Optimal Maintenance Policy for Fixed Operating Time Horizon

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Abstract

We consider an optimal stopping problem for the operation of system that deteriorates with age and fails stochastically until the fixed time limit in advance. When the system fails unexpectedly, we choose one of two actions, repair or stop. The optimal stopping time which minimizes the total expected cost is derived by means of a simple mathematical model and dynamic programming technique. Some numerical examples are presented to illustrate our results in detail when the failure and the repair distributions are given specifically.

1 Introduction

In practice, most system operational periods are fixed in advance. For instance, consider the management of some airline company with B747 jumbo jet. From the view point of running cost, the company takes into consideration of replacing B747 with B787 carbon fiber aircraft. The deliver time of a new aircraft is 3 years from now on. If the B747 jumbo jet fails unexpectedlly, there are two alternatives, repair and revolve service or stop flighting service until the delivery time. It is clear that if the failure such as engine trouble occurs just before the fixed time limit, then it will be better not to repair sevice. Hence, it is an important problem to find a critical point in time between repairing and stopping.

Another example is concerned with the operation of atomic power plants in Japan. As a turning point with the Fukushima's nuclear accident in 2011, the Japanese government has established the operating time limit of all atomic power plants in 2030. In this case, the same problem happens, because the voluntary moratorium on one atomic plant will loss about 1 billion dollar/year. So, one of important problems to the electric power company is to find the optimal operating and stopping policy for existing atomic power plant.

In general, all the system will deteriorate with age and will fail stochastically. When the system fails, it is repaired with a specified repair time distribution or left as it is until the fixed time limit in advance. From the view point of cost, if the system fails close to the time limit, we should stop and not repair the system.

As a result of stopping action, an idle time occurs and a cost is incurred due to the failed system remaining idle[2,7]. It is an interesting problem to find a critical point in time to repair or to leave the failed system as it is. Such problems have been investigated by some authors in the fields of operations research and reliability engineering[1,2,6]. Kijima and et al [4] discussed the periodic replacement problem and Nair and Hopp[5] gave a simple and efficient algorithm for finding the optimal stopping rule of an equipment replacement. A recent survey paper on maintenance strategy has been written by Wang[8].

In the next section, we provide a simple model to derive the optimal operating and stopping rule for the system with arbitrary failure and repair distributions. In section 3, numerical examples with some failure and repair distributions are given to derive the critical point in time explicitly. Section 4 includes our conclusion.

2 Model and Formulation

Consider a system that deteriorates with age and fails stochastically. When the system fails, we can choose one of two actions, repair or stop. If the repair action with repair distribution $R(t)$ is chosen, the setup cost K_2 and the idle time cost per unit time C are incurred. On the other hand, if the stop action is chosen, the system will be idle until the fixed time limit and the fixed cost K_1 (cost of decommissioning) and the idle time cost per unit time C are incurred. Our problem is to find the optimal action in order to minimize the total expected cost and to derive the critical point in time to repair or to stop the failed system.

Concentrating our model, we define the following notation:

- $F(y)$ and $f(y)$ = failure distribution and its density function
- $\lambda(y) = f(y)/(1 - F(y))$ = failure rate. So $\lambda(y)\Delta y$ represents the probability that the system aged y fails between y and $y + \Delta y$.
- $U(x, y)$ = minimum expected cost up to the fixed time limit when there is still a time x to go and the system aged y is in the state of failure
- $V(x, y)$ = minimum expected cost up to the fixed time limit when there is still a time x to go and the system aged y is in the operable state.

Under these notation, consider the situation in which the system aged y is failed when there is still a time x to go and let us compare the system at two closely spaced remaining times x and $x - \Delta x$. In this case, we have two alternatives, repair the system or stop the system. If the repair action is chosen at x, either the system turns out to be an operable state with probability $R(t)$ or the repair action does not finish until the fixed time limit with probability $1 - R(t)$. If we choose stop action, then the next state is still failure state and the cost $K_1 + Cx$ is incurred.

On the other hand, if the current state is operable, then after the small time interval Δy, the state remains as operable with probability $1 - \lambda(y)\Delta y$ and the state will run into the failure state with probability $\lambda(y)\Delta y$. When the repair action is over, the age of the system is A, a given value which may not exceed the system age prior to failure. It should be noted that $A = y$ corresponds to the minimal repair and $A = 0$ major repair. Then, we have the following functional equation:

$$U(x,y) = \min \begin{cases} K_1 + Cx, & \text{: stop} \\ K_2 + \int_0^x \{Ct + V(x-t, A)\} dR(t) & \\ +(K_1 + Cx) \int_x^\infty dR(t), & \text{: repair} \end{cases} \tag{1}$$

For simplicity, we assume that the repair is minimal $A = y$. The first line in the bracket represents the cost of stopping action and the second one the total expected cost of repair service. If x is small enough, it is clear that the stopping action is preferable. Thus, for small x,

$$U(x,y) = K_1 + Cx. \tag{2}$$

On the other hand, for small Δy, $V(x,y)$ is expressed as

$$\begin{cases} V(x,y) & = \lambda(y)\Delta y U(x - \Delta y, y + \Delta y) + (1 - \lambda(y)\Delta y) V(x - \Delta y, y + \Delta y) \\ V(0,y) & = K_1 \end{cases} \tag{3}$$

Using a Taylor expansion for U and V and $\Delta y \to 0$, we have a quasi-linear partial differential equation with the boundary condition $V(0,y) = K_1$.

$$\frac{\partial V(x,y)}{\partial x} - \frac{\partial V(x,y)}{\partial y} = \lambda(y)(K_1 + Cx - V(x,y)). \tag{4}$$

Applying the standard method, the solution for this equation is given by

$$\begin{aligned} V(x,y) &= K_1 + Ce^{-\int_0^x \lambda(x+y-z)dz} \int_0^x \lambda(x+y-z) z e^{\int_0^z \lambda(x+y-\xi)d\xi} dz \\ &= K_1 + C \int_0^x (1 - e^{-\int_y^{y+\xi} \lambda(z)dz}) d\xi \\ &= K_1 + \frac{C}{1 - F(y)} [\int_0^x (F(y+\xi) - F(y)) d\xi]. \end{aligned} \tag{5}$$

Therefore, the functional equation (1) for $U(x,y)$ can be written as

$$U(x,y) = K_1 + Cx + \min\{\ 0\ ;\ G_y(x)\}, \tag{6}$$

where $V(x,y)$ is given by (5) and $G_y(x)$ expresses the optimal stopping time function as

$$\begin{aligned} G_y(x) &= K_2 - K_1 R(x) - C \int_0^x R(t)dt + \int_0^x V(x-t, y) dR(t) \\ &= K_2 - K_1 R(0) + \frac{C}{1 - F(y)} [\int_0^x \int_0^{x-t} F(y+\xi) d\xi dR(t) \\ &\quad - \int_0^x R(t)dt + F(y)R(0)x]. \end{aligned} \tag{7}$$

Note that for each $y > 0$,

$$G_y(0) = K_2 - K_1 R(0).$$

So, if $G_y(0) = K_2 - K_1 R(0) > 0$, then the stopping action should be made, where $K_1 R(0)$ shows the expected stopping cost at $x = 0$. And if $K_2 - K_1 R(0) < 0$, then the repair action is preferential.

Since $R(0)$ means the probability that finishes the repair action in a moment, we assume that $R(0) = 0$ without a special case. Under this assumption, $G_y(x)$ is given by

$$G_y(x) = K_2 + \frac{C}{1 - F(y)}[\int_0^x \int_0^{x-t} F(y + \xi) d\xi dR(t) - \int_0^x R(t) dt]$$

and from this result we can observe that the solution of $G_y(x^*) = 0$ does not depend on K_1.

(Proposition) *If $R(0) = 0$, then the optimal stopping time $x^*(y)$ does not depend on the stopping cost K_1.*

It should be noted that the relation

$$U(x, y) = K_1 + Cx$$

is valid for the preferential region of stopping and

$$U(x, y) = K_1 + Cx + G_y(x)$$

gives the expected cost for repair action, that is

$$G_y(x) > 0 \Rightarrow \text{stop action} \quad G_y(x) < 0 \Rightarrow \text{repair action}$$

Thus, the critical value of x, for which the repair action should be made, is given by the minimum positive root of

$$G_y(x) = 0.$$

Moreover,

$$x^*(y) = \inf_{x>0} \{ \ x \ : \ G_y(x) \leq 0 \}$$

represents the critical value for which the repair action should be made. It is intuitively clear, and can be easily demonstrated, that the optimal region is provided by the simple form as

$$\begin{cases} \text{stop} & \text{for} \quad 0 < x \leq x^*(y) \\ \text{repair} & \text{for} \quad x \geq x^*(y). \end{cases}$$

3 Simple Examples

In this section, we show some simple examples to find a critical value $x^*(y)$ explicitly.

(1) General Failure Distribution and Negligible Repair Time
The first example is shown by an instanteneous repair time distribution $R(0) = 1$ and a genenral failure distribution $F(t)$. By equation (7), we have

$$G_y(x) = K_2 - Cx + \frac{C}{1 - F(y)}[\int_0^x \{F(y+\xi) - F(y)\}d\xi].$$

Especially, if the failure distribution $F(t)$ is given by the exponential distribution $F(t) = 1 - e^{-\lambda t}, \quad \lambda > 0$, then

$$G_y(x) = K_2 + \frac{C}{\lambda}(e^{-\lambda x} - 1).$$

Under the condition $C > \lambda K_2$, we have

$$x^*(y) = -\frac{1}{\lambda}\ln(\frac{C - \lambda K_2}{C}).$$

On the other hand, if the repair is maximal(that is, after the repair the system's age is always $y = 0$) and $F(0) \neq 1$, we have

$$G_y(x) = K_2 - Cx + \frac{C}{1 - F(0)}[\int_0^x \{F(\xi) - F(0)\}d\xi]$$

and

$$G_y(0) = K_2 > 0, \quad G_y(\infty) = -\infty < 0.$$

Therefore, the optimal stopping time equation $G_y(x) = 0$ has at least one root for $x > 0$.

1. If $F(0) = 1$, then the optimal rule is always stop since $G_y(x) = \infty > 0$ and the repaired system fails in a moment.

2. If $F(0) \neq 1$, then

   $$G_y(x) = K_2 - Cx + \frac{C}{1 - F(0)}[\int_0^x F(\xi)d\xi - F(0)x].$$

   So, the optimal stopping time $x^*(y)$ satisfies the following equation:

   $$(1 - F(0))K_2 = C[\int_0^{x^*} (1 - F(\xi))d\xi].$$

   Especially, if $F(0) = 0$, then

   $$G_y(x) = K_2 - C[\int_0^x (1 - F(\xi))d\xi] = K_2 - C[m - T_F(x)]$$

where m is the mean time to failure and

$$T_F(x) = \int_x^\infty (x-\xi)dF(\xi).$$

Note that the transform $T_F(x)$ is a nonnegative convex and strictly decreasing function of x as was pointed out by DeGroot[2]. So, the optimal stopping time x^* is given by

$$x^* = T_F^{-1}(m - \frac{K_2}{C})$$

as shown in Figure 1.

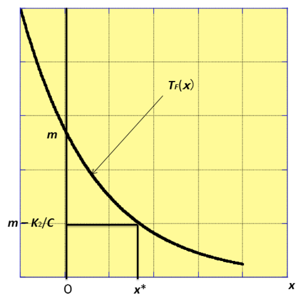

Figure 1: Graph of $T_F(x)$

(2) Gamma Type Failure Distribution

Suppose that the failure distribution $F(t)$ is Gamma type as

$$F(t) = \int_0^t \frac{\lambda^k}{(k-1)!} e^{-\lambda\xi}\xi^{k-1}d\xi.$$

Let

$$\Gamma_k(a,b) = \int_a^b e^{-\lambda t}t^{k-1}dt,$$

then

$$\lambda\Gamma_k(a,b) = k\Gamma_k(a,b) + a^k e^{-\lambda a} - b^k e^{-\lambda b}$$

and equation (5) can be denoted as

$$V(x,y) = K_1 + \frac{C}{\Gamma_k(y,\infty)}[(x+y)\Gamma_k(y,x+y) - \Gamma_{k+1}(y,x+y)].$$

It is difficult to carry out the operation of integral explicitly except for $k=1$. Let $k=1$, then the failure distribution is reduced to an exponential distribution and we have

$$V(x,y) = K_1 + Cx - \frac{C}{\lambda}(1 - e^{-\lambda x}).$$

Thus,

$$G_y(x) = K_2 - Ce^{-\lambda x}\int_0^x e^{-\lambda(x-t)}R(t)dt.$$

From this relationship, the critical value x* is given by the solution of

$$\frac{K_2}{C} = f(x) * R(x)$$

where the symbol * denotes the convolution integral.

In addition to the assumption that the failure distribution is exponential, we suppose that the repair time is subject to an exponential distribution $R(t) = 1 - \exp(-\mu t)$ and $\mu/\lambda = \rho > 1$. Then the optimal stopping time function can be written as

$$G_y(x) = K_2 - \frac{C}{\lambda}(1 - e^{-\mu x}) - \frac{C\mu}{\lambda(\mu - \lambda)}(e^{-\mu x} - e^{-\lambda x}).$$

Letting $e^{-\lambda x} = z$, we can write $G_y(x) = 0$ as

$$(\frac{\rho}{\mu - \lambda} - \frac{1}{\lambda})z^\rho - \frac{\rho}{\mu - \lambda}z = \frac{K_2}{C} - \frac{1}{\lambda}.$$

For $\lambda = 2, \mu = 1, C/K_2 = 8$, this equation yields a quardratic equation in z which has the solution $x^* = \ln 2$.

Especially, if $C > \lambda K_2$, we can easily obtain the analytical form of this value x^* for two extreme cases $\mu = \infty$ and $\mu = 0$. The assumption of $\mu = \infty$ shows a negligible repair time. Thus, the above equation is expressed as

$$G_y(x) = K_2 - \frac{C}{\lambda}(1 - e^{-\lambda x}).$$

Since $G_y(x)$ is a decreasing function of x, there exists the unique value

$$x^* = -\frac{1}{\lambda}\ln(\frac{C - \lambda K_2}{C})$$

as was derived above.

On the other hand, we consider the case of $\mu = 0$. This means that the repair action never finishes in the finite horizon. Then we have $G_y(x) = K_2 > 0$ and

$$U(x, y) = K_1 + Cx + \min\{0 : K_2\} = K_1 + Cx.$$

The result shows that the optimal policy is to be always idle for any x.
As the last example of repair time, we consider it as constant in time D. The distribution function $R(t)$ is written as

$$R(t) = \begin{cases} 0, & \text{for } 0 \le t < D \\ 1, & \text{for } t \ge D \end{cases}$$

Accordingly we have

$$G_y(x) = \begin{cases} K_2 & \text{for } \ 0 \le x < D \\ K_2 - \frac{C}{\lambda}(1 - e^{-\lambda(x-D)}) & \text{for } \ x \ge D \end{cases}$$

From the equation the optimal policy is described as follows:

(i) $C > \lambda K_2$

$$\text{stop for } \ 0 \le x < x^*$$

$$\text{repair for } \ x \ge x^*(> D)$$

where x^* is given by

$$x^* = D - \frac{1}{\lambda} \ln(\frac{C - \lambda K_2}{C})$$

(ii) $C \le \lambda K_2$ idle for all x since the second term is positive for all x.

(3) Linear Failure Distribution

Let

$$F(t) = \begin{cases} \beta t, & 0 \le t \le 1/\beta \\ 1, & t \ge 1/\beta, \end{cases}$$

then the failure rate is given by

$$\lambda(t) = \frac{\beta}{1 - \beta t}, \quad (0 \le t \le 1/\beta)$$

To derive an explicit expression of $V(x, y)$ and $G_y(x)$, we consider the following three cases:

Case(i) $x + y \le 1/\beta$

$$V(x, y) = K_1 + \frac{C\beta x^2}{2(1 - \beta y)}$$

and

$$G_y(x) = K_2 + \frac{C\beta}{1 - \beta y} \int_0^x (x + y - t - \frac{1}{\beta}) R(t) dt.$$

Note that the condition $x + y \le 1/\beta$ suggests that the time remaining until the fixed time limit is short and the system is in the nearly new state.

Case (ii) $x + y \ge 1/\beta$ and $y \le 1/\beta$

This case means that the time remaining is long enough and the system is nearly new. Then we have

$$V(x, y) = K_1 + C\{x + \frac{1}{2}(y - \frac{1}{\beta})\}$$

and

$$G_y(x) = K_2 + \frac{CR(x)}{2}(y - \frac{1}{\beta}).$$

Case(iii) $y \ge 1/\beta$

It is clear that $V(x, y) = \infty$. It follows that the optimal action should be always idle since the system aged $y(\geq 1/\beta)$ fails with probability 1. To study the optimal policy in detail, we specify that distribution of repair time $R(t)$ as follows:

(A) Exponential Repair Distribution $R(t) = 1 - \exp(-\mu t)$

From the results of three cases mentioned above, it follows that

$$G_y(x) = \begin{cases} K_2 + \frac{C\beta}{1-\beta y}[\frac{x^2}{2} + (y - \frac{1}{\beta} - \frac{1}{\mu})\{x - \frac{1}{\mu}(1 - e^{-\mu x})\}] & \text{for } x + y \leq 1/\beta \\ K_2 + \frac{C}{2}(y - \frac{1}{\beta})(1 - e^{-\mu x}) & \text{for } x + y \geq 1/\beta \text{ and } y \leq 1/\beta \\ \infty & \text{for } y \geq 1/\beta \end{cases}$$

It is clear that the critical point x^* depends on x and y. We can find the critical point by the numerical calculation and the following figure 2 and figure 3 are useful.

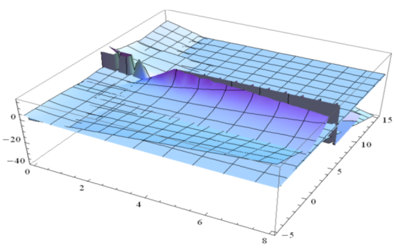

Figure 2: 3 dimensional graph of $G_y(x)$

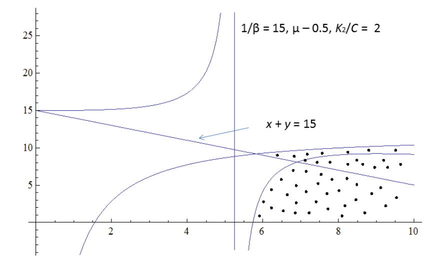

Figure 3: Graph of $G_y(x)$

(B) Straight Line Repair Distribution

Let the repair time distribution be a linear function as

$$R(t) = \begin{cases} \alpha t, & 0 \leq t \leq 1/\alpha \\ 1, & t \geq 1/\alpha \end{cases}$$

To avoid unnecessary complications, we assume that $\alpha \geq \beta$. Then we have the following result:

$$G_y(x) = K_2 + \begin{cases} \frac{C\alpha x^2}{2}\{\frac{\beta x}{3(1-\beta y)} - 1\} & \text{for } x + y \leq 1/\beta, 0 \leq x \leq 1/\alpha \\ C[\frac{1}{2\alpha} - x + \frac{\beta}{2(1-\beta y)}(x^2 - \frac{x}{\alpha} + \frac{1}{3\alpha^2})] & \text{for } x + y \leq 1/\beta, 1/\alpha \leq x \\ \frac{C\alpha x}{2}(y - \frac{1}{\beta}) & \text{for } x + y \geq 1/\beta, y \leq 1/\beta, x \geq 1/\alpha \\ \frac{C}{2}(y - \frac{1}{\beta}) & \text{for } x + y \geq 1/\beta, y \leq 1/\beta, x \leq 1/\alpha \\ \infty & \text{for } y \geq 1/\beta \end{cases}$$

The shaded portion in the figure shows a preferential region of repair service for this example. Note that the optimal stopping time $x^*(y)$ depends on the

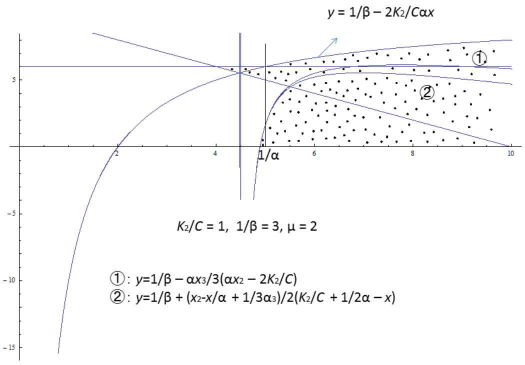

Figure 4: Repair Region

remaining time x and the system age y.

(4) Weibull Failure Distribution
Let

$$f(t) = \alpha\beta(\alpha t)^{\beta-1}e^{-(\alpha t)^{\beta}}.$$

Then

$$\lambda(t) = \beta\alpha^{\beta}t^{\beta-1}$$

1. If $\alpha = 1, \beta = 2$, then the failure distribution shows an increasing failure rate(IFR). In this case we have

$$G_y(x) = K_2 - C\sqrt{\pi}e^{y^2}\int_0^x \{\Phi(\sqrt{2}(x+y-t)) - \Phi(\sqrt{2}y)\}dR(t).$$

2. If $\alpha = 1, \beta = 1/2$, then the failure distribution shows a decreasing failure rate(DFR). In this case we have

$$G_y(x) = K_2 - 2Ce^{\sqrt{y}}\int_0^x \{e^{-y}(1+\sqrt{y}) - e^{-\sqrt{x+y-t}}(1-\sqrt{x+y-t})\}dR(t).$$

Unfortunately, it is difficult to carry out the operation of integrals explicitly.

4 Conclusion

The present paper is concerned with an optimal maintenance policy for the system with repair and idle time during the fixed time limit. An optimal policy and a critical point in time to repair or to leave the failed system as it is are provided by the method of dynamic programming technique. We show that the optimal policy depends not only the time until the fixed time limit but on the system age. It is difficult to obtain an explicit form of optimal policy for arbitrary distributions of failure and repair. The interesting results are that the critical value $x^*(y)$ does not depend on the system's age y for the exponential ditribution family by the memoryless property. Except the exponential distribution, the critical value depends on the remaining time x and the system's age y. A numerical calculation presents a solution to this difficult problem. For some simple examples, convenient figures which specify the critical point and the preferential region of repair action are easily described by the numerical calculation. The results will be useful to solve the practical problems.

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References

[1] Barlow, R. E. and L. Hunter,; Optimum Preventive Maintenance Policies, *Operations Research* 9, 90-100, 1960.
[2] Crookes, P.C. I.; Replacement Strategies, *Operational Research Quarterly* 14, 167-184, 1963.
[3] DeGroot, M. H.; *Optimal Statistical Decisions*, McGraw-Hill, New York, 1970.
[4] Kijima, M., H. Morimura and Y. Suzuki ; Periodic replacement problem without assuming minimal repair, *European Journal of Operational Research* 37,194- 203,1988.
[5] Nair, S. K. and W.J. Hopp; A model for equipment replacement due to technological obsolescence, *European Journal of Operational Research* 63, 207-221, 1992.
[6] Sivazlian, B. D.; On a Discounted Replacement Problem with Arbitrary Repair Time Distribution, *Management Science* 19, 1301-1309, 1973.
[7] Tabata, Y.; A Note on Idle Time Policy with Repair, *Journal of Operations Research Society of Japan* 22, 169-184, 1979.
[8] Wang, H.; A survey of maintenance policies of deteriorating systems, *European Journal of Operational Research* 139, 469-489, 2002.

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