AN INVESTIGATION OF UNITARILY INVARIANT NORM INEQUALITIES OF LÖWNER–HEINZ TYPE

M. Fujii[1], M.S. Moslehian[2], R. Nakamoto[3] and M. Tominaga[4]

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Abstract. We utilize some 2×2 matrix tricks to obtain several unitarily invariant norm inequalities corresponding to the Löwner–Heinz inequality, the arithmetic–geometric mean inequality and the Corach–Porta–Recht inequality. Among others, we establish some norm inequalities for unitarily invariant norms implying an extended Löwner–Heinz inequality.

1 Introduction. Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and let I be its identity. We write $A \geq 0$ if A is a positive operator in the sense that $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. Further, $A \geq B$ if A and B are self-adjoint and $A - B \geq 0$. By a strictly positive operator A, denoted by $A > 0$, we mean a positive operator being invertible. If A, B are operators in $\mathbb{B}(\mathscr{H})$, we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathscr{H} \oplus \mathscr{H}$. Let $\mathbb{K}(\mathscr{H})$ denote the ideal of compact operators on $\mathscr{H}$. For any operator $A \in \mathbb{K}(\mathscr{H})$, let $s_1(A), s_2(A), \cdots$ be the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ in decreasing order and repeated according to multiplicity. If $A \in \mathcal{M}_n$, we take $s_k(A) = 0$ for $k > n$. A unitarily invariant norm in $\mathbb{K}(\mathscr{H})$ is a map $|||\cdot||| : \mathbb{K}(\mathscr{H}) \to [0, \infty]$ given by $|||A||| = g(s(A))$, $A \in \mathbb{K}(\mathscr{H})$, where g is a symmetric gauge function; cf. [12]. The set $\mathcal{I} = \mathcal{I}_{|||\cdot|||} = \{A \in \mathbb{K}(\mathscr{H}) : |||A||| < \infty\}$ is a (two-sided) ideal of $\mathbb{B}(\mathscr{H})$ by the basic property (1) in the below. An operator $A \in \mathbb{K}(\mathscr{H})$ is said to be in the Schatten p-class $\mathcal{C}_p$ $(1 \leq p < \infty)$, if $\sum_j s_j(A)^p < \infty$. The Schatten p-norm of A is defined by $\|A\|_p = \left(\sum_j s_j(A)^p\right)^{\frac{1}{p}}$, which is a typical example of a unitarily invariant norm. Other examples of unitarily invariant norms are the operator norm and the Ky Fan norms $\|A\|_{(k)} := \sum_{j=1}^{k} s_j(A)$, $k \in \mathbb{N}$ under decreasingly arranged on j. Some of basic properties are as follow:

(1) If $B \in \mathcal{I}$, then $|||B||| = |||\,|B|\,||| = |||B^*|||$ and $|||ABC||| \leq \|A\|\,|||B|||\,\|C\|$ for any $A, C \in \mathbb{B}(\mathscr{H})$.

(2) It follows from the Fan dominance principle (see e.g. [1]) that $|||A||| \leq |||B|||$ for all unitarily invariant norms if and only if $|||A \oplus 0||| \leq |||B \oplus 0|||$ for all unitarily invariant norms.

Let $\Lambda^k\mathscr{H}$ be the subspace of the k-fold tensor product $\otimes^k\mathscr{H}$ spanned by antisymmetric tensors. Then the k-fold product $\otimes^k A$ of an operator A on $\mathscr{H}$ leaves this space invariant and the restriction of $\otimes^k A$ to it, denoted by $\Lambda^k A$, is called the exterior power of A. Λ^k is multiplicative, $*$-preserving and unital. We denote the weak-log majorization and the weak majorization , $\prec_{w-log}$ and $\prec_w$, respectively. The following relations among them are known; cf. [1]. Let $X, Y \in \mathbb{K}(\mathscr{H})$. Then

$$\begin{aligned} |X| \prec_{w-log} |Y| \quad &(\text{i.e., } \left\|\Lambda^k X\right\| \leq \left\|\Lambda^k Y\right\| \text{ for any } k \leq n) \\ \Rightarrow |X| \prec_w |Y| \quad &(\text{i.e., } \|X\|_{(k)} \leq \|Y\|_{(k)} \text{ for any } k \leq n). \end{aligned}$$

So the Fan Dominance theorem is rephrased as

$$|X| \prec_{w-log} |Y| \quad \Longrightarrow \quad |||X||| \leq |||Y|||\,.$$

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Now, let us pay attention to some literature reviews. The Heinz inequality states that for $A, B, X \in \mathbb{B}(\mathscr{H})$ with $A, B \geq 0$,

$$\|AX + XB\| \geq \left\|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha\right\|$$

for $0 \leq \alpha \leq 1$, which is one of essential inequalities in operator theory.

McIntosh [11] proved that for all $A, B, X \in \mathbb{B}(\mathscr{H})$,

$$\|A^*AX + XB^*B\| \geq 2\,\|AXB^*\|\,, \tag{1.1}$$

which is called the arithmetic-geometric mean inequality; see also [7]. Bhatia and Kittaneh [4] proved that (1.1) holds for any unitarily invariant norm.

If $A \in \mathbb{B}(\mathscr{H})$ is invertible and self-adjoint, Corach et al. [6] proved that

$$\left\|A^{-1}XA + AXA^{-1}\right\| \geq 2\,\|X\|$$

for every $X \in \mathbb{B}(\mathscr{H})$. It plays a key role in the study of differential geometry of self-adjoint operators, and it has been investigated in [8] as well as [5]. On the other hand, it is known that the Löwner–Heinz inequality

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all } 0 \leq p \leq 1$$

is equivalent to the Araki–Cordes inequality (see [1], [8])

$$\|AB\|^p \geq \|A^pB^p\| \qquad \text{for all} \quad A, B \geq 0 \quad \text{and} \quad 0 \leq p \leq 1. \tag{1.2}$$

In particular, the case $p = \frac{1}{2}$ in (1.2), i.e.

$$\left\|A^2B^2\right\| \geq \left\|AB^2A\right\| \quad \text{for all} \quad A, B \geq 0, \tag{1.3}$$

is essential, which is implied by the Heinz inequality; see [7] and [8].

In this paper, we investigate several unitarily invariant norm inequalities corresponding to the Löwner–Heinz inequality, the arithmetic–geometric mean inequality and the Corach–Porta–Recht inequality. Among others, we propose some norm inequalities for unitarily invariant norms implying an extended Löwner–Heinz inequality.

2 Löwner–Heinz type inequalities. As stated in [1], a Heinz type inequality can be regarded as the arithmetic–geometric mean inequality as follows: Let $A \geq 0$ be a matrix and X a self-adjoint matrix. Then

$$|||\mathrm{Re}\,(\alpha AX + (1-\alpha)XA)||| \geq \left|\left|\left|\mathrm{Re}\,(A^\alpha XA^{1-\alpha})\right|\right|\right| \quad \text{for } \alpha \in [0, 1].$$

We note that the equivalence among Heinz type inequalities for matrices is discussed by Furuta [9]. Now we recall some relations among the Heinz inequality, the Löwner–Heinz inequality and corresponding norm inequalities for the operator norm $\|\cdot\|$; see [7]:

$$\text{Heinz inequality} \Longleftrightarrow \|\mathrm{Re}\,AX\| \geq \|XA\| \ \text{ if } A \geq 0 \text{ and } XA \text{ is self-adjoint,}$$
$$\text{Löwner–Heinz inequality} \Longleftrightarrow \|AX\| \geq \|XA\| \ \text{ if } A \geq 0 \text{ and } XA \text{ is self-adjoint.}$$

In the above inequality, if we take $X = AY$ for any $Y = Y^*$, then we have the inequality $\left\|A^2Y\right\| \geq \|AYA\|$ for $A \geq 0$. In other word, we have

$$\|AX\| \geq \left\|A^{1/2}XA^{1/2}\right\| \quad \text{for } A \geq 0 \text{ and } X = X^*. \tag{2.1}$$

Conversely, if we assume that (2.1) holds for $A \geq 0$ and $X = X^*$, then it implies

$$\|AB\| \geq \left\|A^{1/2}B^{1/2}\right\|^2,$$

that is, (1.3) is obtained and so it ensures the Löwner–Heinz inequality. Namely it is proved that (2.1) is equivalent to the Löwner–Heinz inequality.

We here remark that (2.1) does not hold for nonselfadjoint X in general. As a matter of fact, we have a counterexample as follows: Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

Then $\|A^2X\| = 4$ and $\|AXA\| = 6$.

In succession, we consider the convexity of the function

$$h(t) = \left\|A^tXA^{1-t}\right\| \quad \text{for } t \in [0,1],$$

where $A \geq 0$ and $X = X^*$.

Theorem 2.1. *The function $h(t)$ defined above is convex.*

Proof. For $\mu < \nu$, we take $t = (\mu+\nu)/2$ and $p = t - \mu = \nu - t > 0$. Then

$$\begin{aligned} h(t)^2 &= \left\|A^tXA^{2(1-t)}XA^t\right\| \\ &= r(A^tXA^{2(1-t)}XA^t) = r(A^\mu XA^{1-\mu}A^{1-\nu}XA^\nu) \\ &\leq \left\|A^\mu XA^{1-\mu}A^{1-\nu}XA^\nu\right\| \\ &\leq \left\|A^\mu XA^{1-\mu}\right\| \left\|A^\nu XA^{1-\nu}\right\| = h(\mu)\cdot h(\nu), \end{aligned}$$

where $r(C)$ denotes the spectral radius of the operator $C \in \mathbb{B}(\mathscr{H})$. Therefore

$$h(t) \leq h(\mu)^{1/2}h(\nu)^{1/2} \leq \frac{1}{2}(h(\mu)+h(\nu))$$

so that the continuous function $h(t)$ is convex. □

Next we consider the function

$$g(t) := |||A^tXA^{1-t}||| \quad \text{for } t \in [0,1], \tag{2.2}$$

where $A \geq 0$ and $X \in \mathcal{I}$ with $X = X^*$. Here we remark that every normalized unitarily invariant norm is submultiplicative (see [1, p.94]):

$$|||AB||| \leq |||A||| \cdot |||B||| \quad \text{for all } A, B \in \mathbb{K}(\mathscr{H}). \tag{2.3}$$

Corollary 2.2. *If $|||\cdot|||$ is normalized and $X = X^*$, then the function $g(t)$ defined in* (2.2) *is* log-*convex on* $[0,1]$ *and is symmetric at $\frac{1}{2}$. Consequently, $g(t)$ is convex for arbitrary unitarily invariant norm and so $g(t) \geq g(\frac{1}{2})$.*

Proof. As in the proof of Theorem 2.1, we have, under the same notation,

$$\begin{aligned} \left\|\Lambda^k(A^tXA^{1-t})\right\|^2 &= \left\|(\Lambda^kA^t)(\Lambda^kX)(\Lambda^kA^{1-t})\right\|^2 \\ &\leq \left\|(\Lambda^kA^\mu)(\Lambda^kX)(\Lambda^kA^{1-\mu})(\Lambda^kA^{1-\nu})(\Lambda^kX)(\Lambda^kA^\nu)\right\| \\ &= \left\|\Lambda^k(A^\mu XA^{1-\mu}\cdot A^{1-\nu}XA^\nu)\right\|, \end{aligned}$$

whence

$$|||A^tXA^{1-t}|||^2 \leq |||A^\mu XA^{1-\mu}\cdot A^{1-\nu}XA^\nu|||.$$

Moreover, since every normalized unitarily invariant norm is submultiplicative, we get

$$|||A^tXA^{1-t}|||^2 \leq |||A^\mu XA^{1-\mu}|||\,|||A^{1-\nu}XA^\nu|||,$$

that is, $g(t)^2 \leq g(\mu)g(\nu)$. Therefore $g(t)$ is log-convex and so

$$g(t) \leq \frac{1}{2}(g(\mu)+g(\nu)).$$

Hence the continuous function $g(t)$ is convex. In addition, since the convexity is invariant under positive scalar multiple, $g(t)$ is convex for any arbitrary unitarily invariant norm. □

As a result, the following inequalities are obtained:

Corollary 2.3. (1) *The following inequality holds:*

$$\|\!|AX|\!\| \geq \left|\!\left|\!\left| A^{\alpha} X A^{1-\alpha} \right|\!\right|\!\right| \quad \text{for } A \geq 0,\ X = X^* \in \mathcal{I} \text{ and } 0 \leq \alpha \leq 1. \tag{2.4}$$

(2) *The function $g(t)$ defined in* (2.2) *is monotone decreasing on* $[0, \frac{1}{2}]$ *and monotone increasing on* $[\frac{1}{2}, 1]$ *and consequently*

$$\left|\!\left|\!\left| A^t X A^{1-t} \right|\!\right|\!\right| \geq \left|\!\left|\!\left| A^{\frac{1}{2}} X A^{\frac{1}{2}} \right|\!\right|\!\right| \ (0 \leq t \leq 1) \quad \text{and} \quad \|\!|AX|\!\| \geq \left|\!\left|\!\left| A^{\frac{1}{2}} X A^{\frac{1}{2}} \right|\!\right|\!\right|.$$

Remark 2.4. *We should mention that inequality* (2.4) *follows from [2, Theorem 2], and (2) of Corollary 2.3 follows from the generalized Heinz inequality proved by Bhatia and Davis in [3], but our both approaches are rather different.*

Under these preparations, we have several Löwner–Heinz type inequalities as follows:

Theorem 2.5. *The following mutually equivalent inequalities hold:*

$$\left|\!\left|\!\left| AXA^{-1} \right|\!\right|\!\right| \geq \|\!|X|\!\| \quad \text{for any invertible } A \text{ and } X = X^* \in \mathcal{I}; \tag{2.5}$$

$$\|\!|AX|\!\| \geq \|\!|XA|\!\| \quad \text{for any invertible } A \text{ and } X \in \mathcal{I} \text{ such that } XA \text{ is selfadjoint}; \tag{2.6}$$

$$\|\!|AA^*X|\!\| \geq \|\!|A^*XA|\!\| \quad \text{for any invertible } A \text{ and } X = X^* \in \mathcal{I}. \tag{2.7}$$

Proof. First of all, by putting $\alpha = \frac{1}{2}$ and replacing A by $AA^* = |A^*|^2$ in (2.4), (2.7) is obtained:

$$\|\!|AA^*X|\!\| \geq \|\!||A^*|X|A^*||\!\| = \|\!|A^*XA|\!\|$$

because $A^* = U|A^*|$ with unitary U.

Next we show that (2.5) ⇒ (2.6) ⇒ (2.7) ⇒ (2.5).

(2.5) ⇒ (2.6): Since XA is selfadjoint, it follows from (2.5) that

$$\|\!|XA|\!\| \leq \left|\!\left|\!\left| A(XA)A^{-1} \right|\!\right|\!\right| = \|\!|AX|\!\|.$$

(2.6) ⇒ (2.7): Since a given X is selfadjoint, so is A^*XA. Hence (2.7) is obtained by replacing X by $X_1 = A^*X$ in (2.6), that is,

$$\|\!|AA^*X|\!\| = \|\!|AX_1|\!\| \geq \|\!|X_1A|\!\| = \|\!|A^*XA|\!\|.$$

(2.7) ⇒ (2.5): It is obtained by replacing X by $A^{*-1}XA^{-1}$ in (2.7). □

Theorem 2.6. *For $A, B \geq 0$ and $X \in \mathcal{I}$ it holds that*

$$\|\!|AX \oplus BX^*|\!\| \geq \left|\!\left|\!\left| A^{\alpha} X B^{1-\alpha} \oplus B^{\alpha} X^* A^{1-\alpha} \right|\!\right|\!\right| \quad \text{for } 0 \leq \alpha \leq 1. \tag{2.8}$$

Consequently,

$$\left|\!\left|\!\left| A^{2m+n} X B^{-n} \oplus B^{2m+n} X^* A^{-n} \right|\!\right|\!\right| \geq \left|\!\left|\!\left| A^{2m} X \oplus B^{2m} X^* \right|\!\right|\!\right|, \tag{2.9}$$

where m, n are arbitrary nonnegative integers.

Proof. We note that

$$\|\!|AX \oplus BX^*|\!\| = \left|\!\left|\!\left| \begin{bmatrix} AX & 0 \\ 0 & BX^* \end{bmatrix} \right|\!\right|\!\right|$$

$$
\begin{aligned}
&= \left|\!\left|\!\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \right|\!\right|\!\right| \\
&= \left|\!\left|\!\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right|\!\right|\!\right| \\
&= \left|\!\left|\!\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right|\!\right|\!\right|
\end{aligned}
$$

and

$$
\begin{aligned}
|||A^\alpha X B^{1-\alpha} \oplus B^\alpha X^* A^{1-\alpha}||| &= \left|\!\left|\!\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} B^{1-\alpha} & 0 \\ 0 & A^{1-\alpha} \end{bmatrix} \right|\!\right|\!\right| \\
&= \left|\!\left|\!\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B^{1-\alpha} & 0 \\ 0 & A^{1-\alpha} \end{bmatrix} \right|\!\right|\!\right| \\
&= \left|\!\left|\!\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A^{1-\alpha} & 0 \\ 0 & B^{1-\alpha} \end{bmatrix} \right|\!\right|\!\right| \\
&= \left|\!\left|\!\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^\alpha \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{1-\alpha} \right|\!\right|\!\right|.
\end{aligned}
$$

Hence the desired inequality (2.8) is ensured by (2.4). Inequality (2.9) can be obtained from (2.8) if we replace A, B, X by A^{2m+2n}, B^{2m+2n}, $A^{-n}XB^{-n}$, respectively and put $\alpha = (2m+n)(2m+2n)^{-1}$. □

Remark 2.7. *Related to inequality* (2.8)*, we have the following complementary inequality:*

$$|||A^\beta X B^{1-\beta} \oplus B^\beta X^* A^{1-\beta}||| \geq |||AX \oplus BX^*||| \quad (\beta \notin (0,1)).$$

Indeed, it can be shown by replacing A, B, X and α by $A^{2\beta-1}$, $B^{2\beta-1}$, $A^{1-\beta}XB^{1-\beta}$ and $\frac{\beta}{2\beta-1}$, respectively in (2.8)*.*

If A and B are positive invertible, then (2.7) *holds for $\beta \notin (0,1)$.*

Corollary 2.8. *The following inequalities hold and equivalent:*

(2.10) $\quad |||A^*AX \oplus B^*BX^*||| \geq |||AXB^* \oplus BX^*A^*|||$ *for $A, B \in B(\mathscr{H})$ and $X \in \mathcal{I}$;*

(2.11) $\quad |||AXB^{-1} \oplus BX^*A^{-1}||| \geq |||X \oplus X^*|||$ *for any invertible A, B and $X \in \mathcal{I}$.*

Proof. First of all, we show (2.10) by utilizing (2.8). Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively. We replace A and B by A^*A and B^*B, respectively, in (2.8) and put $\alpha = \frac{1}{2}$. Then we have

$$
\begin{aligned}
|||A^*AX \oplus B^*BX^*||| &\geq |||\,|A|X|B| \oplus |B|X^*|A|\,||| \\
&= \|U \oplus V\| \, |||\,|A|X|B| \oplus |B|X^*|A|\,||| \, \|V^* \oplus U^*\| \\
&\geq |||(U \oplus V)(|A|X|B| \oplus |B|X^*|A|)(V^* \oplus U^*)||| \\
&= |||AXB^* \oplus BX^*A^*||| . \qquad (2.12)
\end{aligned}
$$

Next (2.10) ⇒ (2.11) has been mentioned in [10]. We state its proof for the sake of completeness. Replacing X by $A^{-1}XB^{*-1}$ in (2.10), we have

$$|||A^*XB^{*-1} \oplus B^*X^*A^{*-1}||| \geq |||X \oplus X^*||| ,$$

so that (2.11) is obtained by replacing A^* and B^* by A and B, respectively.

Finally we show (2.11) ⇒ (2.10). Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B. We may assume that $|A|, |B|$ are invertible. It follows from (2.11) that

$$
\begin{aligned}
|||A^*AX \oplus B^*BX^*||| &= |||\,|A|(|A|X|B|)|B|^{-1} \oplus |B|(|B|X^*|A|)|A|^{-1}\,||| \\
&\geq |||\,|A|X|B| \oplus |B|X^*|A|\,||| \\
&\geq |||AXB^* \oplus BX^*A^*|||
\end{aligned}
$$

as we observed in (2.12). □

Remark 2.9. *We comment that* (2.11) *is implied by* (2.7)*. Put* $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *and* $Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$*. It follows from* (2.7) *that*

$$\begin{aligned} |||AXB^{-1} \oplus BX^*A^{-1}||| &= \left|\left|\left| \begin{bmatrix} 0 & AXB^{-1} \\ BX^*A^{-1} & 0 \end{bmatrix} \right|\right|\right| \\ &= |||CYC^{-1}||| = |||C^2(C^{-1}YC^{-1})||| \\ &\geq |||C(C^{-1}YC^{-1})C||| = |||Y||| = |||X \oplus X^*||| . \end{aligned}$$

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Communicated by *Junichi Fujii*

[1] Department of Mathematics, OsakaKyoiku University, Kashiwara, Osaka 582-8582, Japan.

E-mail address, mfujii@cc.osaka-kyoiku.ac.jp

[2] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

E-mail address, moslehian@um.ac.ir and moslehian@member.ams.org

[3] 1-4-13, Daihara-cho, Hitachi, Ibaraki 316-0021, Japan.

E-mail address, r-naka@net1.jway.ne.jp

[4] Practical School Education, Osaka Kyoiku University, 4-88 Minamikawahori-cho, Tennoji-ku, Osaka 543-0054, Japan.

E-mail address, tommy@cc.osaka-kyoiku.ac.jp