(*7)$ **(i)** *Every singleton of* $(\mathbb{Z}, \kappa)$ *is open or closed* (cf. $(*6)$; or $(*1)$ and $(*2)$ above). *This shows that* $(\mathbb{Z}, \kappa)$ *is* $T_{1/2}$ (e.g., [8, Example 4.6]; cf. [22, Definition 5.1], [9, Theorem 2.5]). We recall some topological properties; in general, the class of $T_{1/2}$-spaces is properly placed between the classes of T_0-spaces and T_1-spaces ([22, Corollary 5.6]). Furthermore, Dontchev and Ganster [8, Example 4.6] proved that $(\mathbb{Z}, \kappa)$ *is* $T_{3/4}$; in general, the class of $T_{3/4}$-spaces is properly placed between the classes of T_1-spaces and $T_{1/2}$-spaces ([8, Corollary 4.4 and Corollary 4.7]). For the digital plane $(\mathbb{Z}^2, \kappa^2)$ (cf. Definition 3.4 below), it is well known that $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$ ([26, Section 3]).

• We recall the *semi-openness* (resp. *semi-closedness*) (cf. Section 2) of singletons in $(\mathbb{Z}, \kappa)$ and the *semi-closure* of $\{x\}$, the *semi-interor* of $\{x\}$ and the *semi-kernel* (cf. Definition 2.2(i)) of $\{x\}$ (cf. $(*8)$ and $(*9)$ below): for an integer s,

·**$(*8)$(i)** *every open singleton* $\{2s + 1\}$ *is semi-open and semi-closed in* $(\mathbb{Z}, \kappa)$;

(ii) *every closed singleton* $\{2s\}$ *is semi-closed in* $(\mathbb{Z}, \kappa)$*; but* $\{2s\}$ *is not semi-open in* $(\mathbb{Z}, \kappa)$;

(iii) *the subsets* $\{2s, 2s + 1\}$ *and* $\{2s - 1, 2s\}$ *are semi-open on* $(\mathbb{Z}, \kappa)$.

(*Proof of* $(*8)$). **(i)** Every open set is semi-open and so $\{2s + 1\}$ is semi-open in $(\mathbb{Z}, \kappa)$ (cf. $(*6)$(i) above). And, since $\kappa\text{-Int}(\kappa\text{-Cl}(\{2s+1\}))= \kappa\text{-int}(\{2s, 2s+1, 2s+2\}) = \{2s+1\}$ hold, $\{2s + 1\}$ is semi-closed (cf. $(*5)$(i)(ii) above). **(ii)** Since $\kappa\text{-Int}(\kappa\text{-Cl}(\{2s\})) = \kappa\text{-Int}(\{2s\}) = \emptyset \subset \{2s\}, \{2s\}$ is semi-closed in $(\mathbb{Z}, \kappa)$. And, we have $\text{Cl}(\text{Int}(\{2s\})) = \text{Cl}(\emptyset) = \emptyset \not\supset \{2s\}$ and so $\{2s\}$ is not semi-open in $(\mathbb{Z}, \kappa)$. **(iii)** It is easily shown that $\kappa\text{-Cl}(\kappa\text{-Int}(\{2s, 2s+1\})) = \kappa\text{-Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\} \supset \{2s, 2s+1\}$; and so $\{2s, 2s+1\}$ is semi-open in $(\mathbb{Z}, \kappa)$. Similarly, the subset $\{2s - 1, 2s\}$ is semi-open in $(\mathbb{Z}, \kappa)$. $(\circ)$

·**$(*9)$** For an integer s, we have the following properties:

(i) $\kappa\text{-sCl}(\{2s + 1\}) = \{2s + 1\}$; $\kappa\text{-sCl}(\{2s\}) = \{2s\}$;

(ii) κ-sInt$(\{2s+1\}) = \{2s+1\}$*;* κ-sInt$(\{2s\}) = \emptyset$;

(iii) κ-sKer$(\{2s+1\}) = \{2s+1\}$; κ-sKer$(\{2s\}) = \{2s\}$.

(*Proof of* $(*9)$). **(i)** (resp. **(ii)**) They are proved by $(*8)$(i) (resp. $(*8)$(ii)) above. **(iii)** By $(*8)$(iii) (resp. $(*8)$(i)), it is obtained that κ-sKer$(\{2s\}) = \{2s, 2s+1\} \cap \{2s-1, 2s\} = \{2s\}$ (resp. κ-sKer$(\{2s+1\}) = \{2s+1\}$). $(\circ)$

• We recall more topological properties on $(\mathbb{Z}, \kappa)$:

· **$(*10)$ (i)** *For* $(\mathbb{Z}, \kappa)$*,* $\kappa = PO(\mathbb{Z}, \kappa), PO(\mathbb{Z}, \kappa) \subset SO(\mathbb{Z}, \kappa)$ *and* $\kappa^\alpha = \kappa$ *hold* ([10, Theorem 2.1 (i)(a)(b)]), where $\kappa^\alpha := \{V \mid V$ is α-open in $(\mathbb{Z}, \kappa)\}$. For topological spaces, the concept of the *α-open set* was introduced by Njåstad [31] who called it *the α-set*. A subset A of a topological space (X, τ) is said to be *α-open* in (X, τ) if $A \subset$Int(Cl(Int(A))) holds.

(ii) *The digital line* $(\mathbb{Z}, \kappa)$ *is submaximal.* This fact may be known in folklore; however, we are able to read one of the proof ([10, Theorem 1.1(i)]). Furthermore, it is noted that, by [10, Theorem 1.1(ii)(iii)], the digital plane $(\mathbb{Z}^2, \kappa^2)$ (cf. **(II)** below) is not submaximal but it is *quasi-submaximal.* Al-Nashef [1, Definition 3.2] introduced the concept of *quasi-submaximal* topological spaces which is weaker than one of submaximal spaces (e.g., [3, Definition 1.1], [13, p.137]).

(iii) *The digital line* $(\mathbb{Z}, \kappa)$ *is s-normal* ([11, Section 3, Theorem B]). In 1978, Maheshwari and Prasad [23] introduced the concept of s-normal topological spaces using semi-open sets. The digital plane is also a geometric example of s-normal spaces ([11, Section 5, Theorem D]).

• Using Definition 2.7 for $(X, \tau) = (\mathbb{Z}, \kappa)$, we can define the following subsets $\mathbb{Z}_\kappa := \{x \in \mathbb{Z} \mid \{x\} \in \kappa\}, \mathbb{Z}_\mathcal{F} := \{x \in \mathbb{Z} \mid \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$; for a nonempty subset E of $(\mathbb{Z}, \kappa)$, $E_\kappa := \{x \in E \mid \{x\} \in \kappa\}$ and $E_\mathcal{F} := \{x \in E \mid \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$.

· **$(*11)$ (i)** Let $A \subset \mathbb{Z}$. Then we have that $\mathbb{Z}_\kappa = \{2m+1 \in \mathbb{Z} \mid m \in \mathbb{Z}\}$; $A_\kappa = \{2m+1 \in A \mid m \in \mathbb{Z}\}$ (cf. $(*6)$(i) above);
$\mathbb{Z}_\mathcal{F} = \{2m \in \mathbb{Z} \mid m \in \mathbb{Z}\}$; $A_\mathcal{F} = \{2m \in A \mid m \in \mathbb{Z}\}$ (cf. $(*6)$(ii) above).

(ii) A_κ *is open in* $(\mathbb{Z}, \kappa)$ *for any subset* A *of* $(\mathbb{Z}, \kappa)$*; and* $A_\kappa = \mathbb{Z}_\kappa \cap A$.

(iii) $\mathbb{Z} = \mathbb{Z}_\kappa \cup \mathbb{Z}_\mathcal{F}$ $(\mathbb{Z}_\kappa \cap \mathbb{Z}_\mathcal{F} = \emptyset)$ *and* $A = A_\kappa \cup A_\mathcal{F}$ $(A_\kappa \cap A_\mathcal{F} = \emptyset)$ *for any subset* A *of* $(\mathbb{Z}, \kappa)$ (cf. $(*6)$ above).

(iv) *For any subset* A *of* $(\mathbb{Z}, \kappa)$*,* $A_\mathcal{F} = A \setminus A_\kappa$ *holds and* $A_\mathcal{F}$ *is closed in* $(\mathbb{Z}, \kappa)$*; and* $A_\mathcal{F} = \mathbb{Z}_\mathcal{F} \cap A$.

(v) If $E \subset F \subset \mathbb{Z}$, then $E_\kappa \subset F_\kappa$ and $E_\mathcal{F} \subset F_\mathcal{F}$ hold in $(\mathbb{Z}, \kappa)$.

(*Proof of* $(*11)$) **(iv)**. (*Proof of the closedness of* $A_\mathcal{F}$). Let $x \in \mathbb{Z} \setminus A_\mathcal{F}$.

Case 1. $x = 2s+1$, where $s \in \mathbb{Z}$: for this case, we have $x \in \mathbb{Z}_\kappa$ (cf. $(*6)$(i) above); and so $\{x\} \cap A_\mathcal{F} = \emptyset$ (cf. (iii) above). Thus, there exists an open set $\{x\}$, say U_x, containing x such that $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$.

Case 2. $x = 2t$, where $t \in \mathbb{Z}$: for this case, we have $x \in \mathbb{Z}_\mathcal{F}$ and $x \notin A_\mathcal{F}$ (cf. (iii) above and $(*6)$(ii) above). Hence, for the point $x \in \mathbb{Z}_\mathcal{F} \setminus A_\mathcal{F}$, there exists an open set $\{x-1, x, x+1\}$, say U_x, containing x and $\{x-1, x+1\} \subset \mathbb{Z}_\kappa$; and so $U_x \cap A_\mathcal{F} = \{x-1, x, x+1\} \cap A_\mathcal{F} = \emptyset$, i.e., $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$.

Thus, for each point $x \in \mathbb{Z} \setminus A_\mathcal{F}$, the subset U_x above is an open set containing x such that $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$. We have $\mathbb{Z} \setminus A_\mathcal{F} = \bigcup\{U_x \mid x \in \mathbb{Z} \setminus A_\mathcal{F}\}$ and so $\mathbb{Z} \setminus A_\mathcal{F} \in \kappa$. Namely, $A_\mathcal{F}$ is closed in $(\mathbb{Z}, \kappa)$. $(\circ)$

(II) (digital n-spaces $(n \geq 2)$**):**

• In the final stage of the present section, we recall some structures of the digital n-space $(n \geq 2)$ ([20, Definition 4]; e.g., [26, Section 3], [39], [38], [11]; for $n = 2$, [10], [5, Section 6], [34, Section 5], [7, Section 7], [6], [32, Section 6]).

Definition 3.4 ([20, Definition 4]) Let n be an integer with $n \geq 2$. *The digital n-space* or *Khalimsky n-space* is the Cartesian product of n-copies of the digital line $(\mathbb{Z}, \kappa)$. This topological space is denoted by $(\mathbb{Z}^n, \kappa^n)$, where $\mathbb{Z}^n := \prod_{i=1}^n X_i$, where $X_i = \mathbb{Z}$ for all integers i with $1 \leq i \leq n$, and $\kappa^n := \prod_{i=1}^n \tau_i$, where $\tau_i := \kappa$ for all integers i with $1 \leq i \leq n$. For $n = 2$, $(\mathbb{Z}^2, \kappa^2)$ is called *the digital plane* or *Khalimsky plane.*

Since κ^n is the product topology of n-copies of κ, it is shown that: for a point $x := (x_1, x_2, ..., x_n)$ of $(\mathbb{Z}^n, \kappa^n)$,
· **(∗12)** (a) κ^n-Cl$(\{x\}) = \prod_{i=1}^n \kappa$-Cl$(\{x_i\})$; (b) κ^n-Int$(\{x\}) = \prod_{i=1}^n \kappa$-Int$(\{x_i\})$;
(c) κ^n-Ker$(\{x\}) = \prod_{i=1}^n \kappa$-Ker$(\{x_i\})$.

(*Note on* (c)). Let $(X, \tau) := \prod_{i=1}^n (X_i, \tau_i)$ be a product topological space of topological spaces $(X_i, \tau_i)(1 \leq i \leq n)$. In general, for a point $x := (x_1, x_2, ..., x_n)$ of (X, τ), it is shown that τ-Ker$(\{x\}) = \prod_{i=1}^n (\tau_i$-Ker$(\{x_i\}))$, where $\tau = \prod_{i=1}^n \tau_i$. ∘

We use the following well known property; we recall shortly the proof.

Proposition 3.5 *Let $x := (x_1, x_2, ..., x_n)$ be a point of $(\mathbb{Z}^n, \kappa^n)$.*

(i) *If all the coordinates of the point x is odd, say $x_i = 2s_i + 1 \in \mathbb{Z}$ $(s_i \in \mathbb{Z})$ for each integer i with $1 \leq i \leq n$, then for the point $x = (2s_1 + 1, 2s_2 + 1, ..., 2s_n + 1)$*

(a) κ^n-Cl$(\{x\})= \prod_{i=1}^n \{2s_i, 2s_i + 1, 2s_i + 2\}$.

(b) κ^n-Int$(\{x\})= \prod_{i=1}^n \{2s_i + 1\} = \{x\}$*; and so the singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$.*

(c) κ^n-Ker$(\{x\})= \prod_{i=1}^n \{2s_i + 1\} = \{x\}$.

(ii) *If all the coordinates of the point x is even, say $x_i = 2s_i \in \mathbb{Z}$ $(s_i \in \mathbb{Z})$ for each integer i with $1 \leq i \leq n$, then for the point $x = (2s_1, 2s_2, ..., 2s_n)$*

(a) κ^n-Cl$(\{x\})= \prod_{i=1}^n \{2s_i\} = \{x\}$*; and so the singleton $\{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)$.*

(b) κ^n-Int$(\{x\})= \prod_{i=1}^n \emptyset = \emptyset$.

(c) κ^n-Ker$(\{x\})= \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\} = \prod_{i=1}^n U(2s_i)$.

(iii) (a) *A singleton $\{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)$ if and only if all the coordinates of x, say $x_i (1 \leq i \leq n)$, are even.*
(b) *A singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ if and only if all the coordinates of x, say $x_i (1 \leq i \leq n)$, are odd.*

Proof. **(i) (ii)** The properties are shown by (∗5) in (I), (∗12) in (II) and definitions.

(iii) (a) (Necessity) It follows from assumption that κ^n-Cl$(\{x\}) = \{x\}$. Using (∗12)(a) in (II), it is shown that κ-Cl$(\{x_i\}) = \{x_i\}$ for each integer i with $1 \leq i \leq n$. Then, using (∗6)(ii) in (I), we have that x_i is even for each i with $1 \leq i \leq n$. **(Sufficiency)** It is obtained by (ii)(a) above. **(iii) (b) (Necessity)** By using (∗12)(b) in (II) and (∗6)(i) in (I) above, (iii)(b) is proved. **(Sufficiency)** It is obtained by (i)(b) above. □

Example 3.6 (i) Especially, for the case where $n = 2$, we have the following forms of κ^2-closures of singletons: for integers $s, t \in \mathbb{Z}$,

κ^2-Cl$(\{(2s + 1, 2t + 1)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2t, 2t + 1, 2t + 2\}$;

κ^2-Cl$(\{(2s, 2t)\}) = \{(2s, 2t)\}$;

κ^2-Cl$(\{(2s, 2t + 1)\}) = \{2s\} \times \{2t, 2t + 1, 2t + 2\}$;

κ^2-Cl$(\{(2s + 1, 2t)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2t\}$.

(ii) By the following figure, the closure κ^2-Cl$(\{(2s+1, 2t+1)\})$ is illustrated; the singleton $\{(2s + 1, 2t + 1)\}$ is denoted by a symbol ∘ and the closure κ^2-Cl$(\{(2s + 1, 2t + 1)\})$ contains the 9-points only denoted by the symbols $\circ, \star, \bullet$:

$$\kappa^2\text{-Cl}(\{(2s+1, 2t+1)\}) = \quad \text{Cl}(\circ) = \begin{array}{cccc} \bullet & \star & \bullet & \text{2t+2} \\ \star & \circ & \star & \text{2t+1} \\ \bullet & \star & \bullet & \text{2t} \\ \text{2s} & \text{2s+1} & \text{2s+2} & \end{array}$$

(iii) By the following figure, the closures κ^2-Cl$(\{(2s, 2t+1)\})$ is illustrated:

$$\kappa^2\text{-Cl}(\{(2s, 2t+1)\}) = \quad \text{Cl}(\star) = \begin{array}{cc} \bullet & \text{2t+2} \\ \star & \text{2t+1} \\ \bullet & \text{2t} \\ \text{2s} & \end{array}$$

(iv) By the following figure, the closure κ^2-Cl$(\{(2s+1, 2t)\})$ is illustrated:

$$\kappa^2\text{-Cl}(\{(2s+1, 2t)\}) = \quad \text{Cl}(\star) = \begin{array}{cccc} \bullet & \star & \bullet & \text{2t} \\ \text{2s} & \text{2s+1} & \text{2s+2} & \end{array}$$

We give the concept of the *smallest open set containing a point* of $(\mathbb{Z}^n, \kappa^n)$.

Definition 3.7 (e.g., [39, p.602], [38, p.47], [11, p.47]) For a point $x := (x_1, x_2, ..., x_n)$ of $(\mathbb{Z}^n, \kappa^n)$, the following subset $U^n(x)$ is called *the smallest open set containing the point* x (cf. Theorem 3.9, Definition 3.3):

$U^n(x) := \prod_{i=1}^n U(x_i)$, where $U(x_i)$ is the smallest open set (cf. $(*4)$ in (I)) in $(\mathbb{Z}, \kappa)$ containing the i-th coordinate x_i of $x (1 \leq i \leq n)$.

Example 3.8 (i) For examples, in the case where $n = 2$ of Definition 3.7, we have the following forms $U^2(x)$ for the following points $x \in \mathbb{Z}^2$:
$U^2((2s+1, 2t+1)) = \{(2s+1, 2t+1)\}$;
$U^2((2s, 2t)) = \{2s-1, 2s, 2s+1\} \times \{2t-1, 2t, 2t+1\}$;
$U^2((2s, 2t+1)) = \{2s-1, 2s, 2s+1\} \times \{2t+1\}$ and
$U^2((2s+1, 2t)) = \{2s+1\} \times \{2t-1, 2t, 2t+1\}$.

(ii) In the figure below, a subset $U^2((2s, 2t))$ is illustrated; the singleton $\{(2s, 2t)\}$ is denoted by a symbol $\bullet$ and $U^2((2s, 2t))$ is the set of the 9-points only denoted by the symbols $\bullet, \circ, \star$:

$$U^2((2s, 2t)) = \quad U^2(\bullet) = \begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \circ & \star & \circ & \cdot & \text{2t+1} \\ \cdot & \star & \bullet & \star & \cdot & \text{2t} \\ \cdot & \circ & \star & \circ & \cdot & \text{2t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 2s-1 & 2s & 2s+1 & & \end{array}$$

(iii) In the figure below, a subset $U^2((2s, 2t+1))$ is illustrated; the singleton $\{(2s, 2t+1)\}$ is denoted by a symbol $\star$ and $U^2((2s, 2t+1))$ is the set of the 3-points only denoted by the symbols $\circ$ and $\star$:

$$U^2((2s, 2t+1)) = \quad U^2(\star) = \begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \circ & \star & \circ & \cdot & \text{2t+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \text{2t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \text{2t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 2s-1 & 2s & 2s+1 & & \end{array}$$

(iv) In the figure below, a subset $U^2((2s+1, 2t))$ is illustrated; the singleton $\{(2s+1, 2t)\}$ is denoted by a symbol $\star$ and $U^2((2s+1, 2t))$ is the set of the 3-points only denoted by the symbols $\circ$ and $\star$:

$$U^2((2s+1,2t))= \quad U^2(\star) = \begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \circ & \cdot & \text{2t+1} \\ \cdot & \cdot & \cdot & \star & \cdot & \text{2t} \\ \cdot & \cdot & \cdot & \circ & \cdot & \text{2t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 2s-1 & 2s & 2s+1 & & \end{array}$$

The following property is folklore, but we give its proof. The following theorem shows the well definedness of $U^n(x)$ of Definition 3.7.

Theorem 3.9 *Let x be a point of $(\mathbb{Z}^n, \kappa^n)$ and $U^n(x)$ the subset defined by Definition 3.7. Then, we have the following properties.*

(i) *$x \in U^n(x)$ and $U^n(x) \in \kappa^n$.*

(ii) *If A is an open set containing the point x in $(\mathbb{Z}^n, \kappa^n)$ such that $A \subset U^n(x)$, then $A = U^n(x)$.*

(iii) *If G is any open set containing the point x in $(\mathbb{Z}^n, \kappa^n)$, then $U^n(x) \subset G$.*

Proof. We put $x := (x_1, x_2, ..., x_n)$. **(i)** By Definition 3.7, (i) is shown.
(ii) Since $x \in A$ and $A \in \kappa^n$, there exist open sets $A_i \in \kappa (1 \leq i \leq n)$ such that $\prod_{i=1}^n A_i \subset A$ and $x_i \in A_i$ for each integer i with $1 \leq i \leq n$. Since A_i is open in $(\mathbb{Z}, \kappa)$ such that $x_i \in A_i$, we have $x_i \in U(x_i) \subset A_i$ for each integer i with $1 \leq i \leq n$ (cf. $(*4)$(iii) in (I)); and so $U^n(x) := \prod_{i=1}^n U(x_i) \subset \prod_{i=1}^n A_i \subset A$. Therefore, we have $U^n(x) \subset A$. By using assumption that $A \subset U^n(x)$, it is shown that $A = U^n(x)$ holds. **(iii)** Since $G \in \kappa^n$ and $U^n(x) \in \kappa^n$, we see $G \cap U^n(x) \in \kappa^n$. Put $A := G \cap U^n(x)$. Then, we have $x \in A, A \in \kappa^n$ and $A \subset U^n(x)$. By (ii) above, it is shown that $A = G \cap U^n(x) = U^n(x)$ holds. Namely, we have $U^n(x) \subset G$. □

Remark 3.10 Using Theorem 3.9, we can investigate topological properties of κ^n-Cl(A), κ^n-Int(A) and κ^n-Ker(A), where A is a subset of $(\mathbb{Z}^n, \kappa^n)$.

• **(Some notation)** In the present paper, we use the following notation (cf. Definition 3.11, $(*20)$ below) for $(\mathbb{Z}^n, \kappa^n)(n \geq 2)$ (they are used in [39], [38], [11] for an integer $n \geq 1$); cf. $(*11)$ in (I) for $n = 1$.

Definition 3.11 ([39, Definition 2.1], [38, Section 2], [11, Section 6])

(i) The following subsets $(\mathbb{Z}^n)_{\kappa^n}, (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $(\mathbb{Z}^n)_{mix(r)}$ of $(\mathbb{Z}^n, \kappa^n)$ are well defined, where $r \in \mathbb{Z}$ with $1 \leq r \leq n$:
(i-1) $(\mathbb{Z}^n)_{\kappa^n} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ x_i$ is odd for each integer i with $1 \leq i \leq n\}$; by Proposition 3.5(i)(b) in (II), it is shown that: $(\mathbb{Z}^n)_{\kappa^n} = \{x \in \mathbb{Z}^n | \ \{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)\}$.
(i-2) $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ x_i$ is even for each integer i with $1 \leq i \leq n\}$; by Proposition 3.5(ii)(a), it is shown that: $(\mathbb{Z}^n)_{\mathcal{F}^n} = \{x \in \mathbb{Z}^n | \ \{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)\}$.
(i-3) $(\mathbb{Z}^n)_{mix(r)} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ \#\{i \in \{1, 2, ..., n\} | \ x_i$ is even$\} = r \ \}$, where $1 \leq r \leq n$ and $\#A$ denotes the cardinality of a set A. Especially, for the case where $r = n$, we note $(\mathbb{Z}^n)_{\mathcal{F}^n} = (\mathbb{Z}^n)_{mix(n)}$ holds.

(ii) For a nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$, the following subsets $E_{\kappa^n}, E_{\mathcal{F}^n}$ and $E_{mix(r)}$ of $(\mathbb{Z}^n, \kappa^n)$ are well defined, where $1 \leq r \leq n$:
(ii-1) $E_{\kappa^n} := E \cap ((\mathbb{Z}^n)_{\kappa^n})$ (cf. (i-1) above);
(ii-2) $E_{\mathcal{F}^n} := E \cap ((\mathbb{Z}^n)_{\mathcal{F}^n})$ (cf. (i-2) above);
(ii-3) $E_{mix(r)} := E \cap ((\mathbb{Z}^n)_{mix(r)})$ (cf. (i-3) above); we note $E_{mix(n)} = E_{\mathcal{F}^n}$.

It is well known that: for any nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$,
· **$(*20)$ (i)** $E_{\kappa^n} = \{x \in E \mid \{x\}$ *is open in* $(\mathbb{Z}^n, \kappa^n)\}$ $=\{(x_1, x_2, ..., x_n) \in E \mid x_i$ *is odd for each* $i \in \mathbb{Z}$ *with* $1 \leq i \leq n\}$.

(ii) $E_{\mathcal{F}^n} = \{x \in E \mid \{x\}$ *is closed in* $(\mathbb{Z}^n, \kappa^n)\}$ $=\{(x_1, x_2, ..., x_n) \in E \mid x_i$ *is even for each* $i \in \mathbb{Z}$ *with* $1 \leq i \leq n\}$.

(iii) *The subset* $(\mathbb{Z}^n)_{\kappa^n}$ *and* E_{κ^n} *are open in* $(\mathbb{Z}^n, \kappa^n)$.

(iv) *We have the following decomposition of* $\mathbb{Z}^n$ *and one of a nonempty set* E*, respectively, as follows* (Note: $n \geq 2$),

· $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} \mid 1 \leq r \leq n-1\})$ (disjoint union);

· $E = E_{\kappa^n} \cup E_{\mathcal{F}^n} \cup (\bigcup\{E_{mix(r)} \mid 1 \leq r \leq n-1\})$ (disjoint union).

(Note: in the above decomposition of $\mathbb{Z}^n$ (resp. E), we should take $(\mathbb{Z}^n)_{mix(r)}$ (resp. $E_{mix(r)}$) with $1 \leq r \leq n-1$.)

(v) *Especially, for* $n = 2$ *and* $r = 1$, $E_{mix(1)} = \{(x_1, x_2) \in E \mid x_1$ is even and x_2 is odd$\}$ $\cup\{(x_1, x_2) \in E \mid x_1$ is odd and x_2 is even$\}$; *we have the following decompositions*:

· $\mathbb{Z}^2 = (\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix(1)}$ (disjoint union) and $E = E_{\kappa^2} \cup E_{\mathcal{F}^2} \cup E_{mix(1)}$ (disjoint union).

(vi) *If* $E \subset F \subset \mathbb{Z}^n$*, then* $E_{\kappa^n} \subset F_{\kappa^n}, E_{\mathcal{F}^n} \subset F_{\mathcal{F}^n}$ *and* $E_{mix(r)} \subset F_{mix(r)} (1 \leq r \leq n-1)$ *hold in* $(\mathbb{Z}^n, \kappa^n)$.

In Section 4, we need the following property Theorem 3.12 (cf. Theorem 4.9, Corollary 4.10 below).

Theorem 3.12 ([39, Lemma 2.3]) *Let* $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(a')}$ *and* $y = (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(a)}$*, where* a' *and* a *are integers such that* $a' \leq a, 1 \leq a' \leq n$ *and* $1 \leq a \leq n$*. Suppose that* $U^n(x) \cap U^n(y)$ *contains exactly the* $2^{a'}$ *open singletons, say* $\{q^{(1)}, q^{(2)}, ..., q^{(2^{a'})}\}$*. Then, the following properties holds.*

(i) $\{q^{(1)}, q^{(2)}, ..., q^{(2^{a'})}\} = (U^n(x))_{\kappa^n} = (U^n(x) \cap U^n(y))_{\kappa^n} \subseteq (U^n(y))_{\kappa^n}$.

(ii) $\{i \mid x_i$ *is even* $(1 \leq i \leq n)\} \subseteq \{i \mid y_i$ *is even* $(1 \leq i \leq n)\}$.

(ii)' *If* $a' = a$ *especially, then* $\{i \mid x_i$ *is even* $(1 \leq i \leq n)\} = \{i \mid y_i$ *is even* $(1 \leq i \leq n)\}$.

(iii) $x \in U^n(y)$ *holds.*

(iii)' *If* $a' = a$ *especially, then* $x = y$.

4 ω-closed sets in Sundaram-Sheik John's sense and Λ_s-sets in $(\mathbb{Z}^n, \kappa^n)$

In the present section, we investigate the concept of ω-closed sets (in Sundaram-Sheik John's sense) in $(\mathbb{Z}^n, \kappa^n)$ and we give a characterization of the ω-closedness in the digital n-spaces (cf. Theorem 4.6). In $(\mathbb{Z}^n, \kappa^n)$, we first give an example of a Λ_s-set, say $B(n)$, where $n \geq 2$, (cf. Definition 2.3, Example 4.2) which is not ω-closed (in Sundaram-Sheik John's sense) (cf. Example 4.2(ii-1)); this example informs us general properties on $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.5). In order to explain the example, we prove the following proposition. We use the notations of Definition 3.11 and (II)($*$20) etc in Section 3, i.e., some notation and well known properties in $(\mathbb{Z}^n, \kappa^n)$.

Proposition 4.1 *Let* V *be an open set of* $(\mathbb{Z}^n, \kappa^n)$.

(i) *If* $n \geq 2$*, then* $V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} \mid 1 \leq r \leq n-1\}) \subset \mathrm{Cl}(V_{\kappa^n})$.

(ii) *If* $n = 1$*, then* $V_{\mathcal{F}^n} \subset \mathrm{Cl}(V_{\kappa^n})$.

Proof. **(i)** Let $y \in V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} \mid 1 \leq r \leq n-1\})$ (cf. Definition 3.11(ii), (II)($*$20) etc in Section 3 above). Since $y \in V$ and V is open in $(\mathbb{Z}^n, \kappa^n)$, there exists the smallest open set $U^n(y)$ (cf. Definition 3.7) containing y such that

($*^1$) $U^n(y) \subset V$ (cf. Theorem 3.9(iii)) and so $(U^n(y))_{\kappa^n} \subset V_{\kappa^n}$ (cf. Definition 3.11(ii)(ii-1), (II)($*$20)(vi) above).

Case 1. $y \in V_{\mathcal{F}^n}$, i.e., $y = (2s_1, 2s_2, ..., 2s_n)$ and $y \in V$, where $s_i \in \mathbb{Z}$ $(1 \leq i \leq n)$ (cf. Definition 3.11(ii)(ii-2)): since $U^n(y) = \prod_{i=1}^{n}\{2s_i - 1, 2s_i, 2s_i + 1\}$ for this point y , we have $\prod_{i=1}^{n}\{2s_i - 1, 2s_i, 2s_i + 1\} \subset V$ (cf. Definition 3.7, Theorem 3.9(iii) and (I)($*$4) in Section 3). We pick a point $p(y) := (2s_1 + 1, 2s_2 + 1, ..., 2s_n + 1) \in (U^n(y))_{\kappa^n}$ and so $p(y) \in V_{\kappa^n}$ (cf. Proposition 3.5(iii)(b)). Then, since $\mathrm{Cl}(\{p(y)\}) = \prod_{i=1}^{n}\{2s_i, 2s_i + 1, 2s_i + 2\}$ (cf. Proposition 3.5(i)(a)), we have $y = (2s_1, 2s_2, ..., 2s_n) \in \mathrm{Cl}(\{p(y)\}) \subset \mathrm{Cl}(V_{\kappa^n})$. It is proved that $V_{\mathcal{F}^n} \subset \mathrm{Cl}(V_{\kappa^n})$. We note that the above proof is done for the case where $n \geq 1$ (cf. (I)($*$1), ($*$4), ($*$11)(v) in Section3).

Case 2. $y \in V_{mix(r)}$, where $1 \leq r \leq n-1$ $(n \geq 2)$ (cf. Definition 3.11(ii)(ii-3)): for this point y, we set $y = (y_1, y_2, ..., y_n)$; then by definition, $r = \#\{i \mid y_i$ is an even integer $(1 \leq i \leq n)\}$. We put $I_r := \{i \mid y_i \text{ is even }\} = \{e(1), e(2), ..., e(r)\}$
$(e(1) < e(2) < ... < e(r))$ and $J_{n-r} := \{j \mid y_j \text{ is odd }\} = \{o(1), o(2), ..., o(n-r)\}$ $(o(1) < o(2) < ... < o(n-r))$; then $\{1, 2, ..., n\} = I_r \cup J_{n-r}$ (disjoint union). For the present case, we claim that $y \in Cl(V_{\kappa^n})$. Indeed, we recall that:
($*^2$) $U^n(y) = \prod_{i=1}^{n} U(y_i)$, where $U(y_e) := \{y_e - 1, y_e, y_e + 1\}$ if $e \in I_r$; and $U(y_o) := \{y_o\}$ if $o \in J_{n-r}$ (cf. (I)($*$4) in Section 3, Definition 3.7).
For this point $y \in V_{mix(r)}$ ($1 \leq r \leq n-1$ and $n \geq 2$), we pick a point $p(y) \in U^n(y)$ such that $p(y) \in (U^n(y))_{\kappa^n}$ as follows:
($*^3$) let $p(y) := (p_1, p_2, ..., p_n)$, where $p_e := y_e - 1$ if $e \in I_r$; $p_o := y_o$ if $o \in J_{n-r}$.
Then by ($*^2$) and ($*^3$) above, it is shown that the components of the point $p(y)$ are odd and so ($*^4$) $p(y) \in (U^n(y))_{\kappa^n}$, because the components have the forms of $y_e - 1 \in U(y_e)$ or $y_o \in U(y_o)$.
Thus, using ($*^1$), ($*^4$) above and (II)($*$20)(vi) above, we see that $p(y) \in V_{\kappa^n}$; and so
($*^5$) $\mathrm{Cl}(\{p(y)\}) \subset \mathrm{Cl}(V_{\kappa^n})$.
We note that : $\mathrm{Cl}(\{p(y)\}) = \mathrm{Cl}(\{(p_1, p_2, ..., p_n)\}) = \prod_{i=1}^{n}\mathrm{Cl}(\{p_i\})$ in $(\mathbb{Z}^n, \kappa^n)$, where $\mathrm{Cl}(\{p_e\}) = \{p_e - 1, p_e, p_e + 1\} = \{y_e - 2, y_e - 1, y_e\}$ if $e \in I_r$; and $\mathrm{Cl}(\{p_o\}) = \{p_o - 1, p_o, p_o + 1\} = \{y_o - 1, y_o, y_o + 1\}$ if $o \in J_{n-r}$ (cf. Proposition 3.5). Thus, we have $y = (y_1, y_2, ..., y_n) \in \mathrm{Cl}(\{p(y)\})$. Moreover, using ($*^5$) above, we conclude that $y \in \mathrm{Cl}(V_{\kappa^n})$ for a point $y \in V_{mix(r)}$. Namely, it is proved that $V_{mix(r)} \subset \mathrm{Cl}(V_{\kappa^n})$ for each r with $1 \leq r \leq n-1$ $(n \geq 2)$.

Therefore we have the required inclusion: $V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} |\ 1 \leq r \leq n-1\}) \subset \mathrm{Cl}(V_{\kappa^n})$
.

(ii) For the case where $n = 1$, we may consider the case 1 only of the proof of (i) above; the proof is omitted (cf. (I)($*$1), ($*$4), ($*$11)(v) in Section3). □

Example 4.2 Throughout the present example, let $B(n) := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup \{x(1), x(2), ... , x(s)\}$ be an infinite subset of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$ and s is a positive integer, $\{x(j)\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$ for each integer j with $1 \leq j \leq s$. We have the following properties on the subset $B(n)$: namely,

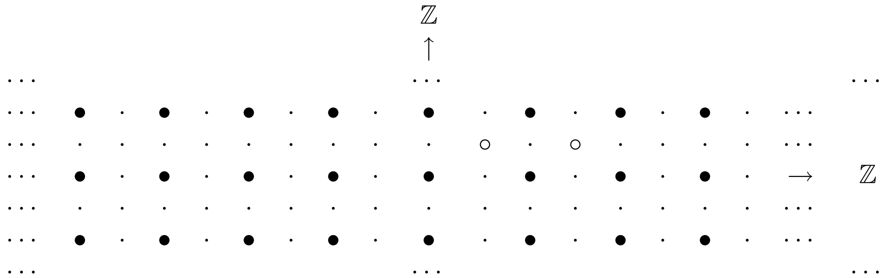
(i) *$B(n)$ is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$ for each $n \geq 1$* (cf. Proof of (i) below and Definition 2.3).

(ii) (ii-1) *If $n \geq 2$, then $B(n)$ is not an ω-closed set (in Sundaram-Sheik John's sense) of $(\mathbb{Z}^n, \kappa^n)$* (cf. Proof of (ii-1) below and Definition 2.1);

(ii-2) *For $n = 1$, $B(n)$ is a closed set of $(\mathbb{Z}, \kappa)$ and so it is an ω-closed set (in Sundaram-Sheik John's sense) in $(\mathbb{Z}, \kappa)$* (cf. Proof of (ii-2) below and Definition 2.1).

(iii) *Let A be a subset of $(\mathbb{Z}^n, \kappa^n)$ such that $B(n) \subset A \subset Cl(B(n))$. Then, A is not semi-open in $(\mathbb{Z}^n, \kappa^n)$.*

For the case where $n = 2$, the following figure illustrates the subset $B = (\mathbb{Z}^2)_{\mathcal{F}^2} \cup \{x(1), x(2)\}$ in $(\mathbb{Z}^2, \kappa^2)$; each symbol $\bullet$ means a point in $(\mathbb{Z}^2)_{\mathcal{F}^2}$ and two symbols $\circ$ mean $x(1) = (1, 1)$ and $x(2) = (3, 1)$ respectively.



In order to prove (i) above, we need the following property $(**)$:

$(**)$ *Suppose* $n \geq 1$. *Let* $F_1(n) := B(n) \cup E_1(n)$ *and* $F_2(n) := B(n) \cup E_2(n)$, *where* $E_1(n) = \{(s_1, s_2, ..., s_n) \in \mathbb{Z}^n \mid s_i \equiv 1 \ mod\ 4\ (1 \leq i \leq n)\}$ *and* $E_2(n) := \{(s_1, s_2, ..., s_n) \in \mathbb{Z}^n \mid s_j \equiv 3\ mod\ 4\ (1 \leq j \leq n)\}$. *Then,* $E_1(n) \cap E_2(n) = \emptyset$ *holds and* $F_1(n)$ *and* $F_2(n)$ *are semi-open sets including* $B(n)$ *such that* $F_1(n) \cap F_2(n) = B(n)$.

Proof of $(**)$. We first recall the following expressions of $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(x_1, x_2, ..., x_n) | \ x_i$ is even $(1 \leq i \leq n)\}$ as follows:
$(*_1)$ $(\mathbb{Z}^n)_{\mathcal{F}^n} = \bigcup\{\prod_{i=1}^n \{x_i\} | \ x_i$ is even $(1 \leq i \leq n)\} = \bigcup\{\prod_{i=1}^n \{s_i - 1, s_i + 1\} | (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\}$; and
$(*_1)'$ $(\mathbb{Z}^n)_{\mathcal{F}^n} = \bigcup\{\prod_{i=1}^n \{s_i - 1, s_i + 1\} | (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 3 \text{ mod } 4\ (1 \leq i \leq n)\}$.

We secondly claim that
$(*_2)$ $\mathrm{Cl}(E_i(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_i(n)$ for each $i \in \{1, 2\}$.
Indeed, we have $\mathrm{Cl}(E_1(n)) = \mathrm{Cl}(\bigcup\{\prod_{i=1}^n \{s_i\} | \ (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\}) \supset \bigcup\{\mathrm{Cl}(\prod_{i=1}^n \{s_i\}) | \ (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\} = \bigcup\{\prod_{i=1}^n \mathrm{Cl}(\{s_i\}) | \ (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\} = \bigcup\{\prod_{i=1}^n \{s_i - 1, s_i, s_i + 1\} | \ (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\} \supset \bigcup\{\prod_{i=1}^n \{s_i - 1, s_i + 1\} | \ (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \text{ mod } 4\ (1 \leq i \leq n)\} = (\mathbb{Z}^n)_{\mathcal{F}^n}$ (cf. $(*_1)$ above, (I)$(*5)$(i) in Section 3) and $\mathrm{Cl}(E_1(n)) \supset E_1(n)$. Hence, we have $\mathrm{Cl}(E_1(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_1(n)$. In the same way, using $(*_1)'$ in place of $(*_1)$, we have $\mathrm{Cl}(E_2(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_2(n)$. Moreover, we claim that
$(*_3)$ $F_i(n)$ is semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each $i \in \{1, 2\}$.
Indeed, by using $(*_2)$ and definitions, it is shown that, for each $i \in \{1, 2\}$, $\mathrm{Cl}\,(\mathrm{Int}(F_i(n))) \supset \mathrm{Cl}\,(\mathrm{Int}((B(n))_{\kappa^n} \cup E_i(n))) = \mathrm{Cl}((B(n))_{\kappa^n} \cup E_i(n)) \supset (B(n))_{\kappa^n} \cup Cl(E_i(n)) \supset \{x(1), x(2), ..., x(s)\} \cup ((\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_i(n)) = B(n) \cup E_i(n) = F_i(n)$. Namely, $F_i(n)$ is semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each $i \in \{1, 2\}$.

Finally, $(*_4)$ $F_1(n) \cap F_2(n) = B(n) \cup (E_1(n) \cap E_2(n)) = B(n)$ hold, because $E_1(n) \cap E_2(n) = \emptyset$. $(\circ)$

Proof of (i). We first claim that $\mathrm{sKer}(B(n)) \subset B(n)$. Indeed, we recall $(**)$ above and so $F_1(n)$ and $F_2(n)$ are semi-open sets in $(\mathbb{Z}^n, \kappa^n)(n \geq 1)$ such that $B(n) \subset F_i(n)$ for each $i \in \{1, 2\}$. Thus, by definitions, it is shown that $\mathrm{sKer}(B(n)) \subset F_1(n) \cap F_2(n)$ (cf. Definition 2.2(i)); and so $\mathrm{sKer}(B(n)) \subset B(n)$, because $F_1(n) \cap F_2(n) = B(n)$ (cf. $(**)$ above). This concludes that $\mathrm{sKer}(B(n)) = B(n)$, because $B(n) \subset sKer(B(n))$ holds. Namely, $B(n)$ is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$.

Proof of (ii)(ii-1). Suppose $n \geq 2$. We first show that:
$(*_5)$ $(\mathrm{Cl}(B(n)))_{mix(r)} \neq \emptyset$, for each integer r with $1 \leq r \leq n-1$. Indeed, since $\mathrm{Cl}(B(n)) = \mathrm{Cl}((\mathbb{Z}^n)_{\mathcal{F}^n}) \cup (\bigcup\{(\mathrm{Cl}(\{x(i)\})) | 1 \leq i \leq s\})$, it is shown that $(\mathrm{Cl}(B(n)))_{mix(r)} \supset (\mathrm{Cl}(\{x(1)\}))_{mix(r)}$ (cf. (II)$(*20)$ in Section 3). We can put $x(1) := (t_1, t_2, ..., t_n)$, where t_j is odd for each j with $1 \leq j \leq n$, because $x(1) \in (\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)). Then, we show $\mathrm{Cl}(\{x(1)\}) = \prod_{j=1}^n \mathrm{Cl}(\{t_j\}) = \prod_{j=1}^n \{t_j - 1, t_j, t_j + 1\}$ (cf. Proposition 3.5(i)(a)) and so
$(\mathrm{Cl}(\{x(1)\}))_{mix(r)} \neq \emptyset$ for each integer r with $1 \leq r \leq n-1$, because we can take a point

$p := (p_1, p_2, ..., p_n)$, where $p_j := t_j - 1$ is even for each j with $1 \leq j \leq r$ and $p_j := t_j$ is odd for each j with $r+1 \leq j \leq n$; and hence $p \in (\mathrm{Cl}(\{x(1)\}))_{mix(r)}$ (cf. Definition 3.11(i)(i-3)) and so $p \in (\mathrm{Cl}(B(n)))_{mix(r)}$ (cf. (II)(∗20) in Section 3). Thus, we prove the property $(*_5)$.

We secondly have the following property: $(*_6)$ $\mathrm{Cl}(B(n)) \not\subset F_1(n)$ holds. Indeed, for a contradiction, we suppose $\mathrm{Cl}(B(n)) \subset F_1(n)$; then $(\mathrm{Cl}(B(n)))_{mix(r)} \subset (F_1(n))_{mix(r)}$ and so $(\mathrm{Cl}(B(n)))_{mix(r)} = \emptyset$ because of $(F_1(n))_{mix(r)} = \emptyset$ for each integer r with $1 \leq r \leq n-1$. This contradicts $(*_5)$ above.

For a contradiction, we finally suppose that $B(n)$ is ω-closed in Sundaram-Sheik John's sense, i.e., $\mathrm{Cl}(B(n)) \subset \mathrm{sKer}(B(n))$ (cf. Theorem 2.5). Then, using $(**)$ above, we have $\mathrm{sKer}(B(n)) \subset F_1(n)$ and so $\mathrm{Cl}(B(n)) \subset F_1(n)$; this contradicts $(*_6)$ above. Therefore, $B(n)$ is not ω-closed (in Sundaram-Sheik John's sense) in $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 2$.

Proof of (ii)(ii-2) Suppose $n = 1$. First, it is shown that $B(n) = B(1)$ is closed in $\mathbb{Z}^n$, where $n = 1$. Indeed, we have $\mathbb{Z} \setminus B(1) = \mathbb{Z}_\kappa \setminus \{x(j) | 1 \leq j \leq s\}$ and so $\mathbb{Z} \setminus B(1) = \bigcup\{\{z\} | z \in \mathbb{Z}_\kappa \text{ and } z \notin \{x(j)|1 \leq j \leq s\}\}$, i.e., $\mathbb{Z} \setminus B(1)$ is the union of some open singletons $\{z\}$, and hence $\mathbb{Z} \setminus B(1) \in \kappa$ (cf. Definition 3.1). Thus, the set $B(1)$ is closed and so it is ω-closed in Sundaram-Sheik John's sense.

Proof of (iii). For a contradiction, we suppose that A is semi-open in $(\mathbb{Z}^n, \kappa^n)$. Then, there exists an open set V such that $V \subset A \subset \mathrm{Cl}(V)$ and so $V \subset \mathrm{Cl}(B(n))$. First we claim that: $(*_7)$ $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n})$ holds for each $n \geq 1$.

Proof of $(*_7)$. **Case (I)**. $n \geq 2$: for this case, we have $V = V_{\kappa^n} \cup V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)}|\ 1 \leq r \leq n-1\})$ (cf. (II)(∗20)(iv) in Section 3). Since V is open, by Proposition 4.1(i), it is shown that $\mathrm{Cl}(V) = \mathrm{Cl}(V_{\kappa^n}) \cup \mathrm{Cl}(V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)}|\ 1 \leq r \leq n-1\})) \subset \mathrm{Cl}(V_{\kappa^n}) \cup \mathrm{Cl}(\mathrm{Cl}(V_{\kappa^n})) = \mathrm{Cl}(V_{\kappa^n})$; and so $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n})$.

Case (II). $n = 1$: for this case, we have $V = V_\kappa \cup V_{\mathcal{F}}$ (cf. (I)(∗11)(iii) in Section 3). Since V is open, by Proposition 4.1(ii), it is shown that

$\mathrm{Cl}(V) = \mathrm{Cl}(V_\kappa) \cup \mathrm{Cl}(V_{\mathcal{F}}) \subset \mathrm{Cl}(V_\kappa) \cup \mathrm{Cl}(\mathrm{Cl}(V_\kappa)) = \mathrm{Cl}(V_\kappa)$; and so $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_\kappa)$. $(\circ)$

We proceed the proof of (iii). We put $V_{\kappa^n} := \{p(k) \in V|\ \{p(k)\} \in \kappa^n, k \in \nu\}$, where $\nu \subset \mathbb{Z}$ is an index set (cf. Definition 3.11(i)(i-1)). Since $p(k) \in V_{\kappa^n} \subset V \subset \mathrm{Cl}(B(n))$ and so $p(k) \in \mathrm{Cl}(B(n))$, it is shown that $\{p(k)\} \cap B(n) \neq \emptyset$ and so $p(k) \in B(n)$ for each $k \in \nu$. Namely, we have:

$(*_8)$ $V_{\kappa^n} \subset (B(n))_{\kappa^n}$ (cf. Definition 3.11(i)(i-1),(ii)(ii-1) and (I)(∗11)(v), (II)(∗20)(vi)). Then, using $(*_7)$ and $(*_8)$ above, we conclude that $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n}) \subset \mathrm{Cl}((B(n))_{\kappa^n}) = \mathrm{Cl}(\{x(1), x(2), ..., x(s)\}) = \bigcup\{\mathrm{Cl}(\{x(j)\}) | 1 \leq j \leq s\}$; and hence $\mathrm{Cl}(V)$ is a finite subset of $(\mathbb{Z}^n, \kappa^n)$, because $\mathrm{Cl}(\{y\})$ is a finite subset of $\mathbb{Z}$ for every point $y \in \mathbb{Z}$ (cf. (I)(∗5)(i) in Section 3) and so $\mathrm{Cl}(\{x(j)\})$ is a finite subset of $\mathbb{Z}^n$ for each j with $1 \leq j \leq s$ (cf. (II)(∗12)(a) in Section 3). Therefore, we have A is a finite subset of $(\mathbb{Z}^n, \kappa^n)$, because of $V \subset A \subset \mathrm{Cl}(V)$; and so $B(n)$ is also finite, because of $B(n) \subset A$; this contradicts the definition of the set $B(n)$ (i.e., $B(n)$ is not finite). Therefore, A is not semi-open in $(\mathbb{Z}, \kappa)$.

In order to state Theorem 4.4, we need the following definition on $I_r(x)$ and $J_{n-r}(x)$, where $x \in \mathbb{Z}^n$.

Definition 4.3 (cf. Definition 3.11(i)(i-3),(II)(∗20)(iv) in Section 3; [39, Definiton 2.1(ii)]) Let $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$, where $n \geq 2$ and r is the cardinality of a set $\{k|\ x_k$ is even$\}$ with $1 \leq r \leq n-1$ (cf. Definition 3.11(i-3),(II)(∗20)(iv) in Section 3; in the present definition, we note the assumption that $1 \leq r \leq n-1$ and $n \geq 2$; and so $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset$). Let $x_{e(1)}, x_{e(2)}, ..., x_{e(r)}$ be all the components of x which are even; and $x_{o(1)}, x_{o(2)}, ..., x_{o(n-r)}$ be all the components of x which are odd, where $e(k)(1 \leq k \leq r)$ and $o(j)(1 \leq j \leq n-r)$ are positive integers with $1 \leq e(1) < e(2) < ... < e(r) \leq n$ and $1 \leq o(1) < o(2) < ... < o(n-r) \leq n$. Then, for this point $x = (x_1, x_2, ..., x_n)$, we define the following subsets $I_r(x)$ and $J_{n-r}(x)$ of $\{1, 2, ..., n\}$ as follows:

• $I_r(x) := \{k|\ x_k$ is even$\}$; and so $I_r(x) = \{e(1), e(2), ..., e(r)\}$ holds;
• $J_{n-r}(x) := \{j|\ x_j$ is odd$\}$; and so
$J_{n-r}(x) = \{o(1), o(2), ..., o(n-r)\}$, $\{1, 2, ..., n\} = I_r(x) \cup J_{n-r}(x)$ $(I_r(x) \cap J_{n-r}(x) = \emptyset)$, $I_r(x) \neq \emptyset$ and $J_{n-r}(x) \neq \emptyset$ hold, where $n \geq 2$ and $1 \leq r \leq n-1$.

We construct some semi-open sets containing a point of $(\mathbb{Z}^n, \kappa^n)$ where $n \geq 1$.

Theorem 4.4 *Let* $x := (x_1, x_2, ..., x_n) \in \mathbb{Z}^n$.

(i) *Suppose* $n \geq 1$. *If* $x \in (\mathbb{Z}^n)_{\kappa^n}$,*i.e., all the components* $x_1, x_2, ..., x_n$ *of the point* x *are odd* (cf. Definition 3.11(i)(i-1)), *then* $\{x\}$ *is a semi-open set containing* x *in* $(\mathbb{Z}^n, \kappa^n)$.

(ii) *Suppose* $n \geq 1$ *and* $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, *i.e., all the components* $x_1, x_2, ..., x_n$ *of the point* x *are even* (cf. Definition 3.11(i)(i-2)). *Then, we have the following properties.*

(ii-1) *We set* $A(x) := \{(x_1 + i_1, x_2 + i_2, ..., x_n + i_n) \in \mathbb{Z}^n | i_k \in \{+1, -1\} (1 \leq k \leq n)\}$ *for the point* $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{\mathcal{F}^n}$. *Then,* $\#A(x) = 2^n$ *holds. And, for each point of* $A(x)$, *say* $p(x, u) (1 \leq u \leq 2^n)$, *the singleton* $\{p(x, u)\}$ *is open in* $(\mathbb{Z}^n, \kappa^n)$.

(ii-2) *In* $(\mathbb{Z}^n, \kappa^n)$, $\{p(x, u)|1 \leq u \leq 2^n\} = (U^n(x))_{\kappa^n}$ *holds, where* $U^n(x)$ *is the smallest open set* (cf. Definition 3.7,Theorem 3.9) *containing the point* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$.

(ii-3) *The subset* $\{x\} \cup \{p(x, u)\}$ *is a semi-open set containing the point* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ *for each* u *with* $1 \leq u \leq 2^n$.

(iii) *Suppose* $n \geq 2$ *and* $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$ *where* $1 \leq r \leq n-1$ (cf. Definition 3.11(i)(i-3),(II)($*$20)(iv) in Section 3). *Let* $I_r(x) = \{e(1), e(2), ..., e(r)\}$ *and* $J_{n-r}(x) = \{o(1), o(2), ..., o(n-r)\}$ (cf. Definition 4.3). *Then, we have the following properties.*

(iii-1) *We set* $B(x) := \{(z_1, z_2, ..., z_n) \in \mathbb{Z}^n|\ z_{e(k)} \in \{x_{e(k)} - 1, x_{e(k)} + 1\}\ (1 \leq k \leq r), z_{o(j)} = x_{o(j)}\ (1 \leq j \leq n-r)\}$ *for the point* $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$. *Then,* $\#B(x) = 2^r$. *And, for each point of* $B(x)$, *say* $p(x, u) (1 \leq u \leq 2^r)$, *the singleton* $\{p(x, u)\}$ *is open in* $(\mathbb{Z}^n, \kappa^n)$.

(iii-2) *In* $(\mathbb{Z}^n, \kappa^n)$, $\{p(x, u)|1 \leq u \leq 2^r\} = (U^n(x))_{\kappa^n}$ *holds, where* $U^n(x)$ *is the smallest open set containing the point* $x \in (\mathbb{Z}^n)_{mix(r)}$.

(iii-3) *The subset* $\{x\} \cup \{p(x, u)\}$ *is a semi-open set containing the point* $x \in (\mathbb{Z}^n)_{mix(r)}$ *for each* u *with* $1 \leq u \leq 2^r$.

Proof. **(i)** For the point $x \in (\mathbb{Z}^n)_{\kappa^n}$, the singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Proposition 3.5(iii)(b)) ; and so it is semi-open.

(ii) (ii-1) Obviously, the cardinality of $A(x)$ is 2^n. The point $p(x, u)$, where $1 \leq u \leq 2^n$, is expressible as $p(x, u) = (x_1 + i_1, x_2 + i_2, ..., x_n + i_n)$ for some integers $i_k \in \{+1, -1\} (1 \leq k \leq n)$ and so all the components of $p(x, u)$ are odd, because all the components $x_1, x_2, ..., x_n$ are even. Thus, $\{p(x, u)\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Proposition 3.5(iii)(b)).

(ii-2) For the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, we set $x = (2s_1, 2s_2, ..., 2s_n)$ for some integers $s_i (1 \leq i \leq n)$. Then, $U^n(x) = \prod_{i=1}^n U(2s_i) = \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}$ is the smallest open set containing x (cf. Definition 3.7 and (I)($*$4)(i) in Section 3). Since $(U^n(x))_{\kappa^n} = \{z \in U^n(x)|\{z\}$ is open in $(\mathbb{Z}^n, \kappa^n)\} = \{(z_1, z_2, ..., z_n) \in \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}| z_1, z_2, ..., z_n$ are odd $\}$, we have $(U^n(x))_{\kappa^n} = \{(2s_1 + i_1, 2s_2 + i_2, ..., 2s_n + i_n) \in \mathbb{Z}^n | i_k \in \{+1, -1\} (1 \leq k \leq n)\} = A(x)$; and so we have $(U^n(x))_{\kappa^n} = \{p(x, u)|1 \leq u \leq 2^n\}$ (cf. Definition 3.11(i)(i-1),(ii)(ii-1) and (ii-1) above).

(ii-3) We first claim that $x \in$Cl$(\{p(x, u)\})$ for each u with $1 \leq u \leq 2^n$. Indeed, we have Cl$(\{p(x, u)\}) = \prod_{k=1}^n$ Cl$(\{x_k + i_k\}) = \prod_{k=1}^n \{x_k + i_k - 1, x_k + i_k, x_k + i_k + 1\}$ (cf. (II)($*$12)(a) in Section 3, Proposition 3.5(i)(a)); and so $x = (x_1, x_2, ..., x_n) \in \prod_{k=1}^n$ Cl$(\{x_k + i_k\}) =$ Cl$(\{p(x, u)\})$. Thus, we show that $\{x\} \cup \{p(x, u)\} \subset$Cl$(\{p(x, u)\}) =$Cl(Int$(\{p(x, u)\})) \subset$Cl(Int

$(\{x\}\cup\{p(x,u)\}))$ (cf. (ii-1) above), i.e., $\{x\}\cup\{p(x,u)\}\subset \mathrm{Cl}(\mathrm{Int}(\{x\}\cup\{p(x,u)\}))$. Namely, $\{x\}\cup\{p(x,u)\}$ is semi-open in $(\mathbb{Z}^n,\kappa^n)$ for each u with $1\le u\le 2^n$.

(iii) (iii-1) By the definition of $B(x)$, it is obviously shown that $\#B(x)=2^r$. A point $p(x,u)$ of $B(x)$ is expressible as $p(x,u)=(z(u)_1,z(u)_2,...,z(u)_n)$, where $z(u)_{e(k)}\in\{x_{e(k)}-1,x_{e(k)}+1\}$ $(1\le k\le r)$ and $z(u)_{o(j)}=x_{o(j)}$ $(1\le j\le n-r)$. We recall that the r components $x_{e(1)},x_{e(2)},...,x_{e(r)}$ are all even and the $n-r$ components $x_{o(1)},x_{o(2)},...,x_{o(n-r)}$ are all odd, because we assume that $x=(x_1,x_2,...,x_n)\in(\mathbb{Z}^n)_{mix(r)}$ where $1\le r\le n-1(n\ge 2)$ and $I_r(x):=\{k|\ x_k$ is even$\}$=$\{e(1),e(2),...,e(r)\}$ $(e(1)<e(2)<...<e(r))$; and
$J_{n-r}(x):=\{j|x_j$ is odd $\}=\{o(1),o(2),...,o(n-r)\}$ $(o(1)<o(2)<...<o(n-r))$ (cf. Definition 3.11(i)(i-3),(II)($*$20)(iv) in Section 3 and Definition 4.3 above). Then, since the integers $x_{e(k)}-1,x_{e(k)}+1$ and $x_{o(j)}$ are odd, all the components $z(u)_1,z(u)_2,...,z(u)_n$ are odd for each u with $1\le u\le 2^r$. We have that the singleton $\{p(x,u)\}=\{(z(u)_1,z(u)_2,...,z(u)_n)\}$ is open in $(\mathbb{Z}^n,\kappa^n)$ (cf. Proposition 3.5(iii)(b)).

(iii-2) We recall that, for this point $x\in(\mathbb{Z}^n)_{mix(r)}$, $U^n(x)=\prod_{i=1}^n U(x_i)$, where $U(x_{e(k)})=\{x_{e(k)}-1,x_{e(k)},x_{e(k)}+1\}(1\le k\le r)$ and $U(x_{o(j)})=\{x_{o(j)}\}(1\le j\le n-r)$ (cf. Definition 4.3,Definition 3.7,(I)($*$4)(i)(ii) in Section 3). Thus, we have that $(z_1,z_2,...,z_n)\in(U^n(x))_{\kappa^n}$ if and only if $z_{e(k)}\in\{x_{e(k)}-1,x_{e(k)}+1\}$ and $z_{o(j)}=x_{o(j)}$ for integers $k,\ j$ with $1\le k\le r$ and $1\le j\le n-r$ (cf. Proposition 3.5(iii)(b), Definition 4.3). Namely, we have $(U^n(x))_{\kappa^n}=B(x)$ for the point $x\in(\mathbb{Z}^n)_{mix(r)}$ and so $(U^n(x))_{\kappa^n}=\{p(x,u)|1\le u\le 2^r\}$ (cf. (iii-1) above).

(iii-3) We first claim that $(*)$ $\{x\}\cup\{p(x,u)\}\subset$Cl$(\{p(x,u)\})$ holds in $(\mathbb{Z}^n,\kappa^n)$ for each u with $1\le u\le 2^r$. Indeed, for the point $p(x,u)$, we set $p(x,u):=(z(u)_1,z(u)_2,...,z(u)_n)$ (cf. (iii-1) above). Then, for each positive integers k,j with $1\le k\le r$ and $1\le j\le n-r$, it is shown that: in $(\mathbb{Z},\kappa)$,
if $z(u)_{e(k)}=x_{e(k)}-1$, then $\mathrm{Cl}(\{z(u)_{e(k)}\})=\{x_{e(k)}-2,x_{e(k)}-1,x_{e(k)}\}$ holds;
if $z(u)_{e(k)}=x_{e(k)}+1$, then $\mathrm{Cl}(\{z(u)_{e(k)}\})=\{x_{e(k)},x_{e(k)}+1,x_{e(k)}+2\}$ holds;
if $z(u)_{o(j)}=x_{o(j)}$, then $\mathrm{Cl}(\{z(u)_{o(j)}\})=\{x_{o(j)}-1,x_{o(j)},x_{o(j)}+1\}$ holds, (cf. (I)($*$5)(i) in Section 3). Thus, we show that $x_{e(k)}\in$Cl$(\{z(u)_{e(k)}\})$ and $x_{o(j)}\in$Cl$(\{z(u)_{o(j)}\})$ $(1\le k\le r$ and $1\le j\le n-r)$; and so $\{x\}\subset\prod_{i=1}^n\mathrm{Cl}(\{z(u)_i\})$ holds in $(\mathbb{Z}^n,\kappa^n)$. Since $\mathrm{Cl}(\{p(x,u)\})=\prod_{i=1}^n\mathrm{Cl}(\{z(u)_i\})$ in $(\mathbb{Z}^n,\kappa^n)$ (cf. (II)($*$12) in Section 3), we show that $\{x\}\subset$Cl$(\{p(x,u)\})$ and $\{x\}\cup\{p(x,u)\}\subset$
Cl$(\{p(x,u)\})$ hold in $(\mathbb{Z}^n,\kappa^n)$.

We finally finish the proof of (iii-3): there exists an open set $\{p(x,u)\}$ such that $\{p(x,u)\}\subset\{x\}\cup\{p(x,u)\}\subset$Cl$(\{p(x,u)\})$, i.e., $\{x\}\cup\{p(x,u)\}$ is a semi-open in $(\mathbb{Z}^n,\kappa^n)$ for each u with $1\le u\le 2^r$. □

Theorem 4.5 *For the digital n-space $(\mathbb{Z}^n,\kappa^n)$ where $n\ge 1$, we have the following properties.*

(i) *For any point x of $(\mathbb{Z}^n,\kappa^n)$, sKer$(\{x\})=\{x\}$.*

(ii) *For any subset E of $(\mathbb{Z}^n,\kappa^n)$, sKer$(E)=E$.*

Proof. **(i)** We first note that: for the case where $n=1$,
$\cdot$ $\mathbb{Z}^n=(\mathbb{Z}^n)_{\kappa^n}\cup(\mathbb{Z}^n)_{\mathcal{F}^n}$ (disjoint union) holds, where $n=1$ (cf. (I)($*$11)(iii) in Section 3); for the case where $n\ge 2$,
$\cdot$ $\mathbb{Z}^n=(\mathbb{Z}^n)_{\kappa^n}\cup(\mathbb{Z}^n)_{\mathcal{F}^n}\cup(\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1\le r\le n-1\})$ (disjoint union) and $(\mathbb{Z}^n)_{mix(r)}\ne\emptyset(1\le r\le n-1)$ hold, where $n\ge 2$ (cf. Definition 3.11, (II)($*$20)(iv) in Section 3).

Let $x\in\mathbb{Z}^n$. It is enough to consider the following three cases for the point $x\in\mathbb{Z}^n$.

Case 1. $x\in(\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)): since $\{x\}$ is open in $(\mathbb{Z}^n,\kappa^n)$, it is semi-open. Then, it is obvious that sKer$(\{x\})=\{x\}$ in $(\mathbb{Z}^n,\kappa^n)$ (cf. Definition 2.2(i)). We note this result is true for the case where $n\ge 1$.

Case 2. $x\in(\mathbb{Z}^n)_{\mathcal{F}^n}$ (cf. Definition 3.11(i)(i-2)): we put $x=(2s_1,2s_2,...,2s_n)$ where

$s_i \in \mathbb{Z}$ $(1 \leq i \leq n)$. Note that, for the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, $U^n(x) := \prod_{i=1}^{n}\{2s_i-1, 2s_i, 2s_i+1\}$ is the smallest open set containing x (cf. Definition 3.7,(I)(∗4)(i) in Section 3,Theorem 3.9). Then, by Theorem 4.4(ii), there exist 2^n semi-open sets $\{x\} \cup \{p(x,u)\}(1 \leq u \leq 2^n)$ containing the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ such that $\{p(x,u)|1 \leq u \leq 2^n\} = (U^n(x))_{\kappa^n} =\{(2s_1+i_1, 2s_2+i_2, ..., 2s_n+i_n)|i_k \in \{+1,-1\}(1 \leq k \leq n)\}$ and $\#((U^n(x))_{\kappa^n}) = 2^n$. Thus, we have $\mathrm{sKer}(\{x\}) \subset \bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^n\}$; moreover, $\bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^n\} = \{x\}$, because $\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^n\} = \emptyset$. We conclude that $\mathrm{sKer}(\{x\}) = \{x\}$ holds for this case. We note the result above is true for the case where $n \geq 1$.

Case 3. $x \in (\mathbb{Z}^n)_{mix(r)}$ where $1 \leq r \leq n-1 (n \geq 2)$ (cf. Definition 3.11(i)(i-3)): for this point x, we set $x = (x_1, x_2, ..., x_n)$; then by definition, $r = \#\{i|\ x_i$ is an even integer $(1 \leq i \leq n)\}$. We recall the following subsets $I_r(x)$ and $J_{n-r}(x)$ as follows (cf. Definition 4.3 above):

$I_r(x) := \{k|\ x_k$ is even$\}$=$\{e(1), e(2), ..., e(r)\}$ $(e(1) < e(2) < ... < e(r))$; and
$J_{n-r}(x) := \{j|x_j$ is odd $\}$= $\{o(1), o(2), ..., o(n-r)\}$ $(o(1) < o(2) < ... < o(n-r))$; and
$\{1, 2, ..., n\} = I_r(x) \cup J_{n-r}(x)$ (disjoint union), $I_r(x) \neq \emptyset, J_{n-r}(x) \neq \emptyset$.

For the point $x \in (\mathbb{Z}^n)_{mix(r)}, U^n(x) = \prod_{i=1}^{n} U(x_i)$ is the smallest open set containing x, where $U(x_{e(k)}) = \{x_{e(k)}-1, x_{e(k)}, x_{e(k)}+1\}(1 \leq k \leq r)$ and $U(x_{o(j)}) = \{x_{o(j)}\}(1 \leq j \leq n-r)$ (cf. Definition 3.7,(I)(∗4) in Section 3,Theorem 3.9). Then, using Theorem 4.4(iii), there exist the 2^r semi-open sets $\{x\} \cup \{p(x,u)\}(1 \leq u \leq 2^r)$ containing the point $x \in (\mathbb{Z}^n)_{mix(r)}$ such that $\{p(x,u)|1 \leq u \leq 2^r\} = (U^n(x))_{\kappa^n} =\{(z_1, z_2, ..., z_n)|z_{e(k)} \in \{x_{e(k)}+1, x_{e(k)}-1\}(1 \leq k \leq r), z_{o(j)} = x_{o(j)}(1 \leq j \leq n-r)\}$ and $\#((U^n(x))_{\kappa^n}) = 2^r$. Thus, it is shown that $\mathrm{sKer}(\{x\}) \subset \bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^r\} = \{x\} \cup (\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^r\}) = \{x\}$, because $\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^r\} = \emptyset$. Then, we show that $\mathrm{sKer}(\{x\}) = \{x\}$ holds for this case.

Therefore, for all cases above, we have proved that $\mathrm{sKer}(\{x\}) = \{x\}$ holds in $(\mathbb{Z}^n, \kappa^n)$, $n \geq 1$.

(ii) Since $E = \bigcup\{\{x\}|x \in E\}$, by Proposition 2.4(i.e., [4, Proposition 3.1]) and (i), it is shown that $\mathrm{sKer}(E) = \bigcup\{\mathrm{sKer}(\{x\})|x \in E\} = \bigcup\{\{x\}|x \in E\} = E$. □

The following result is a characterization of the ω-closed sets in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$.

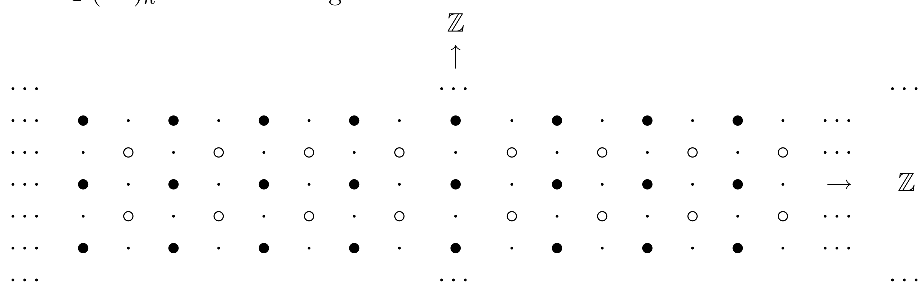
Theorem 4.6 *For a subset A of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$, A is closed in $(\mathbb{Z}^n, \kappa^n)$ if and only if A is an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$.*

Proof. By Theorem 2.5, it is obtained that a subset A is an ω-closed in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ if and only if $\mathrm{Cl}(A) \subset \mathrm{sKer}(A)$. Then, by Theorem 4.5 (ii), it is well known that A=sKer(A) holds. Thus, A is ω-closed in Sundaram-Sheik John's sense if and only if $\mathrm{Cl}(A) \subset A$ (i.e., A is closed in $(\mathbb{Z}^n, \kappa^n)$). □

Remark 4.7 (i) Every subset of $(\mathbb{Z}^n, \kappa^n)$ is a Λ_s-set in $(\mathbb{Z}^n, \kappa^n)$. Indeed, let E be a subset of $(\mathbb{Z}^n, \kappa^n)$. By Theorem 4.5 (ii) and Definition 2.3, it is shown that E=sKer(E) holds, i.e., E is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$.

(ii) By (i) and Proposition 2.6, it is obtained that $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_1 topological space. However, we note that, in 2004, S.I. Nada [30, Theorem 4.2, Theorem 4.1] proved that $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2; the proof is very elegantly done, using the semi-T_2 separation property of $(\mathbb{Z}, \kappa)$ and the product topology of κ; and hence their product space $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2; in 2006, present authors [11, Theorem 2.3, Theorem 4.8 (i)] proved that $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}^2, \kappa^2)$ are semi-T_2. But, in the end of the present paper (Corollary 4.10 below), we shall mention an alternative proof of the result ([30, Theorem 4.2]) using Theorem 4.4 and ideas in [39].

Example 4.8 In general, ω-closed sets in Sundaram-Sheik John's sense of a topological space are placed between closed sets and g-closed sets (cf. Definition 2.1(ii) (i.e.,[35])). The following example shows that there is a g-closed sets which is not an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ (i.e., closed set in $(\mathbb{Z}^n, \kappa^n)$, cf. Theorem 4.6). Suppose $n \geq 2$. Let $A := \mathbb{Z}^n \setminus (\bigcup\{(\mathbb{Z}^n)_{mix(r)} |\ 1 \leq r \leq n-1\})$, i.e., $A = (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\mathbb{Z}^n)_{\kappa^n}$ and $A \neq \emptyset$. We consider the following figure which is shown by the symbols $\bullet \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $\circ \in (\mathbb{Z}^n)_{\kappa^n}$ in $\mathbb{Z}^2$. The figure shows the subset A above for $n = 2$.



Let V be an open set containing A. Then, in below, it is proved that $V = \mathbb{Z}^n$; and hence the set A is g-closed in $(\mathbb{Z}^n, \kappa^n)$ (cf. Definition 2.1(i), i.e., [22, Definition 2.1]).
(*Proof of the property*: $V \supset \mathbb{Z}^n$). Let $x := (x_1, x_2, ..., x_n) \in \mathbb{Z}^n$ such that $x \notin A$. For this point x, we have $x \in (\mathbb{Z}^n)_{mix(r)}$ for some integer r with $1 \leq r \leq n-1$. The component $x_{e(k)}$ is even, where $e(k) \in I_r(x)$ $(1 \leq k \leq r)$ and $x_{o(j)}$ is odd, where $o(j) \in J_{n-r}(x)$ $(1 \leq j \leq n-r)$ (cf. the notation in Definition 4.3, the proof (Case 3) of Theorem 4.5(i) or in the proof (Case 2) of Proposition 4.1(i)). We pick a point $y := (y_1, y_2, ..., y_n)$ as follow: $y_{e(k)} := x_{e(k)}(1 \leq k \leq r)$ and $y_{o(j)} := x_{o(j)} + 1(1 \leq j \leq n-r)$. Then, $y \in (\mathbb{Z}^n)_{\mathcal{F}^n} \subset A$ and $x \in U^n(y)$. Since $y \in A \subset V$ and V is open, we have $U^n(y) \subset V$ (cf. Definition 3.7, (I)($*$4)(i)(ii) in Section 3,Theorem 3.9(iii)); and so $x \in V$. $(\circ)$
Thus, we have $\mathrm{Cl}(A) \subset \mathbb{Z}^n = V$ for an open set V such that $A \subset V$,i.e., A is g-closed. On the other hand, it is shown that $\mathrm{Cl}(A) = \mathbb{Z}^n$ and so A is not closed in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6).

We mention an alternative proof of the result [30, Theorem 4.2] (cf. Remark 4.7(ii) above). For $(\mathbb{Z}^n, \kappa^n)$ $(n \geq 2)$, we can construct directly two disjoint semi-open sets separating two given distinct points (cf. Corollary 4.10). We need the following property Theorem 4.9 on the smallest open sets and Theorem 4.4.

Theorem 4.9 *Let $x, x' \in \mathbb{Z}^n$, where $1 \leq n$. If $x \neq x'$ in $(\mathbb{Z}^n, \kappa^n)$, then $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds.*

Proof. We first recall that $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ (disjoint union) holds and $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset (1 \leq r \leq n-1)$ if $n \geq 2$ (cf. (II)($*$20)(iv) in Section 3). Since $\{x, x'\} \subset \mathbb{Z}^n$, we should check the cases below, Case i ($1 \leq$ i≤ 3), in order to prove $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$. We secondly suppose, for a contradiction, that
($*$1) $(U^n(x))_{\kappa^n} = (U^n(x'))_{\kappa^n}$ holds.

Case 1. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)): for these points x and x', we have that $\{x\}$ and $\{x'\}$ are open singletons and $U^n(x) = \{x\}$ and $U^n(x') = \{x'\}$ (cf. Definition 3.7, (I)($*$4)(ii) in Section 3); and so, by ($*$1) above, $\{x\} = (U^n(x))_{\kappa^n} = (U^n(x'))_{\kappa^n} = \{x'\}$. This contradicts the first setting of the given points x and x' (i.e., $x' \neq x$).

Case 2. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r')} | 1 \leq r' \leq n-1\})$ (cf. Definition 3.11(i)): for this case, $\{x\} = U^n(x)$ holds (cf. Definition 3.7(I)($*$4)(ii) in Section 3); and by Theorem 4.4(ii)(iii), it is obtained that $\#(U^n(x'))_{\kappa^n} = 2^{R'}$, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$

and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1)$. And so, by $(*1)$, we have that $2^{R'} = 1$ holds, i.e., $2^n = 1$ or $2^{r'} = 1$. These contradict the first setting of the given integers n with $n \geq 1$ and r' with $1 \leq r' \leq n-1$.

Case 3. $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1 \leq r \leq n-1\})$ (cf. Definition 3.11(i)(i-2)(i-3)):

By Theorem 4.4(ii) and (iii) for the point x, there exist the open singletons $\{p(x,u)\}(1 \leq u \leq R)$ such that $(U^n(x))_{\kappa^n} = \{p(x,u)|1 \leq u \leq R\}$ holds, where $R := n$ if $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R := r$ if $x \in (\mathbb{Z}^n)_{mix(r)}(1 \leq r \leq n-1, n \geq 2)$. Moreover, for the point x', there exist the open singletons $\{p(x',u')\}(1 \leq u' \leq R')$ such that $(U^n(x'))_{\kappa^n} = \{p(x',u')|1 \leq u' \leq R'\}$ holds, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1$ and $n \geq 2)$. We may assume that $R' \leq R$. Then, $\{p(x',u')|1 \leq u' \leq 2^{R'}\} = (U^n(x'))_{\kappa^n}= (U^n(x))_{\kappa^n} \cap (U^n(x'))_{\kappa^n} = (U^n(x) \cap U^n(x'))_{\kappa^n} \subset U^n(x) \cap U^n(x')$. Namely, $U^n(x) \cap U^n(x')$ contains exactly the $2^{R'}$ open singletons $\{p(x',u')\}$ $(1 \leq u' \leq 2^{R'})$. This shows that the assumptions of Theorem 3.12 (i.e., [39, Lemma 2.3]) are satisfied. And, using $(*1)$ above, we have $2^{R'} = \#((U^n(x'))_{\kappa^n}) = \#((U^n(x))_{\kappa^n}) = 2^R$ and so $R' = R$. Then, under the assumption $(*1)$ above, we do not have the case where that $(R', R) = (r', n)$ or (n, r), because $r, r' \in \{1, 2, ..., n-1\}$ hold. Namely, under $(*1)$, the following case does not occurs : $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1)$ (or $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $x \in (\mathbb{Z}^n)_{mix(r)}(1 \leq r \leq n-1)$). For other all cases where $(R', R) = (n, n)$ (i.e., $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n}$) or $(R', R) = (r', r)$ (i.e., $x \in (\mathbb{Z}^n)_{mix(r)}$ and $x' \in (\mathbb{Z}^n)_{mix(r')}$) with $r, r' \in \{1, 2, ..., n-1\}$, using Theorem 3.12(iii)' (i.e.,[39, Lemma 2.3 (iii)']), we have $x' = x$; this contradicts the first setting of the given points x and x' (i.e., $x' \neq x$).

Therefore, we show the required property that $(*2)$ $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds if $x \neq x'$ in $(\mathbb{Z}^n, \kappa^n)$. □

Corollary 4.10 (Namda [30, Theorem 4.2] for any $n \geq 1$; [11] for $n = 1, 2$) *The digital n-space $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_2-space.*

Proof. Suppose $n \geq 2$ in the present proof; and so we have $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset$ for each integer r with $1 \leq r \leq n-1$ (cf. Definition 3.11(i)(i-3)). We use Theorem 4.4 on the construction of semi-open sets in $(\mathbb{Z}^n, \kappa^n)$ and Theorem 4.9; and we prove that $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2, where $n \geq 2$, as follows.

Let x and x' be any distinct points of $(\mathbb{Z}^n, \kappa^n)$. We set $x = (x_1, x_2, ..., x_n)$ and $x' = (x'_1, x'_2, ..., x'_n)$, where $x_i \in \mathbb{Z}$ and $x'_i \in \mathbb{Z}(1 \leq i \leq n)$. Since $\{x, x'\} \subset \mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1 \leq r \leq n-1\})$ (disjoint union) (cf. (II)$(*20)$(iv) in Section 3), we consider the required proof for the following cases.

For the points x and x', we first use Theorem 4.9; we have that:
$(*2)$ $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds, where $U^n(y)$ is the smallest open set containing each point $y \in \{x, x'\}$. Namely, we have that:

- $(*\text{a})$ there exists a point $z \in (U^n(x))_{\kappa^n}$ and $z \notin (U^n(x'))_{\kappa^n}$; or,
- $(*\text{b})$ there exists a point $z' \in (U^n(x'))_{\kappa^n}$ and $z' \notin (U^n(x))_{\kappa^n}$.

Case 1. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\kappa^n}$: it is obviouse that $\{x\}$ and $\{x'\}$ are the required disjoint semi-open sets, because every open set is semi-open.

Case 2. $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1 \leq r \leq n-1\})$:

• For Case $(*\text{a})$ above, by Theorem 4.4(ii) and (iii) for the point x, it is shown that $z = p(x, u_0)$ holds for some point $p(x, u_0) \in (U^n(x))_{\kappa^n}(1 \leq u_0 \leq 2^R)$, where $R := n$ if $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R := r$ if $x \in (\mathbb{Z}^n)_{mix(r)}$, because $(U^n(x))_{\kappa^n} = \{p(x,u)|1 \leq u \leq 2^R\}$ holds. Moreover, we have that $\{x\} \cup \{z\}$ is a semi-open set containing the point x (cf. Theorem 4.4 (ii-3) and (iii-3)). Using Theorem 4.4 (ii) and (iii) for the point x', we can take any semi-open sets $\{x'\} \cup \{p(x', u')\}$ containing x', where $\{p(x',u')|1 \leq u' \leq 2^{R'}\} = (U^n(x'))_{\kappa^n}$ and the integer R' is defined by $R' := n$ if $x' \in (U^n(x'))_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (U^n(x'))_{mix(r')}$ with $1 \leq r' \leq n-1$. Then, we have that $(\{x\} \cup \{z\}) \cap (\{x'\} \cup \{p(x', u')\}) =(\{x\} \cap \{x'\}) \cup (\{x\} \cap \{p(x', u')\}) \cup (\{z\} \cap \{x'\}) \cup (\{z\} \cap \{p(x', u')\}) \subset (V \cap (\mathbb{Z}^n)_{\kappa^n}) \cup ((U^n(x))_{\kappa^n} \cap V) \cup (\{z\} \cap (U^n(x'))_{\kappa^n}) = \emptyset$, where $V := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$, because of the decomposition of $\mathbb{Z}^n$ and the property in (∗a) (i.e., $z \notin (U^n(x'))_{\kappa^n}$). Thus, for Case (∗a), $\{x\} \cup \{z\}$ and $\{x'\} \cup \{p(x', u')\}$ are the required disjoint semi-open sets containing the points x and x', respectively.

• For Case (∗b) above, by Theorem 4.4(ii) and (iii) for the point x', it is shown that $z' = p(x', u'_0)$ for some point $p(x', u'_0) \in (U^n(x'))_{\kappa^n}$, because $(U^n(x'))_{\kappa^n} = \{p(x', u') | 1 \leq u' \leq R'\}$ holds, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}$ with $1 \leq r' \leq n-1$. Here we note that $z' \notin (U^n(x))_{\kappa^n}$. It is shown that $\{x'\} \cup \{z'\}$ (i.e., $\{x'\} \cup \{p(x', u'_0)\}$) is the required semi-open set containing x' (cf. Theorem 4.4(ii-3) and (iii-3) for the point x'). Using Theorem 4.4 (ii) and (iii) for the point x, we can take any semi-open sets $\{x\} \cup \{p(x, u)\}$ containing x, where $\{p(x, u) | 1 \leq u \leq 2^R\} = (U^n(x))_{\kappa^n}$ for the integer R with $R := n$ if $x \in (U^n(x))_{\mathcal{F}^n}$ and $R := r$ if $x \in (U^n(x))_{mix(r)}$ with $1 \leq r \leq n-1$. Thus, the above semi-open sets $\{x\} \cup \{p(x, u)\}$ and $\{x'\} \cup \{z'\}$ are the required disjoint semi-open sets containing the point x and x', respectively. Indeed, we have that $(\{x\} \cup \{p(x, u)\}) \cap (\{x'\} \cup \{z'\})= (\{x\} \cap \{x'\}) \cup (\{x\} \cap \{z'\}) \cup (\{p(x, u)\} \cap \{x'\}) \cup (\{p(x, u)\} \cap \{z'\}) \subset (V \cap (\mathbb{Z}^n)_{\kappa^n}) \cup ((U^n(x))_{\kappa^n} \cap V) \cup ((U^n(x))_{\kappa^n} \cap \{z'\}) = \emptyset$, where $V := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$, because of the setting that $x \neq x'$, the decomposition of $\mathbb{Z}^n$ and $z' \notin (U^n(x))_{\kappa^n}$ for the Case (∗b).

Case 3. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$: for this case, we have that $\{x\} = U^n(x)$ and $\{x\} \cap (U^n(x'))_{\kappa^n} = \emptyset$ and so $\{x\}$ is the required semi-open set containing the point x. We can construct the required semi-open set containing x' using Theorem 4.4; the construction is done by an argument similar to that in Case 2.

Therefore, by Case 1, Case 2, Case 3 above for distinct points x and x', there exist disjoint semi-open sets containing the point x and x', respectively; and so $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2. □

Remark 4.11 (cf. Remark 4.7(ii)) The digital n-space $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2, where $n \geq 1$ [30]; $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}^2, \kappa^2)$ are semi-T_2 [11]. The results are confirmed directly by Corollary 4.10 above. Moreover, since the semi-T_2 separation axiom implies the semi-T_1 separation axiom, using Proposition 2.6(i), we have an alternative proof of Theorem 4.5(ii) (cf. Definition 2.3). The above proof of Corollary 4.10 is done constructively; the present authors believe that we applies the same method to other topological properties on $(\mathbb{Z}^n, \kappa^n)$ which are not proved by arguments preserving of topological products of $(\mathbb{Z}, \kappa)$ and we have further applications.

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