# Approximately derivative in a vector lattice

Toshiharu Kawasaki

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Abstract. In previous paper we defined the derivative of mappings from a vector lattice into a complete vector lattice. In this paper we define an approximately derivative of mappings from a vector lattice into a complete vector lattice. Moreover we consider a relation between these two derivatives.

**1 Introduction** The purpose of our researches is to consider some derivatives and some integrals of mappings in vector spaces and to study their relations, for instance, the fundamental theorem of calculus, inclusive relations between integrals and so on; see [9–17].

When we consider extending from restricted Denjoy integral to improper Denjoy integral for real valued functions, the derivative is transposed to more general derivative, called approximately derivative. Therefore in this paper we consider approximately derivative for mappings from a vector lattice into a vector lattice.

In [15] we defined the derivative of mappings from a vector lattice into a complete vector lattice. In [12] we defined the approximately derivative in the case where the domain is finite dimension. This derivative seemed to be a subset of bounded linear mappings generally, however in [14] it was proved that the subset consists of a single point. In this paper we consider an approximately derivative of mappings from a vector lattice into a complete vector lattice. Moreover we consider a relation between these two derivatives.

In this paper we use notation and definitions in [15, 16]. Let $X$ be a vector lattice. An element $e \in X$ is said to be a unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of $X$. Let $\mathcal{I}_X$ be the class of intervals of $X$ and $\mathcal{IK}_X$ the class of intervals $[a, b]$ with $b - a \in \mathcal{K}_X$. Let $\mathcal{L}(X, Y)$ be the class of bounded linear mappings from $X$ into a vector lattice $Y$. If $Y$ is complete, then $\mathcal{L}(X, Y)$ is also so [2, 20, 24, 25]. A subset $D \subset X$ is said to be open if for any $x \in D$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset D$. Let $\mathcal{O}_X$ be the class of open subsets of $X$. For an interval $[a, b]$ and $e \in \mathcal{K}_X$ let

$$[a, b]^e = \{x \mid \text{ there exists } \varepsilon \in \mathcal{K}_{\mathbb{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

Let $\Lambda$ be an upward directed set. Then let $\mathcal{U}_X(\Lambda)$ be the class of $\{v_\lambda \mid \lambda \in \Lambda\}$ which satisfies the following conditions:

(U1) $v_\lambda \in X$ with $v_\lambda > 0$;

(U2)$^u$ $v_{\lambda_1} \geq v_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$;

(U3) $\bigwedge_{\lambda \in \Lambda} v_\lambda = 0$.

Moreover we consider the following condition:

(M) There exists an interval function $q : \mathcal{I}_X \longrightarrow [0, \infty)$ such that

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(M1) $q(I_1) \le q(I_2)$ if $I_1 \subset I_2$;

(M2) $q(I) > 0$ if $I \in \mathcal{IK}_X$;

(M3) For any $x \in X$, for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $\delta \in \mathcal{K}_{\mathbb{R}}$ such that $q([x, x+\delta e]) \le \varepsilon$ and $q([x-\delta e, x]) \le \varepsilon$.

*Example* 1.1. Let $X$ be a Banach lattice, that is, it satisfies that $|a| \le |b|$ implies $\|a\| \le \|b\|$. Suppose that $\mathcal{K}_X \neq \emptyset$. For any $a, b \in X$ with $a \le b$ let $q([a,b]) = \|b-a\|$. Then $X$ endowed with $q$ satisfies (M). Indeed, if $[a,b] \subset [c,d]$, then $0 \le b-a \le d-c$ and hence $q([a,b]) = \|b-a\| \le \|d-c\| = q([c,d])$. If $b-a \in \mathcal{K}_X$, then $a \neq b$ and hence $q([a,b]) = \|b-a\| > 0$. Moreover for any $x \in X$, for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$, taking $\delta \le \frac{\varepsilon}{\|e\|}$, then it holds that $q([x, x+\delta e]) = \delta\|e\| \le \varepsilon$ and $q([x-\delta e, x]) = \delta\|e\| \le \varepsilon$. For instance, since $C(K)$, where $K$ is a compact Hausdorff space, and $L^p$, which $1 \le p \le \infty$, are Banach lattices with unit, these spaces endowed with the above $q$ satisfy (M).

*Example* 1.2. Let $X = \mathbb{R}^d \times X_1$, where $X_1$ is any vector lattice with unit. For any $a = ((a_1, \ldots, a_d), a'), b = ((b_1, \ldots, b_d), b') \in X$ we define $a \le b$ whenever $a_i \le b_i$ for any $i = 1, \ldots, d$ and $a' \le b'$. Then $\mathcal{K}_X = \{((e_1, \ldots, e_d), e') \mid e_i > 0$ for any $i = 1, \ldots, d$ and $e' \in \mathcal{K}_{X_1}\}$. Moreover for any $a = ((a_1, \ldots, a_d), a'), b = ((b_1, \ldots, b_d), b') \in X$ with $a \le b$ let $q([a,b]) = \prod_{i=1}^d (b_i - a_i)$. Then $X$ endowed with $q$ satisfies (M). Indeed, if $[a,b] \subset [c,d]$, then $b_i - a_i \le d_i - c_i$ for any $i = 1, \ldots, d$ and hence $q([a,b]) \le q([c,d])$. If $b - a \in \mathcal{K}_X$, then $a_i < b_i$ for any $i = 1, \ldots, d$ and hence $q([a,b]) > 0$. Moreover for any $x \in X$, for any $e = ((e_1, \ldots, e_d), e') \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$, taking $\delta \le \frac{\varepsilon}{\prod_{i=1}^d e_i}$, then it holds that $q([x, x+\delta e]) = \delta \prod_{i=1}^d e_i \le \varepsilon$ and $q([x-\delta e, x]) = \delta \prod_{i=1}^d e_i \le \varepsilon$. For instance, since $\mathbb{R}^S$, where $S$ is an arbitrary nonempty set, is such a space, this space endowed with the above $q$ satisfies (M).

In general a lot of interval functions satisfying (M) in $X$ can be considered. Hereafter in the case of $X = \mathbb{R}^d$ we always consider the Lebesgue measure as an interval function $q$.

## 2 Definitions

**Definition 2.1.** Let $X$ be a vector lattice with unit, $x_0 \in D \in \mathcal{O}_X$ and $E \subset D$. Suppose that $X$ satisfies (M).

$x_0$ is said to be a right density point of $E$ if for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_1 \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \le e_1$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies the following conditions:

(RDS) $E^C \cap [x_0, x_0 + h] \subset \bigcup_{k=1}^\infty [a_k, b_k]^e$.

(RD) $\sum_{k=1}^\infty q([a_k, b_k]) \le \varepsilon q([x_0, x_0 + h])$.

$x_0$ is said to be a left density point of $E$ if for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_1 \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \le e_1$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies the following conditions:

(LDS) $E^C \cap [x_0 - h, x_0] \subset \bigcup_{k=1}^\infty [a_k, b_k]^e$.

(LD) $\sum_{k=1}^\infty q([a_k, b_k]) \le \varepsilon q([x_0 - h, x_0])$.

$x_0$ is said to be a density point of $E$ if it is a right density point and a left density point.

$x_0$ is said to be a right dispersion point of $E$ if for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_1 \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \le e_1$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies (RD) and the following condition:

(RDP) $E \cap [x_0, x_0 + h] \subset \bigcup_{k=1}^{\infty} [a_k, b_k]^e$.

$x_0$ is said to be a left dispersion point of $E$ if for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_1 \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \leq e_1$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies (LD) and the following condition:

(LDP) $E \cap [x_0 - h, x_0] \subset \bigcup_{k=1}^{\infty} [a_k, b_k]^e$.

$x_0$ is said to be a dispersion point of $E$ if it is a right dispersion point and a left dispersion point.

**Definition 2.2.** Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies (M).

For any $l \in \mathcal{L}(X, Y)$ and for any right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$ let

$$\begin{aligned} E^+_{sup}(l; F, x_0) &= \{x \mid x \in D, x - x_0 \in \mathcal{K}_X, F(x) - F(x_0) \not< l(x - x_0)\}, \\ L^+_{sup}(F, x_0) &= \left\{ l \;\middle|\; \begin{array}{l} l \in \mathcal{L}(X, Y), \\ x_0 \text{ is a right dispersion point of } E^+_{sup}(l; F, x_0) \end{array} \right\} \end{aligned}$$

and $o\text{-}\overline{AD}^+ F(x_0)$ the class of $l \in \mathcal{L}(X, Y)$ which satisfies the following conditions:

(a-S1$_R$) For any $l' \in \mathcal{L}(X, Y)$ with $l' > 0$ there exists $l'' \in L^+_{sup}(F, x_0)$ such that $l \leq l'' < l + l'$.

(a-S2$_R$) $l'' \not< l$ for any $l'' \in L^+_{sup}(F, x_0)$.

Let

$$\begin{aligned} E^+_{inf}(l; F, x_0) &= \{x \mid x \in D, x - x_0 \in \mathcal{K}_X, F(x) - F(x_0) \not> l(x - x_0)\}, \\ L^+_{inf}(F, x_0) &= \left\{ l \;\middle|\; \begin{array}{l} l \in \mathcal{L}(X, Y), \\ x_0 \text{ is a right dispersion point of } E^+_{inf}(l; F, x_0) \end{array} \right\} \end{aligned}$$

and $o\text{-}\underline{AD}^+ F(x_0)$ the class of $l \in \mathcal{L}(X, Y)$ which satisfies the following conditions:

(a-I1$_R$) For any $l' \in \mathcal{L}(X, Y)$ with $l' > 0$ there exists $l'' \in L^+_{inf}(F, x_0)$ such that $l \geq l'' > l - l'$.

(a-I2$_R$) $l'' \not> l$ for any $l'' \in L^+_{inf}(F, x_0)$.

For any $l \in \mathcal{L}(X, Y)$ and for any left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$ let

$$\begin{aligned} E^-_{sup}(l; F, x_0) &= \{x \mid x \in D, x_0 - x \in \mathcal{K}_X, F(x_0) - F(x) \not< l(x_0 - x)\}, \\ L^-_{sup}(F, x_0) &= \left\{ l \;\middle|\; \begin{array}{l} l \in \mathcal{L}(X, Y), \\ x_0 \text{ is a left dispersion point of } E^-_{sup}(l; F, x_0) \end{array} \right\} \end{aligned}$$

and $o\text{-}\overline{AD}^- F(x_0)$ the class of $l \in \mathcal{L}(X, Y)$ which satisfies the following conditions:

(a-S1$_L$) For any $l' \in \mathcal{L}(X, Y)$ with $l' > 0$ there exists $l'' \in L^-_{sup}(F, x_0)$ such that $l \leq l'' < l + l'$.

(a-S2$_L$) $l'' \not< l$ for any $l'' \in L^-_{sup}(F, x_0)$.

Let

$$E^-_{inf}(l;F,x_0) = \{x \mid x \in D, x_0 - x \in \mathcal{K}_X, F(x_0) - F(x) \not> l(x_0 - x)\},$$
$$L^-_{inf}(F,x_0) = \left\{ l \;\middle|\; \begin{array}{l} l \in \mathcal{L}(X,Y), \\ x_0 \text{ is a left dispersion point of } E^-_{inf}(l;F,x_0) \end{array} \right\}$$

and $o\text{-}\underline{AD}^-F(x_0)$ the class of $l \in \mathcal{L}(X,Y)$ which satisfies the following conditions:

(a-I1$_L$) For any $l' \in \mathcal{L}(X,Y)$ with $l' > 0$ there exists $l'' \in L^-_{inf}(F,x_0)$ such that $l \geq l'' > l - l'$.

(a-I2$_L$) $l'' \not> l$ for any $l'' \in L^-_{inf}(F,x_0)$.

$F$ is said to be approximately right upper differentiable, approximately right lower differentiable, approximately left upper differentiable and approximately left lower differentiable at $x_0$ if $o\text{-}\overline{AD}^+F(x_0)$, $o\text{-}\underline{AD}^+F(x_0)$, $o\text{-}\overline{AD}^-F(x_0)$ and $o\text{-}\underline{AD}^-F(x_0)$ are not empty, respectively. If $o\text{-}AD^+F(x_0) = o\text{-}\overline{AD}^+F(x_0) \cap o\text{-}\underline{AD}^+F(x_0)$ and $o\text{-}AD^-F(x_0) = o\text{-}\overline{AD}^-F(x_0) \cap o\text{-}\underline{AD}^-F(x_0)$ are not empty, then $F$ is said to be approximately right differentiable and approximately left differentiable at $x_0$, respectively. If $o\text{-}ADF(x_0) = o\text{-}AD^+F(x_0) \cap o\text{-}AD^-F(x_0)$ is not empty, then $F$ is said to be approximately differentiable at $x_0$.

## 3 Properties

**Theorem 3.1.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies* (M).

(1) *If $F$ is approximately right upper differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then any two different elements in $o\text{-}\overline{AD}^+F(x_0)$ are incomparable.*

(2) *If $F$ is approximately right lower differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then any two different elements in $o\text{-}\underline{AD}^+F(x_0)$ are incomparable.*

(3) *If $F$ is approximately left upper differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then any two different elements in $o\text{-}\overline{AD}^-F(x_0)$ are incomparable.*

(4) *If $F$ is approximately left lower differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then any two different elements in $o\text{-}\underline{AD}^-F(x_0)$ are incomparable.*

*Proof.* Assume that $l_1 < l_2$ for $l_1, l_2 \in o\text{-}\overline{AD}^+F(x_0)$. By (a-S1$_R$) for $l_1$ there exists $l'' \in L^+_{sup}(F,x_0)$ such that $l_1 \leq l'' < l_1 + (l_2 - l_1) = l_2$. However it is a contradiction to (a-S2$_R$) for $l_2$, that is, $l'' \not< l_2$ for any $l'' \in L^+_{sup}(F,x_0)$. Therefore any two different elements in $o\text{-}\underline{AD}^+F(x_0)$ must be incomparable. The rest can be proved similarly. □

**Lemma 3.1.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies* (M).

(1) *If $l \in L^\pm_{sup}(F,x_0)$ and $l' \in \mathcal{L}(X,Y)$ with $l' > l$, then $l' \in L^\pm_{sup}(F,x_0)$.*

(2) *If $l \in L^\pm_{inf}(F,x_0)$ and $l' \in \mathcal{L}(X,Y)$ with $l' < l$, then $l' \in L^\pm_{inf}(F,x_0)$.*

*Proof.* It is clear by definition. □

Let $X$ be a vector lattice and $A, B \subset X$. We write $A \leq B$ if $a \leq b$ for any $a \in A$ and for any $b \in B$. Similarly we write $A < B$ and $A \not\leq B$ if $a < b$ and $a \not\leq b$, respectively, for any $a \in A$ and for any $b \in B$, and so on. Moreover we write $A \preceq B$ if for any $a \in A$ there exists $b \in B$ such that $a \leq b$ and if for any $b \in B$ there exists $a \in A$ such that $a \leq b$.

**Lemma 3.2.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies* (M).

*Then*

(1) $L^{\pm}_{inf}(F, x_0) \cap L^{\pm}_{sup}(F, x_0) = \emptyset$.

(2) $L^{\pm}_{inf}(F, x_0) \not\geq L^{\pm}_{sup}(F, x_0)$.

(3) $L^{\pm}_{inf}(F, x_0) \preceq L^{\pm}_{sup}(F, x_0)$.

*Proof.* (1) Assume that $L^{+}_{inf}(F, x_0) \cap L^{+}_{sup}(F, x_0) \neq \emptyset$. Let $l \in L^{+}_{inf}(F, x_0) \cap L^{+}_{sup}(F, x_0)$. Then $x_0$ is a right dispersion point of $E^{+}_{inf}(l; F, x_0)$ and $E^{+}_{sup}(l; F, x_0)$, that is, for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_{inf} \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \leq e_{inf}$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies

$$\begin{aligned} E^{+}_{inf}(l; F, x_0) \cap [x_0, x_0 + h] \quad &\subset \quad \bigcup_{k=1}^{\infty} [a_k, b_k]^e, \\ \sum_{k=1}^{\infty} q([a_k, b_k]) \quad &\leq \quad \varepsilon q([x_0, x_0 + h]), \end{aligned}$$

and there exists $e_{sup} \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \leq e_{sup}$ there exists $\{[c_k, d_k] \mid k = 1, 2, \ldots\}$ which satisfies

$$\begin{aligned} E^{+}_{sup}(l; F, x_0) \cap [x_0, x_0 + h] \quad &\subset \quad \bigcup_{k=1}^{\infty} [c_k, d_k]^e, \\ \sum_{k=1}^{\infty} q([c_k, d_k]) \quad &\leq \quad \varepsilon q([x_0, x_0 + h]). \end{aligned}$$

Let $e_1 = e_{inf} \wedge e_{sup}$. Then the above two inequalities are true for any $h \in \mathcal{K}_X$ with $0 < h \leq e_1$. Since

$$E^{+}_{inf}(l; F, x_0) \cup E^{+}_{sup}(l; F, x_0) = \{x \mid x \in D, x - x_0 \in \mathcal{K}_X\},$$

it holds that

$$\begin{aligned} &(E^{+}_{inf}(l; F, x_0) \cup E^{+}_{sup}(l; F, x_0)) \cap [x_0, x_0 + h] \\ &\quad = \{x \mid x \in D, x - x_0 \in \mathcal{K}_X\} \cap [x_0, x_0 + h]. \end{aligned}$$

Therefore

$$\begin{aligned} \{x \mid x \in D, x - x_0 \in \mathcal{K}_X\} \cap [x_0, x_0 + h] \quad &\subset \quad \bigcup_{k=1}^{\infty} ([a_k, b_k]^e \cup [c_k, d_k]^e), \\ \sum_{k=1}^{\infty} q([a_k, b_k]) + \sum_{k=1}^{\infty} q([c_k, d_k]) \quad &\leq \quad 2\varepsilon q([x_0, x_0 + h]). \end{aligned}$$

It is a contradiction to that $x_0$ is a right density point of $\{x \mid x \in D, x-x_0 \in \mathcal{K}_X\}$. Therefore $L^+_{inf}(F,x_0)\cap L^+_{sup}(F,x_0)=\emptyset$. It can be proved similarly that $L^-_{inf}(F,x_0)\cap L^-_{sup}(F,x_0)=\emptyset$.
(2) Assume that $l_1 \in L^{\pm}_{inf}(F,x_0)$, $l_2 \in L^{\pm}_{sup}(F,x_0)$ and $l_1 > l_2$. By Lemma 3.1 it holds that $l_1 \in L^{\pm}_{sup}(F,x_0)$ and $l_2 \in L^{\pm}_{inf}(F,x_0)$. However it is a contradiction to (1). Therefore $L^{\pm}_{inf}(F,x_0) \not> L^{\pm}_{sup}(F,x_0)$.
(3) Let $l_1 \in L^{\pm}_{inf}(F,x_0)$ and $l_2 \in L^{\pm}_{sup}(F,x_0)$. By Lemma 3.1 it holds that $l_1 \vee l_2 \in L^{\pm}_{sup}(F,x_0)$ and $l_1 \le l_1 \vee l_2$. By Lemma 3.1 it holds that $l_1 \wedge l_2 \in L^{\pm}_{inf}(F,x_0)$ and $l_1 \wedge l_2 \le l_2$. Therefore $L^{\pm}_{inf}(F,x_0) \preceq L^{\pm}_{sup}(F,x_0)$. □

**Theorem 3.2.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies* (M).

(1) *If $F$ is approximately right upper differentiable and approximately right lower differentiable at right density point $x_0$ of $\{x \mid x \in D, x-x_0 \in \mathcal{K}_X\}$, then $o\text{-}\overline{AD}^+F(x_0) \not< o\text{-}\underline{AD}^+F(x_0)$.*

(2) *If $F$ is approximately left upper differentiable and approximately left lower differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0-x \in \mathcal{K}_X\}$, then $o\text{-}\overline{AD}^-F(x_0) \not< o\text{-}\underline{AD}^-F(x_0)$.*

*Proof.* Assume that $l_1 \in o\text{-}\overline{AD}^+F(x_0)$, $l_2 \in o\text{-}\underline{AD}^+F(x_0)$ and $l_1 < l_2$. Let $l=\frac{1}{2}(l_1+l_2)$. Then $l_1 < l < l_2$. By (a-S1$_R$) for $l_1$ there exists $l_1'' \in L^+_{sup}(F,x_0)$ such that $l_1 \le l_1'' < l$. By (a-I1$_R$) for $l_2$ there exists $l_2'' \in L^+_{inf}(F,x_0)$ such that $l_2 \ge l_2'' > l$. By Lemma 3.1 $l$ is belonging to both $L^+_{sup}(F,x_0)$ and $L^+_{inf}(F,x_0)$, however it is a contradiction to Lemma 3.2. Therefore $o\text{-}\overline{AD}^+F(x_0) \not< o\text{-}\underline{AD}^+F(x_0)$. It can be proved similarly that $o\text{-}\overline{AD}^-F(x_0) \not< o\text{-}\underline{AD}^-F(x_0)$. □

**Lemma 3.3.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$, $F, F_1, F_2$ mappings from $D$ into $Y$ and $\alpha \in \mathbb{R}$. Suppose that $X$ satisfies* (M).
*Then*

(1)

$$L^{\pm}_{sup}(\alpha F,x_0) = \begin{cases} \alpha L^{\pm}_{sup}(F,x_0) & \text{if } \alpha \ge 0, \\ \alpha L^{\pm}_{inf}(F,x_0) & \text{if } \alpha < 0. \end{cases}$$

$$L^{\pm}_{inf}(\alpha F,x_0) = \begin{cases} \alpha L^{\pm}_{inf}(F,x_0) & \text{if } \alpha \ge 0, \\ \alpha L^{\pm}_{sup}(F,x_0) & \text{if } \alpha < 0. \end{cases}$$

(2)

$$L^{\pm}_{sup}(F_1,x_0)+L^{\pm}_{sup}(F_2,x_0) \subset L^{\pm}_{sup}(F_1+F_2,x_0),$$
$$L^{\pm}_{inf}(F_1,x_0)+L^{\pm}_{inf}(F_2,x_0) \subset L^{\pm}_{inf}(F_1+F_2,x_0).$$

*Proof.* (1) is clear by definition. We show (2). Let $l_1 \in L^+_{sup}(F_1,x_0)$ and $l_2 \in L^+_{sup}(F_2,x_0)$. If

$$F_1(x)-F_1(x_0)+F_2(x)-F_2(x_0) \not< l_1(x-x_0)+l_2(x-x_0),$$

then

$$F_1(x)-F_1(x_0) \not< l_1(x-x_0) \text{ or } F_2(x)-F_2(x_0) \not< l_2(x-x_0).$$

Therefore

$$E^+_{sup}(l_1 + l_2; F_1 + F_2, x_0) \subset E^+_{sup}(l_1; F_1, x_0) \cup E^+_{sup}(l_2; F_2, x_0).$$

If $x_0$ is a right dispersion point of $E^+_{sup}(l_1; F_1, x_0)$ and of $E^+_{sup}(l_2; F_2, x_0)$, then it is right dispersion point of $E^+_{sup}(l_1 + l_2; F_1 + F_2, x_0)$. Therefore $l_1 + l_2 \in L^+_{sup}(F_1 + F_2, x_0)$. The rest can be proved similarly. □

**Theorem 3.3.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$, $F, F_1, F_2$ mappings from $D$ into $Y$ and $\alpha \in \mathbb{R}$ with $\alpha > 0$. Suppose that $X$ satisfies* (M).

(1) *If $F$ is approximately right upper differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then $\alpha F$ is also so and $o\text{-}\overline{AD}^+(\alpha F)(x_0) = \alpha o\text{-}\overline{AD}^+ F(x_0)$ and $-\alpha F$ is approximately right lower differentiable at $x_0$ and $o\text{-}\overline{AD}^+(-\alpha F)(x_0) = -\alpha o\text{-}\underline{AD}^+ F(x_0)$. If $F$ is approximately right lower differentiable at $x_0$, then $\alpha F$ is also so and $o\text{-}\underline{AD}^+(\alpha F)(x_0) = \alpha o\text{-}\underline{AD}^+ F(x_0)$ and $-\alpha F$ is approximately right upper differentiable at $x_0$ and $o\text{-}\underline{AD}^+(-\alpha F)(x_0) = -\alpha o\text{-}\overline{AD}^+ F(x_0)$.*

(2) *If $F_1, F_2, F_1 + F_2$ are approximately right upper differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then*

$$o\text{-}\overline{AD}^+ F_1(x_0) + o\text{-}\overline{AD}^+ F_2(x_0) \not< o\text{-}\overline{AD}^+(F_1 + F_2)(x_0).$$

*If $F_1, F_2, F_1 + F_2$ are approximately right lower differentiable at $x_0$, then*

$$o\text{-}\underline{AD}^+ F_1(x_0) + o\text{-}\underline{AD}^+ F_2(x_0) \not> o\text{-}\underline{AD}^+(F_1 + F_2)(x_0).$$

(3) *If $F$ is approximately left upper differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then $\alpha F$ is also so and*

$$o\text{-}\overline{AD}^-(\alpha F)(x_0) = \alpha o\text{-}\overline{AD}^- F(x_0)$$

*and $-\alpha F$ is approximately left lower differentiable at $x_0$ and*

$$o\text{-}\overline{AD}^-(-\alpha F)(x_0) = -\alpha o\text{-}\underline{AD}^- F(x_0).$$

*If $F$ is approximately left lower differentiable at $x_0$, then $\alpha F$ is also so and*

$$o\text{-}\underline{AD}^-(\alpha F)(x_0) = \alpha o\text{-}\underline{AD}^- F(x_0)$$

*and $-\alpha F$ is approximately left upper differentiable at $x_0$ and*

$$o\text{-}\underline{AD}^-(-\alpha F)(x_0) = -\alpha o\text{-}\overline{AD}^- F(x_0).$$

(4) *If $F_1, F_2, F_1 + F_2$ are approximately left upper differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then*

$$o\text{-}\overline{AD}^- F_1(x_0) + o\text{-}\overline{AD}^- F_2(x_0) \not< o\text{-}\overline{AD}^-(F_1 + F_2)(x_0).$$

*If $F_1, F_2, F_1 + F_2$ are approximately left lower differentiable at $x_0$, then*

$$o\text{-}\underline{AD}^- F_1(x_0) + o\text{-}\underline{AD}^- F_2(x_0) \not> o\text{-}\underline{AD}^-(F_1 + F_2)(x_0).$$

*Proof.* (1) and (3) are clear by definition. We show (2) and (4). Let $l_1 \in o\text{-}\overline{AD}^+ F_1(x_0)$ and $l_2 \in o\text{-}\overline{AD}^+ F_2(x_0)$. By (a-S1$_R$) for any $l' \in \mathcal{L}(X,Y)$ with $l' > 0$ there exist $l_1'' \in L^+_{sup}(F_1, X_0)$ and $l_2'' \in L^+_{sup}(F_2, X_0)$ such that $l_1 \le l_1'' < l_1 + l'$ and $l_2 \le l_2'' < l_2 + l'$. Since $F_1 + F_2$ is also approximately right upper differentiable at $x_0$, by (a-S2$_R$) it holds that $l'' \not< l$ for any $l \in o\text{-}\overline{AD}^+(F_1 + F_2)(x_0)$ and for any $l'' \in L^+_{sup}(F_1 + F_2, X_0)$. Since by Lemma 3.3 $l_1'' + l_2'' \in L^+_{sup}(F_1 + F_2, x_0)$, it holds that $l_1 + l_2 \le l_1'' + l_2'' \not< l$. Note that $l_1''$ and $l_2''$ can take near $l_1$ and $l_2$ enough. Therefore $l_1 + l_2 \not< l$. Actually assume that $l_1 + l_2 < l$. Then $l_1'' + l_2'' < l_1 + l_2 + 2l' < l$ for any $l' < \frac{1}{2}(l - l_1 - l_2)$. It is a contradiction. Therefore

$$o\text{-}\overline{AD}^+ F_1(x_0) + o\text{-}\overline{AD}^+ F_2(x_0) \not< o\text{-}\overline{AD}^+(F_1 + F_2)(x_0).$$

The rest can be proved similarly. □

**Lemma 3.4.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$. Suppose that $X$ satisfies* (M).

(1) *$l \in o\text{-}AD^+F(x_0)$ if and only if $l$ satisfies* (a-S1$_R$) *and* (a-I1$_R$).

(2) *$l \in o\text{-}AD^-F(x_0)$ if and only if $l$ satisfies* (a-S1$_L$) *and* (a-I1$_L$).

*Proof.* The necessity is clear. We show the sufficiency. We show that if $l \in \mathcal{L}(X,Y)$ satisfies (a-S1$_R$), then it satisfies (a-I2$_R$). Assume that $l$ does not satisfy (a-I2$_R$). Then there exists $l'' \in L^+_{inf}(F, x_0)$ such that $l'' > l$. By (a-S1$_R$) there exists $l''' \in L^+_{sup}(F, x_0)$ such that $l \le l''' < l''$. It is a contradiction to Lemma 3.2. Therefore $l$ satisfies (a-I2$_R$). The rest can be proved similarly. □

**Theorem 3.4.** *Let $X$ be a vector lattice with unit, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $F_1, F_2$ mappings from $D$ into $Y$. Suppose that $X$ satisfies* (M).

(1) *If $F_1$ and $F_2$ are approximately right differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then $F_1 + F_2$ is also so and*

$$o\text{-}AD^+F_1(x_0) + o\text{-}AD^+F_2(x_0) = o\text{-}AD^+(F_1 + F_2)(x_0).$$

(2) *If $F_1$ and $F_2$ are approximately left differentiable at left density point $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then $F_1 + F_2$ is also so and*

$$o\text{-}AD^-F_1(x_0) + o\text{-}AD^-F_2(x_0) = o\text{-}AD^-(F_1 + F_2)(x_0).$$

*Proof.* Let $l_1 \in o\text{-}AD^+F_1(x_0)$ and $l_2 \in o\text{-}AD^+F_2(x_0)$. For any $l' \in \mathcal{L}(X,Y)$ with $l' > 0$ there exist $l_1'' \in L^+_{sup}(F_1, x_0)$ and $l_2'' \in L^+_{sup}(F_2, x_0)$ such that $l_1 \le l_1'' < l_1 + l'$ and $l_2 \le l_2'' < l_2 + l'$. Since by Lemma 3.3 $l_1'' + l_2'' \in L^+_{sup}(F_1 + F_2, x_0)$, $l_1 + l_2$ satisfies (a-S1$_R$) for $F_1 + F_2$. Similarly $l_1 + l_2$ satisfies (a-I1$_R$) for $F_1 + F_2$. Therefore by Lemma 3.4 $F_1 + F_2$ is approximately right differentiable and

$$o\text{-}AD^+F_1(x_0) + o\text{-}AD^+F_2(x_0) \subset o\text{-}AD^+(F_1 + F_2)(x_0).$$

In the above formula we put $-F_1$ into $F_1$ and $F_1 + F_2$ into $F_2$. Then we get

$$o\text{-}AD^+(-F_1)(x_0) + o\text{-}AD^+(F_1 + F_2)(x_0) \subset o\text{-}AD^+F_2(x_0).$$

By Theorem 3.3

$$\begin{aligned} o\text{-}AD^+(F_1+F_2)(x_0) &\subset o\text{-}AD^+F_2(x_0) - o\text{-}AD^+(-F_1)(x_0) \\ &= o\text{-}AD^+F_2(x_0) + o\text{-}AD^+F_1(x_0). \end{aligned}$$

Therefore

$$o\text{-}AD^+F_1(x_0) + o\text{-}AD^+F_2(x_0) = o\text{-}AD^+(F_1+F_2)(x_0).$$

The rest can be proved similarly. □

**4 In the case of $X = \mathbb{R}^d$** Approximately derivative becomes a subset of bounded linear mappings generally. The problem that it consists of a single point is not solved. However it is true to show the following in the case where $X$ is finite dimensional; see [14].

**Lemma 4.1.** *Let $X$ and $Y$ be vector lattices and $l \in \mathcal{L}(X,Y)$.*
*If $\{x_n\}$ is relatively uniformly convergent to 0 in $X$, then $\{l(x_n)\}$ is also so in $Y$.*

*Proof.* Since $\{x_n\}$ is relatively uniformly convergent to 0 in $X$, there exist $\{\varepsilon_n\} \in \mathcal{U}_{\mathbb{R}}(\mathbb{N})$ and $u \in X$ with $u > 0$ such that $|x_n| \le \varepsilon_n u$ for any natural number $n$. Then there exists a monotone sequnce $\{r_n\}$ of real numbers such that it is divergent to infinity and $\{r_n x_n\}$ is relatively uniformly convergent to 0. Actually there exists a monotone sequence $\{N(m)\}$ of natural numbers such that $|x_n| \le \frac{1}{m^2}u$ if $n > N(m)$. Let

$$r_n = \begin{cases} 1 & \text{if } n \le N(1), \\ m & \text{if } N(m) < n \le N(m+1) \ (m = 1, 2, \ldots). \end{cases}$$

Since

$$|r_n x_n| = \begin{cases} |x_n| & \text{if } n \le N(1), \\ m|x_n| & \text{if } N(m) < n \le N(m+1) \ (m = 1, 2, \ldots), \end{cases}$$

and $m|x_n| \le \frac{1}{m}u$, $\{r_n x_n\}$ is relatively uniformly convergent to 0 and $\{r_n\}$ is divergent to infinity. Since $\{r_n x_n\}$ is relatively uniformly convergent to 0, it is bounded. Therefore $\{r_n l(x_n)\}$ is also so, that is, there exists $v \in Y$ with $v > 0$ such that $r_n|l(x_n)| \le v$. For $m$ select $N$ such that $r_{N+1} \ge m$. Then $|l(x_n)| \le \frac{1}{r_n}v \le \frac{1}{m}v$ for any natural number $n > N$. It means that $l(x_n)$ is relatively uniformly convergent to 0. □

**Lemma 4.2.** *Let $X = \mathbb{R}^d$, $Y$ a complete vector lattice, $x_0 \in X$ and $l \in \mathcal{L}(X,Y)$.*
*Then*

$$o\text{-}\overline{AD}^+l(x_0) = o\text{-}\underline{AD}^+l(x_0) = o\text{-}\overline{AD}^-l(x_0) = o\text{-}\underline{AD}^-l(x_0) = \{l\}.$$

*Proof.* We show that $o\text{-}\overline{AD}^+l(x_0) = \{l\}$. The rest can be proved similarly. Since $l \in o\text{-}\overline{AD}^+l(x_0)$ is clear, we show that for any element of $o\text{-}\overline{AD}^+l(x_0)$ it is equals to $l$. First we consider a necessary and sufficient condition for $l'' \in L^+_{sup}(l, x_0)$. Note that $\mathcal{K}_X = \{(e_1, \ldots, e_d) \mid e_i > 0 \text{ for any } i\}$ and $\mathcal{L}(X,Y) \cong Y^d$.
In the case of $l'' > l$:
Since

$$E^+_{sup}(l''; l, x_0) = \{x \mid x - x_0 \in \mathcal{K}_X, l(x - x_0) \not< l''(x - x_0)\} = \emptyset,$$

it holds that $l'' \in L^+_{sup}(l, x_0)$.

In the case of $l'' = l$:

Since

$$E^+_{sup}(l''; l, x_0) = \{x \mid x - x_0 \in \mathcal{K}_X\},$$

it holds that for any $h \in \mathcal{K}_X$

$$E^+_{sup}(l''; l, x_0) \cap [x_0, x_0 + h] = \{x \mid x - x_0 \in \mathcal{K}_X\} \cap [x_0, x_0 + h].$$

Therefore $x_0$ is never a right dispersion point of $E^+_{sup}(l''; l, x_0)$. Then $l'' \notin L^+_{sup}(l, x_0)$.
In the case of $l'' \not\geq l$:

Note that for any $x \in X$ with $x > 0$ there exist $r > 0$ and $0 \leq \theta_i \leq \frac{\pi}{2}$ $(i = 1, \ldots, d-1)$ such that

$$\begin{aligned} x &= f(r, \theta_1, \ldots, \theta_{d-1}) \\ &= r(\cos\theta_1 \cdots \cos\theta_{d-1}, \cos\theta_1 \cdots \sin\theta_{d-1}, \ldots, \sin\theta_1). \end{aligned}$$

Therefore there exists $f(r_0, \theta_{1,0}, \ldots, \theta_{d-1,0})$ with $r_0 > 0$, $0 \leq \theta_{i,0} \leq \frac{\pi}{2}$ $(i = 1, \ldots, d-1)$ such that

$$l''(f(r_0, \theta_{1,0}, \ldots, \theta_{d-1,0})) \not\geq l(f(r_0, \theta_{1,0}, \ldots, \theta_{d-1,0})).$$

Then there exists $\alpha_i$ with $0 < \alpha_i + \theta_{i,0} < \frac{\pi}{2}$, $\alpha_i \neq 0$ such that for any $\theta_i$ with $|\theta_i - \theta_{i,0}| \leq |\alpha_i|$, $0 \leq \theta_i \leq \frac{\pi}{2}$ it holds that

$$l''(f(r_0, \theta_1, \ldots, \theta_{d-1})) \not\geq l(f(r_0, \theta_1, \ldots, \theta_{d-1})).$$

If not, then for $\alpha_{i,1}$ with $0 < \alpha_{i,1} + \theta_{i,0} < \frac{\pi}{2}$, $\alpha_{i,1} \neq 0$ there exists $\theta_{i,1}$ with $0 < |\theta_{i,1} - \theta_{i,0}| \leq |\alpha_{i,1}|$, $0 \leq \theta_{i,1} \leq \frac{\pi}{2}$ such that

$$l''(f(r_0, \theta_{1,1}, \ldots, \theta_{d-1,1})) \geq l(f(r_0, \theta_{1,1}, \ldots, \theta_{d-1,1})).$$

Moreover for $\alpha_{i,2}$ with $0 < \alpha_{i,2} + \theta_{i,0} < \frac{\pi}{2}$, $0 \neq \alpha_{i,2} \leq \frac{1}{2}|\alpha_{i,1}|$ there exists $\theta_{i,2}$ with $0 < |\theta_{i,2} - \theta_{i,0}| \leq |\alpha_{i,2}|$, $0 \leq \theta_{i,2} \leq \frac{\pi}{2}$ such that

$$l''(f(r_0, \theta_{1,2}, \ldots, \theta_{d-1,2})) \geq l(f(r_0, \theta_{1,2}, \ldots, \theta_{d-1,2})).$$

Repeat this way, then we get a sequence $\{f(r_0, \theta_{1,k}, \ldots, \theta_{d-1,k})\}$ such that it is relatively uniformly convergent to $f(r_0, \theta_{1,0}, \ldots, \theta_{d-1,0})$ and

$$l''(f(r_0, \theta_{1,k}, \ldots, \theta_{d-1,k})) \geq l(f(r_0, \theta_{1,k}, \ldots, \theta_{d-1,k})).$$

It is a contradiction to Lemma 4.1. Therefore there exists $\alpha_i$ with $0 < \alpha_i + \theta_{i,0} < \frac{\pi}{2}$, $\alpha_i \neq 0$ such that for any $\theta_i$ with $|\theta_i - \theta_{i,0}| \leq |\alpha_i|$, $0 \leq \theta_i \leq \frac{\pi}{2}$ it holds that

$$l''(f(r_0, \theta_1, \ldots, \theta_{d-1})) \not\geq l(f(r_0, \theta_1, \ldots, \theta_{d-1})).$$

Since $l''$ and $l$ are linear, the above inequality is true for any $r > 0$. Let

$$W = \left\{ f(r, \theta_1, \ldots, \theta_{d-1}) \;\middle|\; r > 0, |\theta_i - \theta_{i,0}| \leq |\alpha_i|, 0 \leq \theta_i \leq \frac{\pi}{2} \ (i = 1, \ldots, d-1) \right\}.$$

Then $\{x \mid x - x_0 \in \mathcal{K}_X\} \cap (x_0 + W) \subset E^+_{sup}(l''; l, x_0)$. Let $El(h) = El(h_1, \ldots, h_d)$ be the intersection of an ellipsoid, which radii are $h_1, \ldots, h_d$, and $\{(x_1, \ldots, x_d) \mid x_i > 0 \text{ for any } i\}$. Then $x_0 + El(h) \subset [x_0, x_0 + h]$. Since $x_0$ is a right density point of $\{x \mid x - x_0 \in \mathcal{K}_X\}$, that is, for any $e \in \mathcal{K}_X$ and for any $\varepsilon \in \mathcal{K}_{\mathbb{R}}$ there exists $e_1 \in \mathcal{K}_X$ such that for any $h \in \mathcal{K}_X$ with $0 < h \le e_1$ there exists $\{[a_k, b_k] \mid k = 1, 2, \ldots\}$ which satisfies

$$\begin{aligned} \{x \mid x - x_0 \in \mathcal{K}_X\}^C \cap [x_0, x_0 + h] &\subset \bigcup_{k=1}^{\infty} [a_k, b_k]^e, \\ \sum_{k=1}^{\infty} q([a_k, b_k]) &\le \varepsilon q([x_0, x_0 + h]), \end{aligned}$$

if $x_0$ is a right dispersion point of $E^+_{sup}(l''; l, x_0)$, that is,

$$\begin{aligned} E^+_{sup}(l''; l, x_0) \cap [x_0, x_0 + h] &\subset \bigcup_{k=1}^{\infty} [c_k, d_k]^e, \\ \sum_{k=1}^{\infty} q([c_k, d_k]) &\le \varepsilon q([x_0, x_0 + h]), \end{aligned}$$

then

$$\begin{aligned} (x_0 + W) \cap (x_0 + El(h)) &\subset \left( \bigcup_{k=1}^{\infty} [a_k, b_k]^e \right) \cup \left( \bigcup_{k=1}^{\infty} [c_k, d_k]^e \right), \\ \sum_{k=1}^{\infty} q([a_k, b_k]) + \sum_{k=1}^{\infty} q([c_k, d_k]) &\le 2\varepsilon q([x_0, x_0 + h]) \end{aligned}$$

proving that $(x_0 + W) \cap (x_0 + El(h))$ is a null set. On the other hand

$$\begin{aligned} q((x_0 + W) \cap (x_0 + El(h))) &\ge \frac{|\alpha_1 \cdots \alpha_{d-1}|}{2\pi^{d-1}} \times \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \times h_1 \cdots h_d \\ &= \frac{|\alpha_1 \cdots \alpha_{d-1}|}{2\pi^{\frac{d}{2}-1} \Gamma(\frac{d}{2} + 1)} h_1 \cdots h_d \\ &= \frac{|\alpha_1 \cdots \alpha_{d-1}|}{2\pi^{\frac{d}{2}-1} \Gamma(\frac{d}{2} + 1)} q([x_0, x_0 + h]), \end{aligned}$$

where $\Gamma$ is $\Gamma$-function. It is a contradiction. Therefore $x_0$ is never a right dispersion point of $E^+_{sup}(l''; l, x_0)$. Then $l'' \notin L^+_{sup}(l, x_0)$.

Therefore $l'' \in L^+_{sup}(l, x_0)$ if and only if $l'' > l$. Let $l_1 \in o\text{-}\overline{AD}^+ l(x_0)$. For any $l' > 0$ there exists $l'' \in L^+_{sup}(l, x_0)$ such that $l_1 \le l'' < l_1 + l'$. Since $l'$ is arbitrary, it holds that $l \le l_1$, moreover by Theorem 3.1 it hold that $l_1 = l$. □

**Theorem 4.1.** *Let $X = \mathbb{R}^d$, $Y$ a complete vector lattice, $x_0 \in D \in \mathcal{O}_X$ and $l \in \mathcal{L}(X, Y)$.*

(1) *If $F$ is approximately right differentiable at right density point $x_0$ of $\{x \mid x \in D, x - x_0 \in \mathcal{K}_X\}$, then $o\text{-}AD^+F(x_0)$ consists of a single point.*

(2) *If $F$ is approximately left differentiable at left density pont $x_0$ of $\{x \mid x \in D, x_0 - x \in \mathcal{K}_X\}$, then $o\text{-}AD^+F(x_0)$ consists of a single point.*

*Proof.* In Theorem 3.4 (1) put $F_1 = F$ and $F_2 = -F$ and by Lemma 4.2

$$o\text{-}AD^+F(x_0) - o\text{-}AD^+F(x_0) = o\text{-}AD^+0(x_0) = \{0\}.$$

Therefore $o\text{-}AD^+F(x_0)$ consists of a single point. Similarly it can be proved that $o\text{-}AD^-F(x_0)$ consists of a single point. □

5 Relation We consider a relation between the approximately derivative and the derivative. However it is not known any desirable relation. In this section we consider the case where $X = \mathbb{R}$ and $Y$ is totally ordered.

**Theorem 5.1.** *Let $X = \mathbb{R}$, $Y$ a complete vector lattice with total ordering, $x_0 \in D \in \mathcal{O}_X$ and $F$ a mapping from $D$ into $Y$.*

(1) *If $F$ is right differentiable at $x_0$, then it is approximately right differentiable and $o\text{-}D^+F(x_0) = o\text{-}AD^+F(x_0)$.*

(2) *If $F$ is left differentiable at $x_0$, then it is approximately right differentiable and $o\text{-}D^-F(x_0) = o\text{-}AD^-F(x_0)$.*

*Proof.* Let $l = o\text{-}D^+F(x_0)$. Then there exists $\{w_{x_0,e}\} \in \mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta_{x_0} \in \mathcal{K}_\mathbb{R}$ such that $|F(x_0+h) - F(x_0) - l(h)| \leq w_{x_0,e}(h)$ for any $h \in X$ with $0 < h \leq \delta_{x_0}e$. Let $l' \in \mathcal{L}(X,Y)$ with $l' > 0$. Since $\mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ is totally ordered, there exists $e \in \mathcal{K}_X$ such that $w_{x_0,e}(h) < \frac{1}{2}l'(h)$ for any $h \in \mathcal{K}_X$ with $0 < h \leq \delta_{x_0}e$. Let $l'' = l + \frac{1}{2}l'$. Then $l \leq l'' < l + l'$ and $l'' \in L^+_{sup}(F, x_0)$. Actually since for any $h \in \mathcal{K}_X$ with $0 < h \leq \delta_{x_0}e$, $x_0 + h \in D$

$$F(x_0 + h) - F(x_0) \leq (l + w_{x_0,e})(h) < l''(h),$$

it holds that $E^+_{sup}(l''; F, x_0) \cap [x_0, x_0 + h] = \emptyset$. Therefore $x_0$ is a right dispersion point of $E^+_{sup}(l''; F, x_0)$. Then $l'' \in L^+_{sup}(F, x_0)$. Therefore $l$ satisfies (a-S1$_R$). Similarly it can be proved that $l$ satisfies (a-I1$_R$). By Lemma 3.4 $F$ is approximately right differentiable at $x_0$. By Theorem 4.1 we obtain that $o\text{-}D^+F(x_0) = o\text{-}AD^+F(x_0)$. The rest can be proved similarly. □

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T. Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail: toshiharu.kawasaki@nifty.ne.jp