More on decompositions of a fuzzy set in fuzzy topological spaces

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Abstract. Using new properties (Theorem B in Section 2) of the concept of fuzzy points in the sense of Pu Pao-Ming and Liu Ying-Ming (Definition 2.1), we first prove that every fuzzy set $\lambda \neq 0$ is decomposed by two fuzzy sets $\lambda_{\mathcal{O}(X,\sigma^f)}$ and $\lambda^*_{\mathcal{PC}(X,\sigma^f)}$ (Theorem A;cf. Theorem 2.5(ii)), where (X,σ^f) is a specified Chang's fuzzy space (Definition 1.2, Remarks 1.3,1.4). Namely, $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$ and $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ hold, and the fuzzy set $\lambda_{\mathcal{O}(X,\sigma^f)}$ is fuzzy open in (X,σ^f) (Theorem 2.5(iii)). Finally, these results are applied to the case where $X = \mathbb{Z}^n (n > 0)$ and $\sigma^f = (\kappa^n)^f$ (Theorem 3.3 and Theorem 3.5), where the topological space (X,σ) is the digital n-space $(\mathbb{Z}^n, \kappa^n)$ (cf. Section 3).

1 Introduction and preliminaries In 1965, Zadeh [26] introduced the fundamental concept of fuzzy sets, which formed the backbone of fuzzy mathematics. After his works, Chang [4] used them to introduce the concept of a fuzzy topology. Throughout the present paper, the symbol I will denote the unit interval [0, 1] and Y a nonempty set. A *fuzzy set* on Y ([26]) is a function with domain Y and values in I, i.e., an element of I^Y.

We recall some concepts and properties as follows. Let (Y,τ_Y) be a Chang's fuzzy topological space [4].

Definition 1.1 (C.L. Chang [4, Definition 2.2]) A *Chang's fuzzy topological space* is a pair (Y,τ_Y), where Y is a non-emptyset and τ_Y is a *Chang's fuzzy topology* on it, where $\tau_Y \subset I^Y$, i.e., a family τ_Y of fuzzy sets satisfying the following three axioms:
(1) $0, 1 \in \tau_Y$;
(2) if $\lambda \in \tau_Y$ and $\mu \in \tau_Y$, then $\lambda \wedge \mu \in \tau_Y$;
(3) let J be an index set. If $\lambda_j \in \tau_Y$ for each $j \in J$, then $\bigvee\{\lambda_j | j \in J\} \in \tau_Y$.
The elements of τ_Y are called *fuzzy open sets* of (X,τ_Y). A fuzzy set μ is called a *fuzzy closed set* of (Y,τ_Y) if the complement $\mu^c \in \tau_Y$.

For a Chang's fuzzy topological space (Y,τ_Y), a fuzzy set μ on Y is said to be *fuzzy preopen* [23] if $\mu \leq$Int(Cl(μ)) holds in (Y,τ_Y). The fuzzy complement of a fuzzy preopen set is said to be *fuzzy preclosed*. Namely, a fuzzy set λ is fuzzy preclosed in (Y,τ_Y) if and only if Cl(Int(λ)) $\leq \lambda$ holds in (Y,τ_Y). A fuzzy set λ is said to be *fuzzy semi-open* [1] in (Y,τ_Y) if there exists a fuzzy open set ν on Y such that $\nu \leq \lambda \leq$Cl(ν) holds in (Y,τ_Y). It is well known that a fuzzy set λ is *fuzzy semi-open* if and only if $\lambda \leq$Cl(Int(λ)). For a subset A of X, χ_A denotes the characteristic function of A, i.e., $\chi_A(y) := 1$ if $y \in A$ and $\chi_A(y) := 0$ if $y \notin A$. The concept of the *ordinary preopen sets* (resp. *ordinary semi-open sets*) was introduced by [21] (resp. [17], [10]).

Definition 1.2 (e.g., [19, Example II, p.244], [8, p.161]) Let (X,σ^f) be a *fuzzy topological space induced by a topological space* (X,σ), where X is a nonempty set and $\sigma^f := \{\chi_U | U \in \sigma\}$; (X,σ^f) is an example of a Chang's fuzzy topological space [4] (cf. Definition 1.1 above).

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There is a bijection, say f, between σ and σ^f which is defined by $f(U) = \chi_U$ for every $U \in \sigma$, because an ordinary subset U is open in (X, σ) (i.e., $U \in \sigma$) if and only if the characteristic function χ_U is fuzzy open in (X, σ^f)(i.e., $\chi_U \in \sigma^f$). However, the below Remark 1.3 and Remark 1.4 show that the fuzzy topology σ^f has some interesting and distinct properties comparing the given ordinary topology σ.

Let $SO(X, \sigma)$ (resp. $FSO(X, \sigma^f)$) denote the family of all ordinary semi-open sets (resp. fuzzy semi-open sets) in (X, σ) (resp. (X, σ^f)); then $\sigma \subset SO(X, \sigma)$ and $\sigma^f \subset FSO(X, \sigma^f)$ hold. An extension of $f : \sigma \to \sigma^f$ to $SO(X, \sigma)$, say $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$, is well defined by $f_s(A) := \chi_A$ for every $A \in SO(X, \sigma)$. The following Remark 1.3 shows that $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto.

Remark 1.3 For the following topological space (X, σ), the correspondence $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto, where $f_s(V) := \chi_V$ for every set $V \in SO(X, \sigma)$. Let $X := \{a, b, c\}$ and $\sigma := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then, we have $SO(X, \sigma) = \sigma \cup \{\{a, c\}, \{b, c\}\}$; and $\{\chi_U | U \in SO(X, \sigma)\} = f_s(SO(X, \sigma))$. Let λ_c be a fuzzy set on X defined by $\lambda_c(a) = 0, \lambda_c(b) = 1, \lambda_c(c) = t$, where t is a real number with $0 < t < 1$. Then, we see that λ_c is fuzzy semi-open in (X, σ^f), i.e., $\lambda_c \in FSO(X, \sigma^f)$). Indeed, there exists a fuzzy open set $\chi_{\{b\}}$ such that $\chi_{\{b\}} \leq \lambda_c \leq \mathrm{Cl}(\chi_{\{b\}})$ hold in (X, σ^f), because $\mathrm{Cl}(\chi_{\{b\}}) = \chi_{Cl(\{b\})} = \chi_{\{b,c\}}$ hold. Since $\lambda_c(c) = t$ and $0 < t < 1$, we see that $\lambda_c \neq \chi_A$ for any set $A \subset X$; and so $\lambda_c \notin f_s(SO(X, \sigma))$. Namely, $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto.

We find an alternative example in [19, (3.5),(III-11)] which is shown on the digital plane $(X, \sigma) = (\mathbb{Z}^2, \kappa^2)$. And, by Remark 3.6 in Section 3, it's general version for the digital n-space $(\mathbb{Z}^n, \kappa^n)$ is given.

The below Remark 1.4 shows that a property for a topological space (X, σ) does not be hereditary to (X, σ^f). In order to explain it, we recall some definitions and properties $(*$ 1)-$(*$ 3) as follows.

In 1970, the concept of $T_{1/2}$-spaces (cf. ($*$3) below) was studied initiately by Levine [18] by introducing the concept of *generalized closed sets* for a topological space. The work on generalized closed sets and their related works are developing by many authors until now. A subset A of (X, σ) is said to be *generalized closed* [18, Definition 2.1] in (X, σ), if $\mathrm{Cl}(A) \subset O$ holds in (X, σ) whenever $A \subset O$ and O is open in (X, σ). The complement of a generalized closed set of (X, σ) is called *generalized open* [18, Definition 4.1] in (X, σ). It is well known that:

($*$1) ([18, Theorem 2.4]) the union of two "generalized closed sets" is "generalized closed"; and

($*$2) ([18, Example 2.5]) the intersection of two "generalized closed sets" is generally not "generalized closed". Moreover, it is well known that every closed set is generalized closed.

($*$3) A topological space (X, σ) is said to be $T_{1/2}$ [18, Definition 5.1] if every "generalized closed set" of (X, σ) is closed in (X, σ). By Dunham [6], it was proved that a topological space (X, σ) is $T_{1/2}$ if and only if, for each point $x \in X, \{x\}$ is open or closed ([6, Theorem 2.5]).

In 1970, E. Khalimsky [11] studied initiately the concept of the digital line $(\mathbb{Z}, \kappa)$ and it is also called the *Khalimsky line* (e.g., Section 3 below; cf. [13] and references there, [12], [14, p.905, line -5],[15, p.175]; e.g., [7]). The digital line $(\mathbb{Z}, \kappa)$ is an interesting and importante example of the $T_{1/2}$-topological space ([5, Example 4.6]) and, moreover, $(\mathbb{Z}, \kappa)$ is a $T_{3/4}$-space ([5, Definition 4, Theorem 4.1]).

Remark 1.4 The digital line $(\mathbb{Z}, \kappa)$ is a $T_{1/2}$-topological space ([5, Example 4.6]); however the induced fuzzy topological space $(\mathbb{Z}, \kappa^f)$ from $(\mathbb{Z}, \kappa)$ is not fuzzy $T_{1/2}$ ([8, Example 4.8]). Here, a fuzzy topological space (Y, τ_Y) is said to be fuzzy $T_{1/2}$ [2] if every fuzzy generalied closed set is fuzzy closed. The above property shows that the property on such separation axiom for a topological space (X, σ) does not be hereditary to the corresponding fuzzy separation axiom for (X, σ^f) even if there is a bijectin $f : \sigma \to \sigma^f$.

One of the purposes in the present paper is to prove the following Theorem A using some properties on (X,σ^f) in Section 2 below. Roughly speaking, when a fuzzy set on X, say λ, is given, then we can consider a decomposition such that $\lambda=\lambda_1\vee\lambda_2(\lambda_1\wedge\lambda_2=0)$ and λ_1 and λ_2 are two fuzzy sets characterized from an induced and specified fuzzy topological space (X,σ^f), where σ is a topology of X. And so, let $\lambda\in I^X$ be a given fuzzy set on X; when we choice many topologies on X, say σ,σ',...., we can get many decompositions of the fuzzy set λ , which are characterized from the induced and specified fuzzy topologies on X, say σ^f, $(\sigma')^f$,...., respectively. Some analogous decomposition properties of a fuzzy set are investigated by [19, Theorem 3.1, Corollary 3.7] and [9, Corollary 2.9, Theorem 3.6].

Theorem A (Theorem 2.5 (ii) in Section 2 below) *Let $\lambda\in I^X$ be a fuzzy set such that $\lambda\neq 0$. Let (X,σ^f) be a fuzzy topological space induced by (X,σ). Then, we have the following decomposition of λ:*
$\lambda=\lambda_{\mathcal{O}(X,\sigma^f)}\vee\lambda^*_{\mathcal{PC}(X,\sigma^f)}$ *and* $\lambda_{\mathcal{O}(X,\sigma^f)}\wedge\lambda^*_{\mathcal{PC}(X,\sigma^f)}=0$.

In Section 3 we have the explicite form of $\lambda_{\mathcal{O}(\mathbb{Z}^n,(\kappa^n)^f)}$ and $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ for the case where $(X,\sigma)=(\mathbb{Z}^n,\kappa^n)$ and $(X,\sigma^f)=(\mathbb{Z}^n,(\kappa^n)^f)$ (cf. Corollary 3.1, Theorem 3.5 below).

2 Proof of Theorem A In the present section we prove Theorem A. We need the concept of fuzzy points in the sense of Pu Pao-Ming and Liu Ying-Ming (Definition 2.1 below), the following notations (Notation I below) and a result (Theorem B below).

In the present paper, for the concept of fuzzy points, we adopt Pu's definition of a fuzzy point in the sense of ([22]).

Definition 2.1 (Pu Pao-Ming and Liu Ying-Ming [22, Definition 2.1], e.g., [19, Definition 1.3]) A fuzzy set on a set Y is said to be *fuzzy point* if it takes the value 0 for all point $y\in Y$ except one point, say $x\in Y$. If it value at x is a $(0<a\le 1)$, we denote this fuzzy point by x_a. We note that $\mathrm{supp}(x_a)=\{a\}$ holds and $0<a\le 1$. Namely, for a point $x\in Y$ and a real number $a\in I$ such that $0<a\le 1$,
- a *fuzzy point* $x_a\in I^Y$ is a fuzzy set defined as, for any point $y\in Y, x_a(y):=a$ if $y=x; x_a(y):=0$ if $y\neq x$.

Notation I. For a Chang's fuzzy topological space (Y,τ_Y),
(i) $FPO(Y,\tau_Y):=\{\lambda\in I^Y|\ \lambda$ is fuzzy preopen in $(Y,\tau_Y)\}$,
$FPC(Y,\tau_Y):=\{\lambda\in I^Y|\ \lambda$ is fuzzy preclosed in $(Y,\tau_Y)\}$.
Namely, by definition, $FPO(Y,\tau_Y)=\{\lambda\in I^Y\ |\lambda\le Int(Cl(\lambda))$ holds in $(Y,\tau_Y)\}$ and $FPC(Y,\tau_Y)=\{\lambda\in I^Y|\ Cl(Int(\lambda))\le\lambda$ holds in $(Y,\tau_Y)\}$.
(ii) For a fuzzy set $\lambda\in I^Y$ such that $\lambda\neq 0$ (i.e., $\mathrm{supp}(\lambda):=\{x\in Y|\lambda(x)\neq 0\}\neq\emptyset$),
$O(\lambda):=\{y\in\mathrm{supp}(\lambda)|\ y_{\lambda(y)}\in\tau_Y\}$,
$PC(\lambda):=\{y\in\mathrm{supp}(\lambda)|\ y_{\lambda(y)}\in FPC(Y,\tau_Y)\}$,
$PC^*(\lambda):=\{y\in\mathrm{supp}(\lambda)|\ y_{\lambda(y)}\in FPC(Y,\tau_Y)$ and $y_{\lambda(y)}\notin\tau_Y\}$.

In the category of fuzzy topological spaces (X,σ^f) induced by topological spaces (X,σ), we know the following theorem [19], say Theorem B in the present paper:
Theorem B (i) ([19, (3.6)(i)]) *Every fuzzy point x_a is fuzzy open or fuzzy preclosed in (X,σ^f). Namely, for every fuzzy point x_a, we have $x_a\in\sigma^f\cup FPC(X,\sigma^f)$.*
(ii) ([19, (3.6)(ii)]) *A fuzzy point x_a is fuzzy open in (X,σ^f) if and only if $a=1$ and $\{x\}$ is open in (X,σ).*
(iii) ([19, (3.2)]) *For a fuzzy set λ on X, $Cl(\lambda)=\chi_{Cl(supp(\lambda))}$ holds in (X,σ^f); and $Int(\lambda)=\chi_{Int(\lambda^{-1}(\{1\}))}$ holds in (X,σ^f).* □

Theorem B (i) above is a fuzzy version of the following property:([3, Lemma 2.4]) *for a topological space (X,σ), every singleton $\{x\}$ is open or preclosed in (X,σ).*

For a fuzzy set λ on Y and a fuzzy topological space (Y, τ_Y), we define three fuzzy sets $\lambda_{\mathcal{O}(Y,\tau_Y)}, \lambda_{\mathcal{PC}(Y,\tau_Y)}$ and $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}$ as follows.

Definition 2.2 Let $\lambda \in I^Y$ be a fuzzy set such that $\lambda \neq 0$ and (Y, τ_Y) a Chang's fuzzy topological space. The following fuzzy sets are well defined: for λ above,

(i) $\lambda_{\mathcal{O}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in \tau_Y\}$ if $O(\lambda) \neq \emptyset$; $\lambda_{\mathcal{O}(Y,\tau_Y)} := 0$ if $O(\lambda) = \emptyset$;

(ii) $\lambda_{\mathcal{PC}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in FPC(Y, \tau_Y)\}$ if $PC(\lambda) \neq \emptyset$; $\lambda_{\mathcal{PC}(Y,\tau_Y)} := 0$ if $PC(\lambda) = \emptyset$,

(iii) $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ \ x_{\lambda(x)} \in FPC(Y, \tau_Y)$ and $x_{\lambda(x)} \notin \tau_Y\}$ if $PC^*(\lambda) \neq \emptyset$; $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} := 0$ if $PC^*(\lambda) = \emptyset$.

Lemma 2.3 *Let λ be a fuzzy set in Y such that $\lambda \neq 0$, i.e., supp$(\lambda) \neq \emptyset$ and (Y, τ_Y) a Chang's fuzzy topological space. Then, we have the following properties:*

(i) *$\lambda_{\mathcal{O}(Y,\tau_Y)} = 0$ holds if and only if $x_{\lambda(x)} \notin \tau_Y$ for each point $x \in$supp(λ)* (i.e., $O(\lambda) = \emptyset$).

(ii) *$\lambda^*_{\mathcal{PC}(Y,\tau_Y)} = 0$ if and only if $x_{\lambda(x)} \notin FPC(Y, \tau_Y)$ or $x_{\lambda(x)} \in \tau_Y$ for each point $x \in$supp(λ)* (i.e., $PC^*(\lambda) = \emptyset$).

(iii) (a) *If $O(\lambda) \neq \emptyset$, then $\lambda_{\mathcal{O}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in O(\lambda)\}$.*

(b) *If $PC(\lambda) \neq \emptyset$, then $\lambda_{\mathcal{PC}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in PC(\lambda)\}$.*

(c) *If $PC^*(\lambda) \neq \emptyset$, then $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$.*

(iv) *$\lambda^*_{\mathcal{PC}(Y,\tau_Y)} \leq \lambda_{\mathcal{PC}(Y,\tau_Y)} \leq \lambda$ hold.*

Proof. **(i)** (**Necessity**) Suppose that there exists a point $z \in$supp(λ) such that $z_{\lambda(z)} \in \tau_Y$. Then, $O(\lambda) \neq \emptyset$. For the point z we set $\mathcal{A}_z := \{x_{\lambda(x)}(z) \in I | x_{\lambda(x)} \in \tau_Y\}$; and so $\mathcal{A}_z \neq \emptyset$. Then, by Definition 2.2 (i), $(\lambda_{\mathcal{O}(Y,\tau_Y)})(z) = \sup \mathcal{A}_z$ and so $\lambda_{\mathcal{O}(Y,\tau_Y)}(z) = \sup\{\lambda(z), 0\} = \lambda(z)$. Indeed, $x_{\lambda(x)}(z) = \lambda(z)$ or 0. Thus we have $\lambda_{\mathcal{O}(Y,\tau_Y)} \neq 0$; this contradicts the assumption. (**Sufficiency**) The proof is obtained by Definition 2.2 (i). **(ii)** The sufficiency is obtained by Definition 2.2 (iii). (**Necessity**) Suppose that there exists a point $z \in$supp(λ) such that $z_{\lambda(z)} \in FPC(Y, \tau_Y)$ and $z_{\lambda(z)} \notin \tau_Y$. Then, $PC^*(\lambda) \neq \emptyset$. For the point z, we set $\mathcal{B}^*_z :=\{x_{\lambda(x)}(z) \in I | x_{\lambda(x)} \in FPC(Y, \tau_Y)$ and $x_{\lambda(x)} \notin \tau_Y\}$ and note $\mathcal{B}^*_z \neq \emptyset$. Then $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) = \sup \mathcal{B}^*_z$. Since $x_{\lambda(x)}(z) = \lambda(z)$ or 0 and $z \in$supp(λ) we have $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) = \sup\{\lambda(z), 0\} = \lambda(z)$ and hence $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) > 0$ for the point z. Namely, we have $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} \neq 0$; this contradicts the assumption. **(iii)** By using definitions (cf. Notation I, Definition 2.2), it is shown that $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in \tau_Y\} = \{x_{\lambda(x)} |\ x \in O(\lambda)\}$, $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(Y, \tau_Y)\}$=$\{x_{\lambda(x)} |\ x \in PC(\lambda)\}$ and $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(Y, \tau_Y), x_{\lambda(x)} \notin \tau_Y\}$=$\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$ hold. Thus we have the required equalities. **(iv)** It is obvious that supp$(\lambda) \supset PC(\lambda) \supset PC^*(\lambda)$ (cf. Notation above). Therefore, we have that $\lambda \geq \lambda_{\mathcal{PC}(Y,\tau_Y)} \geq \lambda^*_{\mathcal{PC}(Y,\tau_Y)}$, because $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in$supp$(\lambda)\}$ holds ([22, Definition 2.2]; e.g., [16, Lemma 2.1], [19, Lemma 2.5(i)]) and the equalities (b) and (c) hold in (iii) above. □

Theorem 2.4 *Let $\lambda \in I^X$ be a fuzzy set such that $\lambda \neq 0$. For a fuzzy topological space (X, σ^f) induced by a topological space (X, σ), $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ if and only if $\lambda = \lambda^*_{\mathcal{PC}(X,\sigma^f)} = \lambda_{\mathcal{PC}(X,\sigma^f)}$ hold.*

Proof. (**Necessity**) It follows from assumption and Lemma 2.3(i) that $x_{\lambda(x)} \notin \sigma^f$ for every point $x \in$supp(λ). Thus, by Theorem B(i) above, it is shown that, for every point $x \in$supp$(\lambda), x_{\lambda(x)}$ is fuzzy preclosed in (X, σ^f). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in$supp$(\lambda)\} = \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(X, \sigma^f)$ and $x_{\lambda(x)} \notin \sigma^f\} = \lambda^*_{\mathcal{PC}(X,\sigma^f)}$. Therefore, using Lemma 2.3(iv), we conclude that $\lambda = \lambda^*_{\mathcal{PC}(X,\sigma^f)} = \lambda_{\mathcal{PC}(X,\sigma^f)}$ hold. (**Sufficiency**) Assume that $\lambda = \lambda_{\mathcal{PC}(X,\sigma^f)}$=$\lambda^*_{\mathcal{PC}(X,\sigma^f)}$ hold. We recall that $\lambda^*_{\mathcal{PC}(X,\sigma^f)}$= $\bigvee\{x_{\lambda(x)} | x_{\lambda(x)} \in FPC(X, \sigma^f)$ and $x_{\lambda(x)} \notin \sigma^f\}$= $\bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$ (cf. Lemma 2.3 (iii)). Suppose $PC^*(\lambda) = \emptyset$.

Then, $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(iii)); and so we have $\lambda = 0$; this contradicts the assumption on λ (i.e., supp$(\lambda) \neq \emptyset$). Thus, we consider the case where $PC^*(\lambda) \neq \emptyset$ for λ. We claime that supp$(\lambda) \subset PC^*(\lambda)$. Indeed, let w be any point such that $w \notin PC^*(\lambda)$. Then, for each point $x \in PC^*(\lambda)$, we have $x_{\lambda(x)}(w) = 0$, because of $w \neq x$. Here, we put $\mathcal{B}^*_w := \{x_{\lambda(x)}(w) \in I | x \in PC^*(\lambda)\}$; then $\mathcal{B}^*_w = \{0\}$; and so we have $(\lambda^*_{\mathcal{PC}(X,\sigma^f)})(w) =$sup $\mathcal{B}^*_w = 0$. By using the assumption of the present Sufficiency, it is shown that $\lambda(w) = 0$ and so $w \notin$supp(λ). Therefore, we show supp$(\lambda) \subset PC^*(\lambda)$. Therefore, we have $x_{\lambda(x)} \notin \sigma^f$ for every point $x \in$supp(λ), because of $x \in PC^*(\lambda)$. By Lemma 2.3(i), it is obtained that $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$. □

We shall prove Theorem A as follows; Theorem A is included in Theorem 2.5 below (i.e., Theorem 2.5 (ii)). First we recall the following notation:
Notation II: for a topological space (X, σ) and a subset E of X,
let $X_\sigma := \{x \in X | \ \{x\} \in \sigma\}$; and $E_\sigma := E \cap X_\sigma$. It is obvious that E_σ is open in (X, σ) for any subset $E \subset X$.
Notation III : for a fuzzy set λ on X and a topological space (X, σ),
(i) $\lambda^{-1}(\{1\}) := \{y \in X | \ \lambda(y) = 1\}$; then $\lambda^{-1}(\{1\})$ is a subset of X, because $\lambda \in I^X$;
(ii) $(\lambda^{-1}(\{1\}))_\sigma := \lambda^{-1}(\{1\}) \cap X_\sigma$ (i.e., $(\lambda^{-1}(\{1\}))_\sigma = \{y | \ y \in \lambda^{-1}(\{1\}), \{y\}$ is open in $(X, \sigma)\}$).

Theorem 2.5 *Let $\lambda \in I^X$ be a fuzzy set such that $\lambda \neq 0$. Let (X, σ) be a topological space and (X, σ^f) a fuzzy topological space induced by (X, σ). Then, we have the following properties of λ:*
(i) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.
(ii) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$ *and* $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.
(iii) $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$, *where* $E := X_\sigma \cap \lambda^{-1}(\{1\}) = (\lambda^{-1}(\{1\}))_\sigma$; $\lambda_{\mathcal{O}(X,\sigma^f)}$ *is fuzzy open in* (X, σ^f).

Proof. We first recall the following $(*^1)$ with Notation I and we claim the following properties $(*^2)$ and $(*^3)$:
$(*^1)$ supp$(\lambda) \supset PC(\lambda) \supset PC^*(\lambda)$ and supp$(\lambda) \supset O(\lambda)$ hold in (X, σ) (cf. Notation I);
$(*^2)$ supp$(\lambda) = O(\lambda) \cup PC(\lambda)$ holds in (X, σ);
$(*^3)$ supp$(\lambda) = O(\lambda) \cup PC^*(\lambda)$ and $O(\lambda) \cap PC^*(\lambda) = \emptyset$ hold in (X, σ).

Proof of $(*^2)$. By Theorem B, it is shown that, for a point $x \in$ supp(λ), the fuzzy point $x_{\lambda(x)}$ is fuzzy open or fuzzy preclosed in (X, σ^f), i.e., $x_{\lambda(x)} \in \sigma^f$ or $x_{\lambda(x)} \in FPC(\lambda)$. Thus, for a point $x \in$supp $(\lambda), x \in O(\lambda)$ or $x \in PC(\lambda)$; and so we have supp$(\lambda) \subset O(\lambda) \cup PC(\lambda)$. Since $O(\lambda) \subset$supp(λ) and $PC(\lambda) \subset$supp(λ), we have the required equality $(*^2)$. $(\diamond)$

Proof of $(*^3)$. By definition, it is easily shown that $PC^*(\lambda) \subset PC(\lambda)$. And, we have $PC^*(\lambda) = \{y \in$supp$(\lambda) | \ y_{\lambda(y)} \in FPC(X, \sigma^f)\} \cap \{y \in$supp$(\lambda) | \ y_{\lambda(y)} \notin \sigma^f\} =PC(\lambda) \cap$[supp $(\lambda) \setminus O(\lambda)]$; and so $PC^*(\lambda) = PC(\lambda) \cap$ [supp $(\lambda) \setminus O(\lambda)]$. Thus, we have $PC^*(\lambda) \cup O(\lambda) = [PC(\lambda) \cap ($supp$(\lambda) \setminus O(\lambda)] \cup O(\lambda) =$supp$(\lambda)$ (cf. $(*^2)$) and $PC^*(\lambda) \cap O(\lambda) \subset PC(\lambda) \cap [X \setminus O(\lambda)] \cap O(\lambda) = \emptyset$. $\diamond$
In the finnal stage, we prove (i), (ii) and (iii) as follows.

(i). For the proof of (i) we consider the following three cases. And it is well known that $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in$supp$(\lambda)\}$ holds (cf. [22, Definition 2.2], e.g., [16, lemma 2.2],[19, Lemma 2.5(i)]).

Case 1. $O(\lambda) \neq \emptyset, PC(\lambda) \neq \emptyset$: for this case, using $(*^2)$ above and Lemma 2.3 (iii), we have $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in$supp$(\lambda)\}$= $(\bigvee\{x_{\lambda(x)} | \ x \in O(\lambda)\}) \vee (\bigvee\{x_{\lambda(x)} | \ x \in PC(\lambda)\}$=$\lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.

Case 2. $O(\lambda) \neq \emptyset, PC(\lambda) = \emptyset$: for this case, we have $\lambda_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(ii)) and supp$(\lambda) = O(\lambda)$ (cf. $(*^2)$ above). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in$supp$(\lambda)\}$=$\bigvee\{x_{\lambda(x)} | \ x \in O(\lambda)\}$=$\lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$, because $\lambda_{\mathcal{PC}(X,\sigma^f)} = 0$.

Case 3. $O(\lambda) = \emptyset$: for this case, by ($*^2$) above and Lemma 2.3(i), it is shown that $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ and $\mathrm{supp}(\lambda) = PC(\lambda)$; and so $PC(\lambda) \neq \emptyset$, because of $\lambda \neq 0$. Thus, we have $\lambda = \bigvee\{x_{\lambda(x)}|\ x \in \mathrm{supp}(\lambda)\} = 0 \vee (\bigvee\{x_{\lambda(x)}|\ x \in PC(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.
Therefore, we show that the equality (i) holds for all cases.

(ii). Since $\mathrm{supp}(\lambda) = O(\lambda) \cup PC^*(\lambda)$ (cf. ($*^3$)), we are able to conclude that (ii-1) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$; and (ii-2) $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

Proof of (ii-1). We consider the following three cases for the proof.

Case 1. $O(\lambda) \neq \emptyset, PC^*(\lambda) \neq \emptyset$: for this case, using ($*^3$) above and Lemma 2.3 (iii), we have $\lambda = \{x_{\lambda(x)}|\ x \in \mathrm{supp}(\lambda)\} = (\bigvee\{x_{\lambda(x)}|\ x \in O(\lambda)\}) \vee (\bigvee\{x_{\lambda(x)}|\ x \in PC^*(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$.

Case 2. $O(\lambda) \neq \emptyset, PC^*(\lambda) = \emptyset$: for this case, we have $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(iii)) and $\mathrm{supp}(\lambda) = O(\lambda)$ (cf. ($*^3$) above). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)}|\ x \in \mathrm{supp}(\lambda)\} = \bigvee\{x_{\lambda(x)}|\ x \in O(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$, because $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

Case 3. $O(\lambda) = \emptyset$: for this case, we have $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ (cf. Definition 2.2(i)). By ($*^3$), it is shown that $\mathrm{supp}(\lambda) = PC^*(\lambda)$; and so $PC^*(\lambda) \neq \emptyset$, because of $\lambda \neq 0$. Thus, we have $\lambda = \bigvee\{x_{\lambda(x)}|\ x \in \mathrm{supp}(\lambda)\} = 0 \vee (\bigvee\{x_{\lambda(x)}|x \in PC^*(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PO}(X,\sigma^f)}$. ($\diamond$)

Proof of (ii-2). For a point $y \in X$, we claim that $(\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)})(y) = 0$; i.e., $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$. For the point y, we consider the following two cases.

Case 1. $y \in O(\lambda)$: for this point y, we have $y \notin PC^*(\lambda)$ (cf. ($*^3$) before the proof of (i) above). Then, we have that $y \neq x$ for each $x \in PC^*(\lambda)$, i.e., $x_{\lambda(x)}(y) = 0$ for each $x \in PC^*(\lambda)$. Thus, if $PC^*(\lambda) \neq \emptyset$, then $\lambda^*_{\mathcal{PC}(X,\sigma^f)}(y) = (\bigvee\{x_{\lambda(x)}|\ x \in PC^*(\lambda)\})(y)$ $=\sup\{x_{\lambda(x)}(y)|\ x \in PC^*(\lambda)\} = \sup\{0\} = 0$ (cf. Lemma 2.3(iii)(c)). And, if $PC^*(\lambda) = \emptyset$, then $\lambda^*_{\mathcal{PC}(X,\sigma^f)}(y) = 0$ (cf. Definition 2.2(iii)). Thus, for this Case 1, we show that $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$.

Case 2. $y \notin O(\lambda)$: for the point y, we have that $x \neq y$ for each point $x \in O(\lambda)$; and so $x_{\lambda(x)}(y) = 0$ for each point $x \in O(\lambda)$. Thus, if $O(\lambda) \neq \emptyset$, then $\lambda_{\mathcal{O}(X,\sigma^f)}(y) = (\bigvee\{x_{\lambda(x)}|\ x \in O(\lambda)\})(y) = \sup\{x_{\lambda(x)}(y)|\ x \in O(\lambda)\} = \sup\{0\} = 0$ (cf. Lemma 2.3(iii)(a)). And, if $O(\lambda) = \emptyset$, then $\lambda_{\mathcal{O}(X,\sigma^f)}(y) = 0$ (cf. Definition 2.2(i)). Thus, for this Case 2, we show that $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$.
Therefore we prove $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

(iii). By Theorem B(ii) in the top of the present section, it is well known that a fuzzy point x_a is fuzzy open in (X, σ^f) if and only if $a = 1$ and $\{x\}$ is open in (X, σ). For a point $x \in \mathrm{supp}(\lambda), \lambda(x) > 0$ and so a fuzzy point $x_{\lambda(x)}$ is well defined. Thus, we have that $x_{\lambda(x)}$ is fuzzy open in (X, σ^f) (i.e., $x_{\lambda(x)} \in \sigma^f$) if and only if $\lambda(x) = 1$ and $\{x\}$ is open in (X, σ) (i.e., $x \in E := \lambda^{-1}(\{1\}) \cap X_\sigma$, cf. Notation II, Notation III). Therefore, if $E \neq \emptyset$, then we have that $\lambda_{\mathcal{O}(X,\sigma^f)} = \bigvee\{x_{\lambda(x)}|\ x_{\lambda(x)} \in \sigma^f\} = \bigvee\{x_{\lambda(x)}|\ x \in \lambda^{-1}(\{1\}) \cap X_\sigma\} = \bigvee\{x_1|\ x \in E\}$ $=\bigvee\{\chi_{\{x\}}|\ x \in E\} = \chi_F = \chi_E$, where $F = \bigcup\{\{x\}|\ x \in E\}$, and hence $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$.
If $E = \emptyset$, then $O(\lambda) := \{y \in \mathrm{supp}(\lambda)|\ y_{\lambda(y)} \in \sigma^f\} = \{y \in \mathrm{supp}(\lambda)|\ \lambda(y) = 1$ and $\{y\} \in \sigma\} = \{y \in \mathrm{supp}(\lambda)|y \in E = \emptyset\} = \emptyset$ and so $\lambda_{\mathcal{O}(X,\sigma^f)} = 0 = \chi_\emptyset$. Therefore, we prove $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$. For the proof of $\lambda_{\mathcal{O}(X,\sigma^f)} \in \sigma^f$, it is evident from the openness of $E := \lambda^{-1}(\{1\}) \cap X_\sigma = (\lambda^{-1}(1))_\sigma$ and the definition of σ^f. □

3 Decompositions of fuzzy sets on $(\mathbb{Z}^n, (\kappa^n)^f)$

Let $(\mathbb{Z}^n, \kappa^n)$ be the digital n-space and $(\mathbb{Z}^n, (\kappa^n)^f)$ a Chang's fuzzy topological space induced from $(\mathbb{Z}^n, \kappa^n)$ (cf. Definition 1.2). In the present section, we have the following decomposition theorem (Corollary 3.1) of a fuzzy set λ on $\mathbb{Z}^n$ by two fuzzy sets χ_E and $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ with fuzzy topological properties in $(\mathbb{Z}^n, (\kappa^n)^f)$ and the precise form of $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ (Theorem 3.5).

We recall that:

• the *digital n-space* $(\mathbb{Z}^n, \kappa^n)$ (e.g., [15, Definition 4],[7]) is the topological product of n-copies of the *digital line* $(\mathbb{Z}, \kappa)$ (cf. this is called the *Khalimsky line* in the contents between

Remark 1.4 and (∗3) in Section 1), where n is an integer with $n \geq 2$. The *digital line* $(\mathbb{Z}, \kappa)$ is the set of the integers, $\mathbb{Z}$, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} \mid m \in \mathbb{Z}\}$ as a subbace (e.g., [15, p.175]). Some joint papers by the one of the present authors include a short survey or frequently used properties on $(\mathbb{Z}^n, \kappa^n)$ where $n \geq 1$ (cf. [20, Section 3], [25], [7]). It is well known that a singleton $\{2m\}$ is closed and not open and $\{2m+1\}$ is open and not closed in $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$; moreover $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$ holds and $\mathrm{Int}(\{2s\}) = \emptyset$ holds in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. We use the following notation (cf. [7, Section 6], [24, Section 2], [25, Definition 2.1], [20, Definition 3.11]): for $n \geq 1$,

- $(\mathbb{Z}^n)_{\kappa^n} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n \mid y_i$ is odd for each integer i with $1 \leq i \leq n\}$; for any element x of $(\mathbb{Z}^n)_{\kappa^n}$, $\{x\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$ (cf. Notation II in Section 2 for $X := \mathbb{Z}^n$ and $\sigma := \kappa^n$);
- $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n \mid y_i$ is even for each integer i with $1 \leq i \leq n\}$; for any element x of $(\mathbb{Z}^n)_{\mathcal{F}^n}$, $\{x\}$ is a closed singleton of $(\mathbb{Z}^n, \kappa^n)$;
- $(\mathbb{Z}^n)_{mix(r)} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n \mid r = \#\{i \in \{1, 2, ..., n\} \mid y_i$ is even$\}\}$, where $1 \leq r \leq n$ and $\#A$ denotes the cardinality of a set A. Especially, for the case where $r = n$, we note $(\mathbb{Z}^n)_{mix(n)} = (\mathbb{Z}^n)_{\mathcal{F}^n}$.
- For a nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$, the following subsets $E_{\kappa^n}, E_{\mathcal{F}^n}$ and $E_{mix(r)}$ are well defined as follows: $E_{\kappa^n} := E \cap (\mathbb{Z}^n)_{\kappa^n}$, $E_{\mathcal{F}^n} := E \cap (\mathbb{Z}^n)_{\mathcal{F}^n}$, $E_{mix(r)} := E \cap (\mathbb{Z}^n)_{mix(r)}$ $(1 \leq r \leq n)$. Namely, we have that $E_{\kappa^n} := \{x \in E \mid \{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)\} \subset E$ and $E_{\mathcal{F}^n} := \{x \in E \mid \{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)\} \subset E$; and E_{κ^n} is an open subset of $(\mathbb{Z}^n, \kappa^n)$.

First we apply Theorem 2.5 to the digital n-space $(\mathbb{Z}^n, \kappa^n)$; then we have the following corollary of Theorem 2.5.

Corollary 3.1 *Let $\lambda \in I^{\mathbb{Z}^n}$ be a fuzzy set on $\mathbb{Z}^n$ such that $\lambda \neq 0$. Then, we have the following properties.*

(i) $\lambda_{\mathcal{O}(\mathbb{Z}^n, (\kappa^n)^f)} = \chi_E$, *where* $E := (\lambda^{-1}(\{1\}))_{\kappa^n}$.

(ii) *Any fuzzy set λ has a decomposition:* $\lambda = \chi_E \vee \lambda^*_{\mathcal{PC}(\mathbb{Z}^n, (\kappa^n)^f)}$ *and* $\chi_E \wedge \lambda^*_{\mathcal{PC}(\mathbb{Z}^n, (\kappa^n)^f)} = 0$, *where* $E := (\lambda^{-1}(\{1\}))_{\kappa^n}$.

Proof. **(i)** (resp. **(ii)**) By Theorem 2.5(iii) (resp. Theorem 2.5(ii)) for $(X, \sigma) = (\mathbb{Z}^n, \kappa^n)$, (i) (resp. (ii)) is obtained. □

In the below, we shall show an exlicite expression of the fuzzy set $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n, (\kappa^n)^f)}$ above (cf. Theorem 3.5).

Theorem 3.2 *For a fuzzy topological space $(\mathbb{Z}^n, (\kappa^n)^f)$ induced by the digital n-space $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$, and a fuzzy point x_a in $\mathbb{Z}^n$, where $x \in \mathbb{Z}^n$ and $0 < a \leq 1$, we have the following properties.*

(i) (i-1) *Let* $x \in (\mathbb{Z}^n)_{\kappa^n}$ (i.e., $x = (2m_1 + 1, 2m_2 + 1, ..., 2m_n + 1)$, *where* $m_i \in \mathbb{Z} (1 \leq i \leq n)$). *Then,*

$\mathrm{Cl}(x_a) = \chi_{E^o_x}$, *where* $E^o_x := \prod_{i=1}^n \{2m_i, 2m_i + 1, 2m_i + 2\}$.

(i-2) *Let* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ (i.e, $x = (y_1, y_2, ..., y_n)$ *for some even integers* $y_i (1 \leq i \leq n)$). *Then,*

$\mathrm{Cl}(x_a) = \chi_{\{x\}}$.

(i-3) *Suppose that $n \geq 2$. Let* $x := (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(r)} (1 \leq r \leq n-1)$ *and* $E^m(y_i) = \{y_i\}$, *if y_i is even in* $\mathbb{Z}(1 \leq i \leq n)$; $E^m(y_i) = \{y_i - 1, y_i, y_i + 1\}$, *if y_i is odd in* $\mathbb{Z}(1 \leq i \leq n)$. *Then,*

$\mathrm{Cl}(x_a) = \chi_{E^m_x}$, *where* $E^m_x := \prod_{i=1}^n E^m(y_i)$.

(ii) (ii-1) *If* $x \in (\mathbb{Z}^n)_{\kappa^n}$ *and* $a = 1$, *then* $\mathrm{Int}(x_a) = \chi_{\{x\}} = x_a$ *holds.*

(ii-2) *If* $x \in (\mathbb{Z}^n)_{\kappa^n}$ *and* $a \neq 1$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

(ii-3) *If* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

(ii-4) *If* $x \in (\mathbb{Z}^n)_{mix(r)}$ *with* $1 \leq r \leq n-1$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

Proof. **(i) (i-1)** It is well known that $\{x\}$ is an open singleton in $(\mathbb{Z}^n, \kappa^n)$ and $\mathrm{Cl}(\{x\}) = \prod_{i=1}^n \mathrm{Cl}(\{2m_i+1\}) = \prod_{i=1}^n \{2m_i, 2m_i+1, 2m_i+2\} = E_x^o$ in $(\mathbb{Z}^n, \kappa^n)$. Thus, we have $\mathrm{Cl}(x_a) = \chi_{\mathrm{Cl}(\{x\})} = \chi_{E_x^o}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ for a point $x \in (\mathbb{Z}^n)_{\kappa^n}$, because $\mathrm{supp}(x_a) = \{x\}$ (cf. Theorem B (iii)).
(i-2) We have $\mathrm{Cl}(x_a) = \chi_{\mathrm{Cl}(\{x\})} = \chi_{\{x\}}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)) for a point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ (i.e., $\{x\}$ is a closed singleton of $(\mathbb{Z}^n, \kappa^n)$).
(i-3) Let $x = (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(r)} (1 \le r \le n-1)$ (i.e., $r = \#\{i|\ y_i$ is even $\}$). Since $\mathrm{Cl}(\{x\}) = \prod_{i=1}^n \mathrm{Cl}(y_i) = \prod_{i=1}^n E^m(y_i) = E_x^m$ in $(\mathbb{Z}^n, \kappa^n)$, it is shown that $\mathrm{Cl}(x_a) = \chi_{E_x^m}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B(iii)).

(ii) (ii-1) Since $a = 1$, we have $x_a = \chi_{\{x\}}$ and $(x_a)^{-1}(\{1\}) = \{x\}$. And, since $\{x\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$, it is shown that $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(x_1)^{-1}(\{1\})} = \chi_{\mathrm{Int}(\{x\})}$ (cf. Theorem B (iii)).
(ii-2) For this fuzzy point x_a, where $a \ne 1$, we have $(x_a)^{-1}(\{1\}) = \emptyset$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\emptyset)} = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)).
(ii-3) For this fuzzy point x_a, we have $(*)$ $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}((x_a)^{-1}(\{1\}))} = \chi_{\mathrm{Int}(\{x\})}$ if $a = 1$; $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}((x_a)^{-1}(\{1\}))} = \chi_\emptyset = 0$ if $a \ne 1$ (cf. Theorem B (iii)).
Thus, we show (ii-3) for the case where $a = 1$ only. Since $\mathrm{Int}(\{x\}) = \emptyset$ in $(\mathbb{Z}^n, \kappa^n)$ for this point x. we have $\mathrm{Int}(x_1) = \chi_{\mathrm{Int}(\{x\})} = \chi_\emptyset = 0$ (cf. Theorem B (iii)).
(ii-4) For this point x, say $x = (y_1, y_2, ..., y_n)$, there exists even integers, say $y_{i(e)} (1 \le e \le r)$, where $\{i(1), i(2), ..., i(r)\} \subset \{1, 2, ..., n\}$, because $1 \le r \le n-1$ and $r = \#\{i|1 \le i \le n, y_i$ is even$\}$; and $\mathrm{Int}(\{y_{i(e)}\}) = \emptyset$ for each e with $1 \le e \le r$ in $(\mathbb{Z}, \kappa)$. Then, we have $\mathrm{Int}(\{x\}) = \prod_{j=1}^n \mathrm{Int}(y_j) = \emptyset$ in $(\mathbb{Z}^n, \kappa^n)$. Thus, if $a = 1$, then $\mathrm{supp}(x_a) = (x_1)^{-1}(\{1\}) = \{x\}$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\mathrm{supp}(x_1))} = \chi_{\mathrm{Int}(\{x\})} = \chi_\emptyset = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$; if $a \ne 1$, then $\mathrm{supp}(x_a) = (x_a)^{-1}(\{1\}) = \emptyset$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\mathrm{supp}(x_a))} = \chi_\emptyset = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)). Therefore, for this fuzzy point x_a, we show $\mathrm{Int}(x_a) = 0$. □

Theorem 3.3 *A fuzzy point x_a is fuzzy open, otherwise x_a is fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$.*

Proof. In general, by Theorem B(i) in Section 2, every fuzzy point is fuzzy open or fuzzy preclosed in (X, σ^f), where (X, σ) is a topological space. Then we prove only that non-existence of fuzzy point x_a which is fuzzy open and fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$. Suppose that there exists a fuzzy point x_a such that $x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and $x_a \in (\kappa^n)^f$. Since x_a is fuzzy open in $(\mathbb{Z}^n, (\kappa^n)^f)$, we have $a = 1$ and $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem B(ii) in Section 2). Thus, we can put $x := (2m_1+1, 2m_2+1, ..., 2m_n+1) \in (\mathbb{Z}^n)_{\kappa^n}$. For this point x and fuzzy singleton x_a, where $a = 1$, by Theorem 3.2, Cl(Int(x_a))=$\mathrm{Cl}(x_a) = \chi_{E_x^O}$, where $E_x^O := \prod_{i=1}^n \{2m_i, 2m_i+1, 2m_i+2\}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$. Put $x^+ := (2m_1+2, 2m_2+2, ..., 2m_n+2)$. Then, we have $x \ne x^+$ and so $\mathrm{Cl}(\mathrm{Int}(x_1))(x^+) = \chi_{E_x^O}(x^+) = 1 \not\le x_1(x^+) = 0$; this contradicts $x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Notation I in Section 2). □

Since $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1 \le r \le n-1\})$ (disjoint union), we see obviously that $\mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n} = (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)}|1 \le r \le n-1\})$ holds in the digital n-space $(\mathbb{Z}^n, \kappa^n)$, where $n \ge 2$. And, we see $\mathbb{Z} \setminus \mathbb{Z}_\kappa = \mathbb{Z}_\mathcal{F}$ hold in the digital line $(\mathbb{Z}, \kappa)$.

Corollary 3.4 *Let x_a be a fuzzy point on $\mathbb{Z}^n$, where $0 < a \le 1$. The following properties are equivalent:*
(1) $x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$;
(2) $x \in E$ *or* $0 < a < 1$, *where* $E := \mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n}$;
(2)' $x \notin (\mathbb{Z}^n)_{\kappa^n}$ *or* $a \ne 1$;
(3) $x_a \notin (\kappa^n)^f$ (i.e., x_a *is not fuzzy open in* $(\mathbb{Z}^n, (\kappa^n)^f)$).

Proof. **(1)⇒(2)** Suppose that $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $a = 1$. Then, by Theorem B(ii) in Section 2, x_a is fuzzy open; and hence by Theorem 3.3, x_a is not fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$; this

contradicts the assumption (1). Therefore, we showed that $x \in E$ or $0 < a < 1$. **(2)⇔(2)'** It is obvious.
(2)⇒(3) By Theorem B(ii) in Section 2 for $(X, \sigma) = (\mathbb{Z}^n, \kappa^n), x_a$ is not fuzzy open in $(\mathbb{Z}^n, (\kappa^n)^f)$. **(3)⇒(1)** It is proved by Theorem 3.3. □

Finally we show some explicite forms of $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$.

Theorem 3.5 *Let λ be a fuzzy set on $\mathbb{Z}^n$ with $\lambda \neq 0$. Then, we have the following properties:*
(i) $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ *holds.*
(ii) *If* $\mathrm{supp}(\lambda) \cap (\mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n}) \neq \emptyset$, *then*
(ii-1) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} \neq 0$;
(ii-2) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^n} \mid x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$; *and*
(ii-3) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \mathcal{A}(\lambda)_0 \vee (\bigvee\{\mathcal{A}(\lambda)_r \mid 1 \leq r \leq n\})$, *where*
$\mathcal{A}(\lambda)_0 := \bigvee\{x_{\lambda(x)} \mid x \in (\,\mathrm{supp}(\lambda) \setminus \lambda^{-1}(\{1\}))_{\kappa^n}\}$ *and* $\mathcal{A}(\lambda)_r := \bigvee\{x_{\lambda(x)} \mid x \in (\mathrm{supp}(\lambda))_{mix(r)}\}$ *for each integer r with $1 \leq r \leq n$.*

Proof. **(i)** We consider the following two cases for the proof.
Case 1. $PC^*(\lambda) \neq \emptyset$: by Definition 2.2(iii) and Corollary 3.4(1)⇔(3), it is obtained that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := \bigvee\{x_{\lambda(x)} \mid x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and $x_{\lambda(x)} \notin (\kappa^n)^f\} = \bigvee\{x_{\lambda(x)} \mid x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)\}$. And so, we have $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$, because $PC^*(\lambda) \subset PC(\lambda)$ and $PC(\lambda) \neq \emptyset$ hold.
Case 2. $PC^*(\lambda) = \emptyset$: for this case, $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := 0$ (cf. Notation I in Section 2, Definition 2.2(iii)). We claim that $PC(\lambda) = \emptyset$ holds under the assumption of Case 2 (i.e.,$PC^*(\lambda) = \emptyset$). Suppose that $PC(\lambda) \neq \emptyset$ (cf. Notation I in Section 2, Definition 2.2(ii)). Then, there exists a point of $\mathbb{Z}^n$, say $z \in PC(\lambda)$, and so $z_{\lambda(z)} \in PC(\mathbb{Z}^n, (\kappa^n)^f)$ and, by Theorem 3.3, $z_{\lambda(z)} \notin (\kappa^n)^f$. The above result shows that $z_{\lambda(z)} \in PC^*(\mathbb{Z}^n, (\kappa^n)^f)$ holds, i.e., $z \in PC^*(\lambda)$ (cf. Notation I in Section 2); this contradicts the assumption of Case 2 (i.e., $PC^*(\lambda) = \emptyset$). Thus, we claimed that if $PC^*(\lambda) = \emptyset$ then $PC(\lambda) = \emptyset$. And, under the assumption of Case 2, we show that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := 0 = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ hold.

Therefore, by Case 1 and Case 2, it is proved that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ holds.

(ii) (ii-1) It follows from the assumption of (ii) that there exists a point $z \in \mathrm{supp}(\lambda)$ (i.e., $\lambda(z) > 0$) and $z \notin (\mathbb{Z}^n)_{\kappa^n}$. By Corollary 3.4(2)'⇔ (1), it is obtained that $z_{\lambda(z)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and so $z \in PC(\lambda) \neq \emptyset$ (cf. Notation I). We have that $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} \mid x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)\}$ (cf. Definition 2.2(ii)) and $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}(z) \neq 0$ for the point z, i.e., $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} \neq 0$.

(ii-2) For a fuzzy point $x_{\lambda(x)}$, we have that $\lambda(x) > 0$, i.e., $x \in \mathrm{supp}(\lambda)$. Then, by using definitions and Corollary 3.4 (1)⇔(2)', it is shown that: $x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ if and only if $x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(1))_{\kappa^n}$. By (ii-1) and Definition 2.2(ii), it is shown that: $PC(\lambda) \neq \emptyset$ and so $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} \mid x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$.

(ii-3) We use the well known decomposition of $\mathbb{Z}$: $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} \mid 1 \leq r \leq n\})$(disjoint union) and $(\mathbb{Z}^n)_{mix(n)} = (\mathbb{Z}^n)_{\mathcal{F}^n}$. It follows from assumption that $\mathrm{supp}(\lambda) \neq \emptyset$. We consider the decomposition of $\mathrm{supp}(\lambda)$ in $(\mathbb{Z}^n, (\kappa^n)^f)$:
$\mathrm{supp}(\lambda) = (\mathrm{supp}(\lambda))_{\kappa^n} \cup (\bigcup\{(\mathrm{supp}(\lambda))_{mix(r)} \mid 1 \leq r \leq n\})$; then, we have the following equality in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. the right hand side equality in the end of the proof of (ii-2)):

(•) $\mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n} = (\mathrm{supp}(\lambda) \setminus \lambda^{-1}(1))_{\kappa^n} \cup (\bigcup\{(\mathrm{supp}(\lambda))_{mix(r)})\mid 1 \leq r \leq n\}$.
Then, using (ii-2), the equality (•) above and a property of fuzzy union of fuzzy points (e.g. [19, Lemma 2.5(ii)]), we have that:
$\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} \mid x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$
$= [\bigvee\{x_{\lambda(x)} \mid x \in (\,\mathrm{supp}(\lambda) \setminus \lambda^{-1}(\{1\}))_{\kappa^n}] \vee [\bigvee\{x_{\lambda(x)} \mid x \in (\,\mathrm{supp}(\lambda))_{mix(r)} \mid 1 \leq r \leq n\}]$
$= \mathcal{A}(\lambda)_0 \vee (\bigvee\{\mathcal{A}(\lambda)_r) \mid 1 \leq r \leq n\})$; and hence (ii-3) is proved. □

The following remark is pre-announced in Remark 1.3.

Remark 3.6 (cf. Remark 1.3, [19, (III-12) in Section 3]) The following example also shows that the correspondence $f_s : SO(\mathbb{Z}^n, \kappa^n) \to FSO(\mathbb{Z}^n, (\kappa^n)^f)$ is not onto, even if $f : \kappa^n \to (\kappa^n)^f$ is bijective, where $f_s(U) := \chi_U$ and $f(V) := \chi_V$ for every $U \in SO(\mathbb{Z}^n, \kappa^n)$ and every $V \in \kappa^n$. We choice the follwing subset A as follows:

$A := \{y^{(1)}, y^{(2)}\} \subset \mathbb{Z}^n$, where $y^{(1)} := (2m_1, 2m_2, ..., 2m_n)$ and $y^{(2)} = (2m_1 + 1, 2m_2 + 1, ..., 2m_n + 1)$ for some integers $m_i (1 \le i \le n)$; and so $y^{(1)} \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $y^{(2)} \in (\mathbb{Z}^n)_{\kappa^n}$. Using the subset A, we define the fuzzy set $\lambda_A \in I^{\mathbb{Z}^n}$ as follows:

$\lambda_A(y^{(2)}) := 1, \lambda_A(y^{(1)}) := 1/2$ and $\lambda_A(y) := 0$ for every point $y \in \mathbb{Z}^n$ with $y \notin A$.
Then, we have that $\lambda_A \in FSO(\mathbb{Z}^n, (\kappa^n)^f)$; indeed, $\mathrm{Cl}(\mathrm{Int}(\lambda_A)) = \chi_{Cl(\{y^{(2)}\})} \ge \lambda_A$ hold (cf. Theorem B(iii)). However, $\lambda_A \notin f_s(SO(\mathbb{Z}^n, \kappa^n))$; indeed, it follows from the definition of f_s that $f_s(SO(\mathbb{Z}^n, \kappa^n)) = \{\chi_U | U \in SO(\mathbb{Z}^n, \kappa^n)\}$ and $\lambda_A \ne \chi_U$ for each $U \in SO(\mathbb{Z}^n, \kappa^n)$.

Remark to [19, Definition 1.2 (i)]: the authors of the present paper have this opportunity of taking notice the following typographical correction in [19, Definition 1.2 (i)].

(•) line +3 from the top of the text of [19, Definition 1.2]:
" if $\lambda \le \mathrm{Int}(\mathrm{Cl}(\tau_Y))$ " should be replaced by " if $\lambda \le \mathrm{Int}(\mathrm{Cl}(\lambda))$ ".

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