THE EDGEWORTH EXPANSION FOR THE NUMBER OF DISTINCT OBSERVATIONS WITH THE MIXTURE OF DIRICHLET PROCESSES

HAJIME YAMATO AND MASAO KONDO

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Abstract. Let a random distribution $\mathcal{P}$ on the real line $\mathbb{R}$ have the mixture of Dirichlet processes. Let $S^{(n)} = (S_1, \cdots, S_n)$ be the random partition of the positive integer n based on a sample of size n from $\mathcal{P}$. For the number $K_n = S_1 + \cdots + S_n$ of distinct observations among the sample, Yamato (2012) gives the asymptotic distribution of K_n and the rate $O(1/\log^{1/3} n)$ of its convergence. In this pager we give the Edgeworth expansion for K_n with the rate $O(1/\log^{2/5} n)$ and the rate $O(1/\log^{3/7} n)$.

1 Introduction. Let H_0 be a continuous distribution on the real line $\mathbb{R}$ and $\mathcal{B}$ be the σ-field which consists of the subsets of $\mathbb{R}$. Let θ be a positive random variable having the distribution γ. Given θ, let the random distribution $\mathcal{P}$ have the Dirichlet process $\mathcal{D}(\theta H_0)$ on $(\mathbb{R}, \mathcal{B})$ with parameters θ and H_0. Then this random distribution $\mathcal{P}$ has the mixture of Dirichlet processes $\mathcal{D}(\theta H_0)$ with the mixing distribution γ (Antoniak (1974)). For a sample of size n from the random distribution $\mathcal{P}$, S_1 denotes the number of observations which occur only once, S_2 the number of observations which occur exactly twice, ... and so on. Then $K_n = S_1 + \cdots + S_n$ denotes the number of distinct observations among the sample. For the convergence of K_n, Yamato (2012) gives

$$\sup_{-\infty<x<\infty}\left|P\left(\frac{K_n}{\log n} \le x\right) - \gamma(x)\right| = O\left(\frac{1}{\log^{1/3} n}\right).$$

In case the distribution γ is degenerate at θ_0, that is the θ equals to a positive constant θ_0, the random distribution $\mathcal{P}$ has the Dirichlet process $\mathcal{D}(\theta_0 H_0)$. Then, K_n has the well-known Ewens sampling formula and the asymptotic normality, whose Edgeworth expansion is given by

$$P\left(\frac{K_n - \theta_0 \log n}{\sqrt{\theta_0 \log n}} \le x\right) = \Phi(x) - \frac{1}{6\sqrt{\theta_0 \log n}}\phi(x)\left[x^2 - 1 - 6\theta_0\psi(\theta_0)\right] + O\left(\frac{1}{\log n}\right),$$

which holds uniformly in $x \in \mathbb{R}$ (Yamato (2013)). Here Φ and ϕ are the distribution function and the density function of the standard normal distribution, respectively, and ψ is the digamma function defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$, where $\Gamma(x)$ is the gamma function. The purpose of this paper is to give the Edgeworth expansion for K_n, in case $\mathcal{P}$ has the mixture of Dirichlet processes $\mathcal{D}(\theta H_0)$ with the mixing distribution γ which is not degenerate. We denote the distribution function (d.f) of the distribution γ by $G(x)$. Let g be the bounded density function of the d.f. G.

In the section 2 , we give the Edgeworth expansion for K_n with the rate $O(1/\log^{2/5} n)$. In the section 3, we give it with the rate $O(1/\log^{3/7} n)$. In the section 4, we show numerical examples.

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2 The Edgeworth expansion with the rate $1/\log^{2/5} n$. We first note the random variable P_n^*, which has the Poisson distribution with the mean $\theta(\log n - \psi(\theta))$, given θ. By Lemma 2.1 of Yamato (2013), we have:

Lemma 2.1 *Under the condition that $E_\gamma \theta$ and $E_\gamma \theta^{-1}$ are finite ,*

$$\sup_{B \subset \mathbb{Z}_+} \big| P(K_n \in B) - P(P_n^* \in B) \big| = O\Big(\frac{1}{\log n}\Big), \quad n \to \infty, \tag{2.1}$$

where where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and E_γ denotes the expectation with respect to the distribution γ.

In this section 2, we suppose that $E_\gamma(\theta^{-1})$, $E_\gamma(\theta^2)$, and $E_\gamma[\theta^2\psi(\theta)^2]$ exist. The following conditions are necessary for the proof of the proposition 2.2 using the smoothing lemma (see, for example, Petrov (1995; Theorem 5.2)); (i) $g(x)$ is twice differentiable, (ii) $x\psi(x+1)g(x)$, $g(x)$ and $xg'(x)$ are the functions of bounded variation, (iii) $g'(x)$, $xg''(x)$ and $[x\psi(x+1)g(x)]'$ are bounded, and (iv) $g(x) \to 0$, $xg'(x) \to 0$ as $x \to 0$, and $x\psi(x+1)g(x) \to 0$, $xg'(x) \to 0$ as $x \to +\infty$. Note that for $x \geq 0$, $\psi(x+1)$ is monotone increasing and $\psi(x+1) \geq \psi(1)$, where $-\psi(1)$ equals Euler's constant $(= 0.57721 \cdots)$. Then we have:

Proposition 2.2 *For $n > 3$, we have*

$$\sup_{-\infty < x < \infty} \left| P\Big(\frac{K_n}{\log n} \leq x\Big) - \Big[G(x) + \frac{1}{2\log n}\Big\{\big[2x\psi(x+1) - 1\big]g(x) + xg'(x)\Big\}\Big] \right| = O\Big(\frac{1}{\log^{2/5} n}\Big). \tag{2.2}$$

In the following proof, we use the well-known relations for any complex number z such that

$$e^z = 1 + z + \frac{c_1}{2} \mid z \mid^2, \tag{2.3}$$

$$= 1 + z + \frac{1}{2}z^2 + \frac{c_2}{6} \mid z \mid^3, \tag{2.4}$$

where for $i = 1, 2$ c_i is a complex number satisfying $\mid c_i \mid \leq 1$.

Proof of Proposition 2.2. Given θ, the characteristic function of $P_n^*/\log n$ is given by the following; For $-\infty < t < \infty$,

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\} \mid \theta\Big] = \exp\big\{\theta[\log n - \psi(\theta)]\big[e^{it/\log n} - 1\big]\big\}, \tag{2.5}$$

which is written as

$$= \exp\theta\Big\{\big[\log n - \psi(\theta)\big]\Big[\frac{it}{\log n} - \frac{t^2}{2\log^2 n} + \frac{c_{1n}}{6}\frac{\mid t \mid^3}{\log^3 n}\Big]\Big\}$$

by (2.4), where c_{1n} is a complex number such that $\mid c_{1n} \mid \leq 1$. Thus we have

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\} \mid \theta\Big] = \exp\theta\{it + A_1\} = e^{it\theta} \times e^{\theta A_1} \tag{2.6}$$

where

$$A_1 = -\psi(\theta)\frac{it}{\log n} - \frac{t^2}{2\log n} + \psi(\theta)\frac{t^2}{2\log^2 n} + \frac{c_{1n}}{6}\frac{\mid t \mid^3}{\log^2 n} - \psi(\theta)\frac{c_{1n}}{6}\frac{\mid t \mid^3}{\log^3 n}.$$

By using (2.3) to the term $e^{\theta A_1}$ of the right-hand side of (2.6), we have

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\} \mid \theta\Big] = e^{it\theta}\Big\{1+\theta A_1 + \frac{c_{2n}(\theta)}{2}\theta^2 \mid A_1\mid^2\Big\},$$

where $c_{2n}(\theta)$ is a complex number such that $\mid c_{2n}(\theta)\mid\le 1$. This is written as

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\} \mid \theta\Big] = e^{it\theta}\Big\{1-\theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n} + B_1\Big\},$$

where

$$B_1 = \theta B_{10} + \frac{c_{2n}(\theta)}{2}\theta^2 \mid A_1\mid^2,\quad B_{10} = \Big[\psi(\theta)\frac{t^2}{2\log^2 n} + \frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^2 n} - \psi(\theta)\frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^3 n}\Big]. \tag{2.7}$$

Thus we get

$$\left|E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\} \mid \theta\Big] - e^{it\theta}\Big\{1-\theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n}\Big\}\right| \le \mid B_1\mid. \tag{2.8}$$

About B_{10}, for $\mid t\mid < \log^{2/5} n$ $(n>3)$ we have

$$\mid B_{10}\mid \le \frac{\mid t\mid}{\log^{4/5} n}\Big[\frac{1}{6}+\frac{2}{3}\mid\psi(\theta)\mid\Big], \tag{2.9}$$

because of the following relations,

$$\frac{\mid t\mid}{\log^2 n} < \frac{1}{\log^{8/5} n} < \frac{1}{\log^{4/5} n},\quad \frac{t^2}{\log^2 n} < \frac{1}{\log^{6/5} n} < \frac{1}{\log^{4/5} n}\quad \text{and}\quad \frac{t^2}{\log^3 n} < \frac{1}{\log^{4/5} n}.$$

The similar inequalities to these are used, hereafter. About $|A_1|$, for $|t| < \log^{2/5} n$ $(n>3)$, by $t^2 < \log n$ and $\log n > 1$ we have

$$\begin{aligned}
\mid A_1\mid^2 &\le \Big\{\mid\psi(\theta)\mid\frac{\mid t\mid}{\log n} + \frac{t^2}{2\log n} + \mid\psi(\theta)\mid\frac{t^2}{2\log n} + \frac{\mid t\mid}{6\log n} + \mid\psi(\theta)\mid\frac{\mid t\mid}{6\log n}\Big\}^2\\
&= \Big\{\Big(\frac{7}{6}\mid\psi(\theta)\mid + \frac{1}{6}\Big)\frac{\mid t\mid}{\log n} + \big(\mid\psi(\theta)\mid+1\big)\frac{t^2}{2\log n}\Big\}^2\\
&\le 2\Big\{\Big(\frac{7}{6}\mid\psi(\theta)\mid + \frac{1}{6}\Big)^2\frac{t^2}{\log^2 n} + \big(\mid\psi(\theta)\mid+1\big)^2\frac{t^4}{4\log^2 n}\Big\}\\
&\le 2\Big\{\Big(\frac{7}{6}\mid\psi(\theta)\mid + \frac{1}{6}\Big)^2 + \frac{1}{4}\big(\mid\psi(\theta)\mid+1\big)^2\Big\}\frac{\mid t\mid}{\log^{4/5} n}.
\end{aligned} \tag{2.10}$$

By applying (2.9) and (2.10) to (2.7), for $|t| < \log^{2/5} n$ $(n>3)$,

$$\mid B_1\mid \le \frac{\mid t\mid}{\log^{4/5} n}\Big[\frac{1}{6}\theta + \frac{2}{3}\theta\mid\psi(\theta)\mid + \Big\{\Big(\frac{7}{6}\theta\mid\psi(\theta)\mid + \frac{1}{6}\theta\Big)^2 + \frac{1}{4}\big(\theta\mid\psi(\theta)\mid+\theta\big)^2\Big\}\Big]. \tag{2.11}$$

Therefore, for $|t| < \log^{2/5} n$ $(n>3)$, by $E_\gamma(\theta^2)$ and $E_\gamma[\theta^2\psi(\theta)^2] < \infty$, (2.8) and (2.11) yield

$$\left|E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\}\Big] - \Big\{E_\gamma e^{it\theta} - \frac{it}{\log n}E_\gamma\big[\theta\psi(\theta)e^{it\theta}\big] - \frac{t^2}{2\log n}E_\gamma\big[\theta e^{it\theta}\big]\Big\}\right| \le d_{11}\frac{\mid t\mid}{\log^{4/5} n}, \tag{2.12}$$

where d_{11} is a positive constant. Since $E_\gamma e^{it\theta}$ is the characteristic function of the distribution function G, that is, it is the Fourier transform of the distribution function $G(x)$. Similarly, $-itE_\gamma[\theta\psi(\theta)e^{it\theta}]$ is the Fourier transform of the function $x\psi(x)g(x)$, and $-t^2E_\gamma[\theta e^{it\theta}]$ is the Fourier transform of the function $\{xg(x)\}' = g(x) + xg'(x)$. Therefore, by applying the smoothing lemma to (2.12), we have the following.

$$\sup_x \left| P\left(\frac{P_n^*}{\log n} \le x\right) - \left[G(x) + \frac{1}{\log n} x\psi(x)g(x) + \frac{1}{2\log n}\{g(x) + xg'(x)\}\right]\right| \le \frac{d_{11}}{\log^{4/5} n}\int_0^{\log^{2/5} n} dt \; + \; \frac{d_{12}}{\log^{2/5} n} = O\left(\frac{1}{\log^{2/5} n}\right), \tag{2.13}$$

where d_{12} is positive constant depending only on d_{11}. We get (2.2) by (2.1) and (2.13), using the relation $x\psi(x) = x\psi(x+1) - 1$.

3 The Edgeworth expansion with the rate $1/\log^{3/7} n$. In addition to the assumption of the section 2, we assume that $g(x)$ is differentiable four times, $\{x^2g(x)\}^3$ is the function of bounded variation and $\{x^2g(x)\}^4$ is bounded. Besides, we suppose $E_\gamma(\theta^3)$ and $E[\theta^3|\psi(\theta)|^3]$ exist. Then we have:

Proposition 3.1 *For $n > 3$, we have*

$$\sup_{-\infty<x<\infty} \left| P\left(\frac{K_n}{\log n} \le x\right) - \left[G(x) + \frac{1}{2\log n}\Big\{\big[2x\psi(x+1) - 1\big]g(x) + xg'(x)\Big\} + \frac{1}{8\log^2 n}\{x^2g(x)\}^{(3)}\right]\right| = O\left(\frac{1}{\log^{3/7} n}\right). \tag{3.1}$$

In the following proof, in addition to (2.4) we use the well-known relation for any complex number z such that

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{c_3}{24} \mid z \mid^4, \tag{3.2}$$

where c_3 is a complex number satisfying $\mid c_3 \mid \le 1$.

Proof of Proposition 3.1. Given θ, the characteristic function (2.5) of $P_n^*/\log n$ is written as

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \;\Big|\; \theta\right] = \exp\left\{\theta[\log n - \psi(\theta)]\left[\frac{it}{\log n} - \frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^4 n}\right]\right\},$$

by (3.2), where $-\infty < t < \infty$ and c_{3n} is a complex number satisfying $\mid c_{3n} \mid \le 1$. Thus we can write

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \;\Big|\; \theta\right] = \exp\theta\{it + A_2\} = e^{it\theta} \times e^{\theta A_2}, \tag{3.3}$$

where

$$A_2 = -\psi(\theta)\frac{it}{\log n} - \frac{t^2}{2\log n} + \psi(\theta)\frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^2 n} + \psi(\theta)\frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^3 n} - \frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n}.$$

By using (2.4) to the term $e^{\theta A_2}$ of the right-hand side of (3.3), we have

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\}\ \Big|\ \theta\Big] = e^{it\theta}\Big\{1+\theta A_2+\frac{1}{2}\theta^2 A_2^2+\frac{c_{4n}(\theta)}{6}\theta^3 \mid A_2\mid^3\Big\},$$

where $c_{4n}(\theta)$ is a complex number such that $\mid c_{4n}(\theta)\mid\leq 1$. This is written as

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\}\ \Big|\ \theta\Big] = e^{it\theta}\Big\{1-\theta\psi(\theta)\frac{it}{\log n}-\frac{\theta t^2}{2\log n}+\frac{\theta^2 t^4}{8\log^2 n}+B_2\Big\}, \tag{3.4}$$

where

$$\begin{aligned}
B_2 &= \theta B_{21}+\frac{\theta^2}{2}B_{22}+\frac{c_{4n}(\theta)}{6}\theta^3\mid A_2\mid^3,\\
B_{21} &= \psi(\theta)\frac{t^2}{2\log^2 n}-\frac{it^3}{6\log^2 n}+\psi(\theta)\frac{it^3}{6\log^3 n}+\frac{c_{3n}}{24}\frac{t^4}{\log^3 n}-\frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n},\\
B_{22} &= A_2^2-\frac{t^4}{4\log^2 n}.
\end{aligned}$$

For $|t|<\log^{3/7} n$ $(n>3)$, by $|t|<\log^{1/2} n$ we have

$$\mid B_{21}\mid\leq\Big(\frac{5}{24}+\frac{17}{24}\mid\psi(\theta)\mid\Big)\frac{\mid t\mid}{\log n} < \Big(\frac{5}{24}+\frac{17}{24}\mid\psi(\theta)\mid\Big)\frac{\mid t\mid}{\log^{6/7} n}. \tag{3.5}$$

About $\mid A_2\mid$, at first for $|t|<\log^{3/7} n$ $(n>3)$, by $|t|<\log^{1/2} n$ we have

$$\begin{aligned}
\mid A_2\mid &\leq \mid\psi(\theta)\mid\frac{\mid t\mid}{\log n}+\frac{t^2}{2\log n}+\mid\psi(\theta)\mid\frac{t^2}{2\log^2 n}+\frac{\mid t\mid^3}{6\log^2 n}\\
&\qquad +\mid\psi(\theta)\mid\frac{\mid t\mid^3}{6\log^3 n}+\frac{t^4}{24\log^3 n}+\mid\psi(\theta)\mid\frac{t^4}{24\log^4 n}\\
&\leq \mid\psi(\theta)\mid\frac{\mid t\mid}{\log n}+\frac{t^2}{2\log n}+\mid\psi(\theta)\mid\frac{\mid t\mid}{2\log^{3/2} n}+\frac{\mid t\mid}{6\log n}\\
&\qquad +\mid\psi(\theta)\mid\frac{\mid t\mid}{6\log^2 n}+\frac{\mid t\mid}{24\log^{3/2} n}+\mid\psi(\theta)\mid\frac{\mid t\mid}{24\log^{5/2} n}.
\end{aligned}$$

Thus, for $|t|<\log^{3/7} n$ $(n>3)$, we have

$$\mid A_2\mid\leq\frac{\mid t\mid}{\log n}\eta(\theta)+\frac{t^2}{2\log n}\quad\text{where}\quad \eta(\theta)=\frac{41}{24}\mid\psi(\theta)\mid+\frac{5}{24}. \tag{3.6}$$

Therefore, for $|t|<\log^{3/7} n$ $(n>3)$ we have

$$\mid A_2\mid^3\leq 4\Big\{\frac{\mid t\mid^3}{\log^3 n}\eta(\theta)^3+\frac{t^6}{8\log^3 n}\Big\}\leq 4\Big\{\eta(\theta)^3+\frac{1}{8}\Big\}\frac{\mid t\mid}{\log^{6/7} n}. \tag{3.7}$$

For the evaluation of B_{22}, at first we write A_2 as

$$A_2=-\frac{t^2}{2\log n}+A_{21}$$

where

$$A_{21} = -\psi(\theta)\frac{it}{\log n} + \psi(\theta)\frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^2 n} + \psi(\theta)\frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^3 n} - \frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n}.$$

Then we

$$| B_{22} | \le \frac{t^2}{\log n} | A_{21} | + | A_{21} |^2 . \tag{3.8}$$

We note that A_{21} is obtained by deleting $-t^2/(2\log n)$ from A_2. Similarly to (3.6), for $|t| < \log^{3/7} n$ $(n > 3)$, we have

$$| A_{21} | \le \frac{| t |}{\log n}\eta(\theta). \tag{3.9}$$

Applying (3.9) to (3.8), for $|t| < \log^{3/7} n$ $(n > 3)$, we have

$$| B_{22} | \le \frac{| t |^3}{\log^2 n}\eta(\theta) + \frac{t^2}{\log^2 n}\eta(\theta)^2 \le \{\eta(\theta) + \eta(\theta)^2\}\frac{| t |}{\log n}. \tag{3.10}$$

From (3.4) we get

$$\left| E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \Big| \theta\right] - e^{it\theta}\left\{1 - \theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n} + \frac{\theta^2 t^4}{8\log^2 n}\right\}\right| \le B_2, \tag{3.11}$$

and from (3.5), (3.7) and (3.10) we have

$$| B_2 | \le \theta\left(\frac{5}{24} + \frac{17}{24} | \psi(\theta) | \right)\frac{| t |}{\log^{6/7} n} + \frac{1}{2}\theta^2\{\eta(\theta) + \eta(\theta)^2\}\frac{| t |}{\log n} + \frac{2}{3}\theta^3\left\{\eta(\theta)^3 + \frac{1}{8}\right\}\frac{| t |}{\log^{6/7} n}. \tag{3.12}$$

Therefore, for $|t| < \log^{3/7} n$ $(n > 3)$, under the condition $E_\gamma(\theta^3),\ E_\gamma[\theta^3|\psi(\theta)|^3] < \infty$, (3.11) and (3.12) give

$$\left| E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\}\right] - \left\{E_\gamma e^{it\theta} - \frac{it}{\log n}E_\gamma\left[\theta\psi(\theta)e^{it\theta}\right] - \frac{t^2}{2\log n}E_\gamma\left[\theta e^{it\theta}\right] + \frac{t^4}{8\log^2 n}E_\gamma\left[\theta^2 e^{it\theta}\right]\right\}\right| \le d_{21}\frac{|t|}{\log^{6/7} n},$$

where d_{21} is a positive constant. Since $t^4 E_\gamma[\theta^2 e^{it\theta}]$ is the Fourier transform of the function $\{xg(x)\}^{(3)}$, by the reason similar to (2.13) we have

$$\sup_x \left| P\left(\frac{P_n^*}{\log n} \le x\right) - \left[G(x) + \frac{1}{\log n}x\psi(x)g(x) + \frac{1}{2\log n}\{g(x) + xg'(x)\} + \frac{1}{8\log^2 n}\{x^2 g(x)\}^{(3)}\right]\right| = O\left(\frac{1}{\log^{3/7} n}\right). \tag{3.13}$$

Therefore we get (3.1) by (2.1) and (3.13), using the relation $x\psi(x) = x\psi(x+1) - 1$.

4 numerical examples. We examine the Propositions 2.2 and 3.1 graphically by using the gamma distribution as γ whose density is given by $g_c(x) = x^{c-1}e^{-x}/\Gamma(c)$. The distribution function of $K_n/\log n$ is obtained approximately by using the random numbers of R and described by the step function.

At first, for the examination of (2.2) by taking $c > 1$. Then, the conditions of the Propositions 2.2 are satisfied. The approroximate function $G_1(x) = G(x) + \big\{[2x\psi(x+1) - 1]g(x) + xg'(x)\big\}/(2\log n)$ is described by the broken curve. The distribution function G_c of $g_c(x)$ is described by the dotted curve. For $n = 50$, the Figure's 1, 2, 3 and 4 give the cases of $c = 1.1$, $c = 1.5$, $c = 2$ and $c = 3$. If c is small and near 1, then the function $G_1(x)$ is good approximation to the distribution function of $K_n/\log n$. Even if c increases, the function $G_1(x)$ is better than $G_c(x)$ as the approximation to the distribution function of $K_n/\log n$. But, the tail is not good approximation, similar to the usual Edgeworth expansion.

Figure 1: c=1.1, n=50

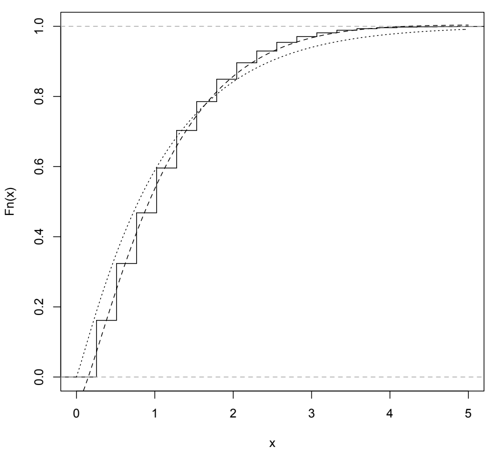


Figure 2: c=1.5, n=50

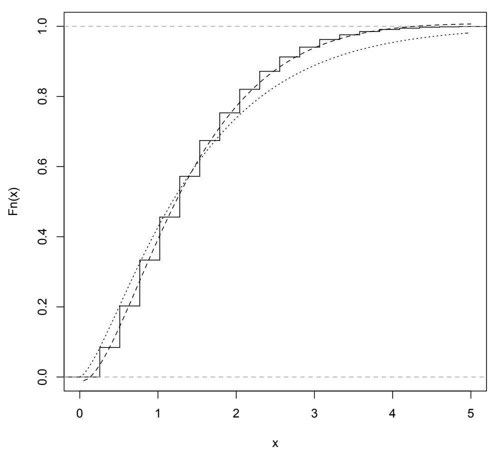


Figure 3: c=2.0, n=50

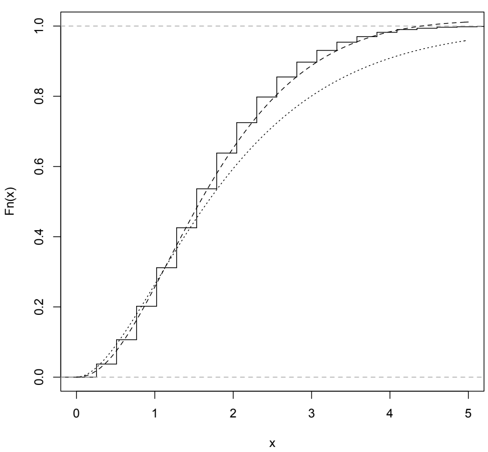
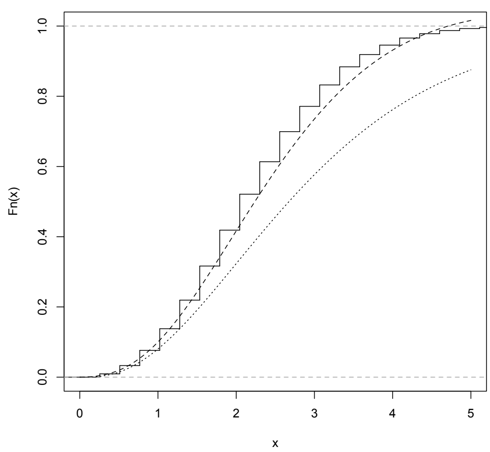


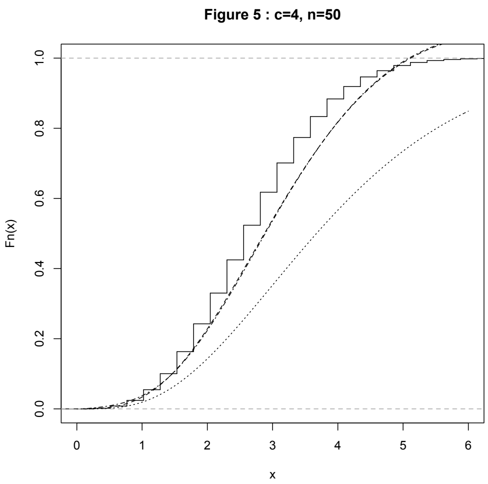
Figure 4: c=3.0, n=50

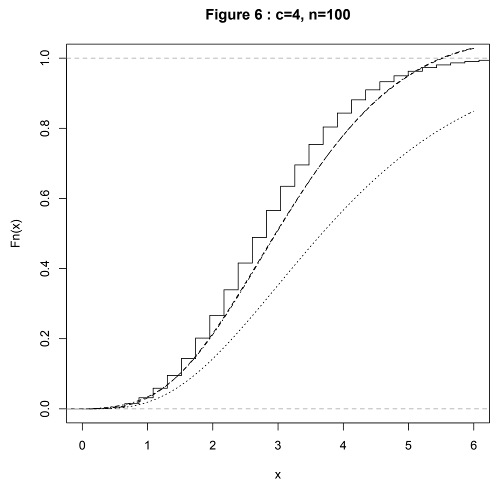


Next, we examine the relation (3.5) by taking $c = 4$. Then, the conditions of the Propositions 3.1 are satisfied.

The approximate distribution G_1 is described by the broken curve. The approximate function $G_2(x) = G(x) + \big\{[2x\psi(x+1) - 1]g(x) + xg'(x)\big\}/(2\log n) + \{x^2 g(x)\}^{(3)}/(8\log^2 n)$

is described by the dot-broken curve. The distribution function G_c of $g_c(x)$ is described by the dotted curves. For $c = 4$, the Figure's 5 and 6 give the cases of $n = 50$ and $n = 100$, respectively. Both the functions G_1 and G_2 give a little good approximate to the distribution function of $K_n/\log n$. But there are no obvious difference between G_1 and G_2, becuase the value of $G_2(x) - G_1(x) = \{x^2 g(x)\}^{(3)}/(8\log^2 n)$ is small. The little difference may be seen at the left tail.





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Communicated by *Hisao Nagao*

First author:
Emeritus of Kagoshima University, Take 3-32-1-708 Kagoshima 890–0045, Japan
Second author:
Department of Mathematics and Computer Science, Kagoshima University, Kagosima 890-0065, Japan