A NOTE ON RATIONAL OPERATOR MONOTONE FUNCTIONS

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Received May 19, 2014 ; revised April 10, 2014

Abstract. Let f be oeprator monotone for some open interval I of $\mathbb{R}$. It is known that f has the analytic continuation on $\mathbb{H}_+ \cup I \cup \mathbb{H}_-$, where $\mathbb{H}_+$ (resp. $\mathbb{H}_-$) is the upper (resp. the lower) half plane of $\mathbb{C}$. In this note, we determine the form of rational operator monotone functions by using elementary argument, and prove the operator monotonicity of some meromorphic functions.

1 Introduction.

We denote the set of all $n \times n$ matrices over $\mathbb{C}$ by M_n and set

$$H_n = \{A \in M_n \mid A^* = A\} \text{ and } H_n^+ = \{A \in H_n \mid A \geq 0\},$$

where $A \geq 0$ means that A is non-negative, that is, the value of inner product

$$(Ax, x) \geq 0 \quad \text{for all } x \in \mathbb{C}^n.$$

Let I be an open interval of the set $\mathbb{R}$ of real numbers. We also denote by $H_n(I)$ the set of $A \in H_n$ with its spectra $\mathrm{Sp}(A) \subset I$. A real continuous function f defined on the open interval I is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for any $n \in \mathbb{N}$ and $A, B \in H_n(I)$. In this note, we assume that an operator monotone function is not a constant function.

Let f be a real-valued continuous function on the interval I. We call f a Pick function if f has an analytic continuation on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ into itself. It also has an analytic continuation to the lower half plane $\mathbb{H}_-$, obtained by reflections across I.

We denote by $\mathbb{P}(I)$ the set of all Pick functions on I. It is well known that $f \in \mathbb{P}(I)$ is equivalent to that f is operator monotone on I ([1], [4], [5]).

We characterize the rational Pick function (rational operator monotone function) by an elementary method in Section 2 and give some examples using this characterization in Section 3.

2 Rational operator monotone functions.

Let I be an open interval and $f(t) = \dfrac{at+b}{ct+d}$ $(a, b, c, d \in \mathbb{R},\ ad - bc \geq 0)$. It is well known that f is operator monotone on $(-\infty, -\frac{d}{c})$ or $(-\frac{d}{c}, +\infty)$ (see [1], [5]). So the following rational function is also operator monotone on I:

$$b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i},$$

where $b_0 \in \mathbb{R}$, $a_0, a_1, \dots, a_n \geq 0$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \setminus I$.

2000 *Mathematics Subject Classification.* 47A63 .

Key words and phrases. Monotone matrix function, Operator monotone function, Pick function, Rational operator monotone function.

Let $g \in \mathbb{P}(I)$ be rational. Then there exists polynomials $p(t)$ and $q(t)$ with real coefficients such that

$$g(t) = \frac{p(t)}{q(t)} \qquad (t \in I),$$

where common devisors of $p(t)$ and $q(t)$ are only scalars and a coefficient of the highest degree term of $q(t)$ is 1. The polynomial $q(t)$ with real coefficients is represented as products of the following factors:

$$t - a, \quad t^2 + at + b \qquad (a, b \in \mathbb{R}).$$

Since g has the analytic continuation to the upper half plane $\mathbb{H}_+$ and the lower half plane $\mathbb{H}_-$,

$$g(z) = \frac{p(z)}{q(z)} \qquad (z \in \mathbb{H}_+ \cup I \cup \mathbb{H}_+)$$

and g has no poles on $\mathbb{H}_+ \cup I$. So we may assume that $g(z)$ has the following form:

$$g(z) = \frac{p(z)}{(z - c_1)^{n(1)}(z - c_2)^{n(2)} \cdots (z - c_k)^{n(k)}},$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R} \cap I^c$ and each $n(i)$ $(i = 1, 2, \ldots, k)$ is a positive integer with $n(1) + n(2) + \cdots + n(k) = \deg q(z)$. By the partially fractional decomposition of $g(z)$,

$$g(z) = r(z) + \sum_{i=1}^{k} \sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z - c_i)^j},$$

where $r(z)$ is the remainder of $p(z)$ by $q(z)$ and $\{b_{i.j}\} \subset \mathbb{R}$.

Lemma 2.1. *In above setting, $g \in \mathbb{P}(I)$ satisfies the following conditions:*

(1) *There exist $r_0, r_1 \in \mathbb{R}$ such that $r_1 \geq 0$ and $r(z) = r_0 + r_1 z$.*

(2) *$n(i) = 1$ and $b_{i,1} \leq 0$ for all $i = 1, 2, \ldots, k$.*

Proof. (1) We set

$$r(z) = r_0 + r_1 z + \cdots + r_d z^d,$$

where $d = \deg r(z)$. Put

$$\theta = \begin{cases} \frac{3\pi}{2d} & \text{if } d \geq 2 \text{ and } r_d > 0 \\ \frac{\pi}{2d} & \text{if } d \geq 1 \text{ and } r_d < 0 \end{cases}.$$

For a sufficiently large $R > 0$ and $z = Re^{\theta\sqrt{-1}} \in \mathbb{H}_+$, we may assume that

$$|r_d|R^d = |r_d z^d| > |\sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k} \sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z - c_i)^j}|.$$

Then we have

$$\begin{aligned} \mathrm{Im} g(z) &= \mathrm{Im}(-|r_d|R^d\sqrt{-1} + \sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k} \sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z - c_i)^j}) \\ &\leq -|r_d|R^d + |\sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k} \sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z - c_i)^j}| < 0. \end{aligned}$$

This contradicts to $g(z) \in \mathbb{H}_+$. So we have that $r(z) = r_0 + r_1 z$ and $r_1 \geq 0$.

(2) In a suitable neighborhood of c_i in $\mathbb{H}_+ \cup I \cup \mathbb{H}_-$, $g \in \mathbb{P}(I)$ has the form

$$g(z) = \frac{b_{i,1}}{z - c_i} + \cdots + \frac{b_{i,n(i)}}{(z - c_i)^{n(i)}} + h(z),$$

where $h(z)$ is holomorphic on the neighborhood of c_i. Put

$$\theta = \begin{cases} \dfrac{\pi}{2n(i)} & \text{if } n(i) \geq 2 \text{ and } b_{i,n(i)} > 0 \\ \dfrac{3\pi}{2n(i)} & \text{if } n(i) \geq 1 \text{ and } b_{i,n(i)} < 0 \end{cases}.$$

For a sufficiently small $r > 0$, $z = c_i + re^{\theta\sqrt{-1}} \in \mathbb{H}_+$ and we may assume that

$$\frac{|b_{i,n(i)}|}{r^{n(i)}} = |\frac{b_{i,n(i)}}{(z - c_i)^{n(i)}}| > |\sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)|.$$

Then we have

$$\begin{aligned} \mathrm{Im} g(z) &= \mathrm{Im}(-\frac{|b_{i,n(i)}|}{r^{n(i)}}\sqrt{-1} + \sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)) \\ &\leq -\frac{|b_{i,n(i)}|}{r^{n(i)}} + |\sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)| < 0. \end{aligned}$$

This contradicts to $g(z) \in \mathbb{H}_+$. So we have that $n(i) = 1$ and $b_{i,1} \leq 0$ for all $i = 1, 2, \ldots, k$. □

We can now prove the following theorem:

Theorem 2.2. *The following are equivalent:*

(1) $f \in \mathbb{P}(I)$ *is rational.*

(2) *There exist* $b_0 \in \mathbb{R}$, *non-negative numbers* $a_0, a_1, \ldots, a_n$ *and real numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *such that*

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i}.$$

(3) *There exist* $a_0, c \geq 0$, $b_0 \in \mathbb{R}$, $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *and* $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \mathbb{R}$ *satisfying that*

$$f(t) = b_0 + a_0 t - \frac{c(t - \beta_1)(t - \beta_2)\cdots(t - \beta_{n-1})}{(t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_n)}$$

and $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n$.

Proof. (1) ⇔ (2) This is proved by Lemma 2.1.

(2) ⇒ (3) We assume

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i},$$

where $b_0 \in \mathbb{R}$, $a_0, a_1, \ldots, a_n \geq 0$, $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. We define $g(t)$ as follows:

$$\sum_{i=1}^{n} \frac{a_i}{t - \alpha_i} = \frac{g(t)}{(t - \alpha_1) \cdots (t - \alpha_n)},$$

that is,

$$g(t) = \sum_{i=1}^{n} a_i (t - \alpha_1) \cdots (t - \alpha_{i-1})(t - \alpha_{i+1}) \cdots (t - \alpha_n).$$

Since

$$g(\alpha_i) = (\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \cdots (\alpha_i - \alpha_n),$$

we have

$$\text{sign } g(\alpha_i) = (-1)^{n-i} \qquad (i = 1, 2, \ldots, n).$$

By the fact $\deg g(t) = n - 1$ and the continuity of g, there exist a positive number c and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ such that

$$g(t) = c(t - \beta_1)(t - \beta_2) \cdots (t - \beta_{n-1})$$

and $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n$.

(3) $\Rightarrow$ (2) Set

$$g(t) = \frac{c(t - \beta_1)(t - \beta_2) \cdots (t - \beta_{n-1})}{(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)},$$

where $c > 0$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n$. Then $g(t)$ has the following form:

$$g(t) = \sum_{i=1}^{n} \frac{b_i}{t - \alpha_i},$$

for some $b_i \in \mathbb{R} \setminus \{0\}$. It suffices to show that $b_i > 0$ for $i = 1, 2, \ldots, n - 1$.

When we choose t such that $\beta_{i-1} < t < \alpha_i$ and $\alpha_i - t$ is sufficiently small, we have

$$\text{sign } g(t) = -\text{sign } b_i.$$

Because $\alpha_1 < \cdots < \beta_{i-1} < t < \alpha_i < \cdots < \alpha_n$,

$$\text{sign } g(t) = (-1)^{(n-1)-(i-1)+n-(i-1)} = -1.$$

So we have $b_i > 0$. □

For a rational function $f(t)$, we can choose polynomials $p(t)$ and $q(t)$ such that

$$f(t) = \frac{p(t)}{q(t)}$$

and common devisors of $p(t)$ and $q(t)$ are only scalars. Then we call f of order n if

$$n = \max\{\deg p(t),\ \deg q(t)\}.$$

Corollary 2.3. *The followings are equivalent:*

(1) *$f \in \mathbb{P}(I)$ is rational of order n.*

(2) *f has one of the following forms:*

$$(a)\ f(t) = \frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_{n+1})}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)},$$
$$(b)\ f(t) = \frac{a(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1})},$$
$$(c)\ f(t) = -\frac{a(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)}$$

or

$$(d)\ f(t) = -\frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)},$$

where $a > 0$, $\alpha_i \notin I$ and

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_n < \beta_{n+1}.$$

Proof. (1) $\Rightarrow$ (2) When $f(t)$ has the form

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n-1} \frac{a_i}{(t-\alpha_i)},$$

where $a_1, a_2, \ldots, a_{n-1} > 0$. Since f is rational of order n, we have $a_0 > 0$. We set

$$\begin{aligned} g(t) &= (b_0 + a_0 t)(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1}) \\ &\quad - \sum_{i=1}^{n-1} a_i (t-\alpha_1)\cdots(t-\alpha_{i-1})(t-\alpha_{i+1})\cdots(t-\alpha_{n-1}), \end{aligned}$$

that is,

$$f(t) = \frac{g(t)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1})}.$$

Then we have

$$\begin{gathered} \operatorname{sign}(\lim_{t\to\infty} g(t)) = 1, \quad \operatorname{sign} g(\alpha_{n-1}) = -1, \quad \operatorname{sign} g(\alpha_{n-2}) = 1, \\ \cdots, \quad \operatorname{sign} g(\alpha_1) = (-1)^{n-1}, \quad \operatorname{sign}(\lim_{t\to-\infty} g(t)) = (-1)^n. \end{gathered}$$

So f has the form (b).

When $f(t)$ has the form

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{(t-\alpha_i)},$$

where $a_1, a_2, \ldots, a_n > 0$. Since f is rational of order n, we have $a_0 = 0$. We set

$$\begin{aligned} g(t) &= b_0 (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n) \\ &\quad - \sum_{i=1}^{n} a_i (t-\alpha_1)\cdots(t-\alpha_{i-1})(t-\alpha_{i+1})\cdots(t-\alpha_n), \end{aligned}$$

that is,

$$f(t) = \frac{g(t)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)}.$$

Using the same argument as above, f has the form (d) if $b = 0$, the form (a) if $b > 0$ and the form (c) if $b < 0$.

(2) $\Rightarrow$ (1) When f has the form (a),(b),(c) or (d), f is rational of order n.

When f has the form (d), $f \in \mathbb{P}(I)$ by Theorem 2.2.

When f has the form (a), f is represented as the following form:

$$f(t) = \sum_{i=1}^{n} \frac{b_i}{t-\alpha_i} + a,$$

where $a > 0$ and some $b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Since

$$\lim_{t\to\alpha_i+0} f(t) = \lim_{t\to\alpha_i+0} \frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_{n+1})}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)} = -\infty,$$

we get $b_i < 0$ from the fact

$$\lim_{t\to\alpha_i+0} \sum_{i=1}^{n} \frac{b_i}{t-\alpha_i} + a = -\infty.$$

So $f \in \mathbb{P}(I)$.

By the similar reason, $f \in \mathbb{P}(I)$ if f has the form (b) or (c). □

3 Examples. The following Example 3.1 has been announced by M. Uchiyama in many Conferences (cf. [7], [8]).

Example 3.1. *Let* $\{p_n(x)\}$ *be the orthogonal polynomials on a closed interval* $[a, b]$ *whose leading coefficient is positive. It is well known that the zeros* $\{c_1, c_2, \ldots, c_n\}$ *of* $p_n(x)$ *satisfies that*

$$a = c_0 < c_1 < c_2 \cdots < c_n < c_{n+1} = b,$$

and each interval (c_i, c_{i+1}) $(i = 0, 1, \ldots, n)$ *contains exactly one zeros of* $p_{n+1}(x)$ *([6]). So* $p_{n+1}(x)/p_n(x)$ *has the form (b) in Corollary 2.3. This means that* $p_{n+1}(x)/p_n(x)$ *is operator monotone on any interval which does not contain any zeros of* $p_n(x)$*.*

Example 3.2. *Let* $0 = a_0 < a_1 < a_2 < \cdots < a_{2n-1} < a_{2n} = \pi$*. Then*

$$f(x) = \frac{\cos(x-a_1)\cos(x-a_3)\cdots\cos(x-a_{2n-1})}{\cos(x-a_0)\cos(x-a_2)\cdots\cos(x-a_{2n-2})}$$

is operator monotone on any interval I *contained in* $\mathbb{R} \setminus \{\frac{(2m+1)\pi}{2} + a_{2i} \mid m \in \mathbb{Z},\ i = 0, 1, \ldots, n-1\}$*.*

In particular, $\tan x$ *is operator monotone on any interval contained in* $\mathbb{R} \setminus \{m\pi - \frac{\pi}{2} \mid m \in \mathbb{Z}\}$ *(when* $n = 1$, $a_0 = 0$, $a_1 = \frac{\pi}{2}$*).*

Proof. The function $\cos x$ is represented by the infinite product as follows:

$$\cos x = \lim_{m\to\infty} f_m(x),$$

where

$$f_m(x) = \prod_{k=-m}^{m-1} (1 - \frac{2x}{(2k+1)\pi}).$$

Remarking the fact

$$f_m(x) = \frac{(-1)^m 2^{4m-2}((m-1)!)^2}{((2m-1)!)^2} \prod_{k=-m}^{m-1} (x - \frac{2k+1}{2}\pi),$$

we have that

$$\begin{aligned} g_m(x) &= \frac{f_m(x-a_1)f_m(x-a_3)\cdots f_m(x-a_{2n-1})}{f_m(x-a_0)f_m(x-a_2)\cdots f_m(x-a_{2n-2})} \\ &= \prod_{k=-m}^{m-1} \frac{(x-(\frac{(2k+1)\pi}{2}+a_1))(x-(\frac{(2k+1)\pi}{2}+a_3))\cdots(x-(\frac{(2k+1)\pi}{2}+a_{2n-1}))}{(x-(\frac{(2k+1)\pi}{2}+a_0))(x-(\frac{(2k+1)\pi}{2}+a_2))\cdots(x-(\frac{(2k+1)\pi}{2}+a_{2n-2}))} \end{aligned}$$

belongs to $\mathbb{P}(I)$ by Corollary 2.3. Since

$$f(x) = \lim_{m\to\infty} g_m(x),$$

$f(x)$ is operator monotone on I. □

Example 3.3. *Let $a_0 < a_1 < a_2 < \cdots < a_{2n-1} < a_0 + 1$ and $k(1), k(2), \ldots, k(n) \in \mathbb{Z}$. Then*

$$f(x) = \frac{\Gamma(x-a_0-k(1))\Gamma(x-a_2-k(2))\cdots\Gamma(x-a_{2n-2}-k(n))}{\Gamma(x-a_1-k(1))\Gamma(x-a_3-k(2))\cdots\Gamma(x-a_{2n-1}-k(n))}$$

is operator monotone on any interval I contained in $\mathbb{R}\backslash\{a_{2i-1}+k(i)-m \mid i = 1, 2, \ldots, n,\ m = 0, 1, 2, \ldots\}$, where $\Gamma(x)$ is the Gamma function, i.e.,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \qquad (x > 0).$$

Proof. We use Gauss's Formula of $\Gamma(x)$ as follows:

$$\Gamma(x) = \lim_{m\to\infty} g_m(x),$$

where $g_m(x) = \frac{m^x m!}{x(x+1)\cdots(x+m)}$ and the convergence is uniformly on any compact subset of $\mathbb{R} \setminus \{0, -1, -2, \ldots\}$ ([3]). For $a < b < a+1$,

$$\frac{g_m(x-a)}{g_m(x-b)} = m^{b-a}\frac{(x-b)(x-(b-1))\cdots(x-(b-m))}{(x-a)(x-(a-1))\cdots(x-(a-m))}$$

is operator monotone on any interval contained in $\mathbb{R} \setminus \{a, a-1, \ldots, a-m\}$ by Corollary 2.4. Then we have that

$$h_m(x) = \frac{g_m(x-a_0-k(1))g_m(x-a_2-k(2))\cdots g_m(x-a_{2n-2}-k(n))}{g_m(x-a_1-k(1))g_m(x-a_3-k(2))\cdots g_m(x-a_{2n-1}-k(n))}$$

also has the form (a) in Corollary 2.3, and is operator monotone on I. So is $f(x)$, because $f(x) = \lim_{m\to\infty} h_m(x)$. □

Acknowledgement. This work was partially supported by Grant-in-Aid for Scientific Research (C)22540220.

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Communicated by *Hiroyuki Osaka*

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