Fixed points of multifunctions on COTS with end points *

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Abstract. We prove that if F and G are multifunctions from X to Y, with connected values, where X is connected, Y a space admitting a continuous bijection to a connected space Z with endpoints, and Z is T_0 whenever $|Z| = 2$ such that both F, G are either upper semicontinuous with compact values, or, are lower semicontinuous with one of F and G onto, then $F(w) \cap G(w) \neq \emptyset$ for some $w \in X$. We proved that if a multifunction F on a connected space X with endpoints such that X is T_0 whenever $|X| = 2$, has a connected multigraph, then there exists some $w \in X$ such that $w \in F(w)$.

1 Introduction COTS (=connected ordered topological space), defined by Khalimsky, Kopperman and Meyer [6], is an integral part of any study of cut points. Topological spaces are assumed to be connected for any consideration of cut points. By Theorem 2.7 of [6], there are two total orders (or linear orders) on every COTS and each of these orders is the reverse of the other. A COTS can have at most two endpoints [6, Proposition 2.5]. A set with a total order has a topology called interval topology. A topological space is a LOTS (=linearly ordered topological space) if its topology equals some interval topology. Multifunctions are considered on LOTS by Park in [8]. The main result (Theorem 1) of Park [8] about fixed point requires the space to be a connected LOTS having two end points. It can be seen that every LOTS is Hausdorff (without assuming it to be connected). As noted in Proposition 2.9 of [6], the topology of a T_1 COTS is finer than the interval topology given by any of its two orders, so a COTS need not be a LOTS. The concept of COTS does not require any separation axiom. In view of the applications of cut points (see e.g. [6]) and the fact that the many connected topological spaces used for cut points like the Khalimsky line, are not T_1, the assumption of separation axioms is avoided as far as possible. There is the concept of strong cut points for connected topological spaces. Without assuming cut points to be strong cut points, a topological space with endpoints is defined in [2]. Since by Theorem 3.4 of [2], $H(i)$ connected topological spaces have at least two non-cut points, it follows from Remark 4.5 of [2] that such topological spaces with at most two non-cut points turn out to be COTS with endpoints. It is shown in [3] that a connected topological space with endpoints is a COTS with endpoints. It is proved in [4] that a connected topological space is a COTS with endpoints iff it admits a continuous bijection onto a topological space with endpoints. In [4] and [5] there are obtained several classes of connected topological spaces where the members are COTS with endpoints. In this paper, we study multifunction on COTS with endpoints.

Notation, definitions and preliminaries are given in Section 2. The main results of the paper appear in Section 3. In Section 3, we prove that if F and G are multifunctions from X to Y, with connected values, where X is connected, Y a space admitting a continuous bijection to a connected space Z with endpoints and Z is T_0 whenever $|Z| = 2$ has only two points such that both F, G are either upper semicontinuous with compact values, or, are

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lower semicontinuous with one of F and G onto, then $F(w) \cap G(w) \neq \emptyset$ for some $w \in X$. It is proved that if, for a connected space X with endpoints such that X is T_0 whenever $|X| = 2$, F is a multifunction from X to X with connected multigraph, then there exists some $w \in X$ such that $w \in F(w)$. This gives a sort of fixed point theorem. Some results are obtained in the presence of a connected space with endpoints and/or multifunctions.

2 Notation, definitions and preliminaries Some of the standard notation and definitions have been included here for completeness sake. Let X be a space. X is called $T_{1/2}$([6]) if every singleton set is either open or closed. Let $\Delta = \{(x, x) : x \in X\}$ and $\Delta(O) = \{(x, x) : x \in X, \{x\}$ is open in $X\}$. Let $A \subset X$. For $K \subset X$, if need be, A^{+K} is used for the set $A \cup K$, and A^{-K} for the set $A - K$. If X is disconnected, a *separation* of X is denoted by $A|B$, and each one of A and B is a called a *separating set* of X. If A is a separating set of X and $K \subset X$ is connected, if need be, we write $A(K)$ for A if $K \subset A$, and $A(-K)$ for A if $K \subset X - A$. If $K = \{x\}$ for some $x \in X$, then $A^{+x}, A^{-x}, A(x)$ and $A(-x)$ are respectively used for $A^{+K}, A^{-K}, A(K)$ and $A(-K)$. For $x \in X$, if the dependence of a separation $A|B$ of X^{-x} on x is to be specified, then $A|B$ is denoted by $A_x|B_x$. Let $x \in X$. x is called a *cut point* of X if X^{-x} is disconnected. x is called *strong cut point* of X, if X^{-x} has a separation with connected separating sets. ctX is used to denote the set of all cut points of X. A space X is called COTS (=*connected ordered topological space*) ([6]) if it is connected and has the property: if Y is a three-point subset of X, then there is a point x in Y such that Y meets two connected components of X^{-x}. Let X be a space. Let $a, b \in X$. A point $x \in X - \{a, b\}$, is said to be a *separating point* between a and b or x *separates* a and b if there exists a separation $A|B$ of X^{-x} with $a \in A$ and $b \in B$. $S(a, b)$ is used to denote the set of all separating points between a and b. Clearly $S(a, b) \subset ctX$. If we adjoin the points a and b to $S(a, b)$, then the new set is denoted by $S[a, b]$. A space X is called a *space with endpoints* if there exist a and $b \in X$ such that $X = S[a, b]$. For $x \in S(a, b)$, we shall write $X^{-x} = A(a) \cup B(b)$ for a separation $A|B$ of X^{-x}.

For spaces X and Y, a *multifunction* ([7]) from X to Y is a function F from X to $P(Y)$ (= the set of all subsets of Y) with $F(x) \neq \emptyset$ for every $x \in X$, (written as $F : X\text{-}\circ Y$). Let $F : X\text{-}\circ Y$ be a multifunction. F has *compact* (*connected*) *values* if $F(x)$ is compact (connected) for every $x \in X$. For $V \subset Y, F^{\subset}(V)$ (resp. $F^{\cap}(V)$) denotes the set $\{x \in X : F(x) \subset V\}$ (resp. $\{x \in X : F(x) \cap V \neq \emptyset\}$). For $A \subset X, F(A)$ denotes the subset $\cup\{F(x) : x \in A\}$ of Y. For a subset A of X, *multigraph of F over A* is the subset $\{(x, y) \in X \times Y : x \in A, y \in F(x)\} = \bigcup\{\{x\} \times F(x) : x \in A\}$, it is denoted by $mgrA$, or $F\text{-}mgrA(F\text{-}mgrA(Y))$ if the dependence on $F(F$ and $Y)$ is to be specified; multigraph of F over X is called the *multigraph* of F. F is said to be *lower* (resp *upper*) *semicontinuous* ([7]) if for each open (resp. closed) set V of Y, the set $F^{\cap}(V)$ is open (resp. closed) in X. F is called a *connectivity multifunction* ([8]) if its multigraph over each connected subset of X is a connected set. F is called *closed* ([8]) if multigraph of F is closed in $X \times Y$; F is called compact ([8]) if $cl_Y(F(X))$ is a compact subset of Y. For sets X and Y, let $p_1 : X \times Y \to X$, and $p_2 : X \times Y \to Y$ be the projection maps. Let $T \subset X \times Y$. For a multifunction F from X to Z (resp. G from Y to Z), F^1 (resp. G^2) denotes the multifunction $F \circ p_1$ from T to Z (resp. $G \circ p_2$ from T to Z).

For a set X, a multifunction F from X to X is called a multifunction on X. A multifunction F on X is said to have a *fixed point* if there exists some $w \in X$ such that $w \in F(w)$. The multifunction on X taking $x \in X$ to $\{x\}$ is denoted by i_X.

Remark 2.1 Let F be multifunction from X to Y. (i) For $A \subset Y, F^{\cap}(\{A\}) = \bigcup\{F^{\cap}(\{y\}) : y \in A\}$. (ii) For $A \subset X, p_2(F\text{-}mgrA) = F(A)$.

Let $h : Y \to Z$. Define $h^p : P(Y) \to P(Z)$ as $h^p(A) = h(A)$ for $A \in P(Y)$. Let F be a multifunction from X to Y. $h^p \circ F$ is a multifunction from X to Z.
Let X, Y and Z be spaces. Let F be a multifunction from X to Y and G a multifunction from X to Z. For $x \in X$, if we define $(F \times G)(x) = F(x) \times G(x) (\in P(Y) \times P(Z) \subset P(Y \times Z))$, then $F \times G$ is a multifunction from X to $Y \times Z$.

Let F and G be multifunctions from X to Y. $(h^p \circ F) \times (h^p \circ G) : X \to P(Z \times Z)$.
The following lemma is a modified version of some results (i.e., Theorems 7.3.12, 7.3.14 and 7.4.4) of [7] in our notation.

Lemma 2.2 *Let X, Y and Z be spaces. For a function $h : Y \to Z$ and multifunctions F, G from X into Y, let $H = (h^p \circ F) \times (h^p \circ G)$. Let h be continuous.*
(a) *If F, G are lower semicontinuous, then $F \times G$ and H are lower semicontinuous.*
(b) *If F and G are upper semicontinuous with compact values, then $F \times G$ and H are upper semicontinuous with compact values.*
(c) *Let F and G be with connected values. Then H has connected values.*

Let X and Y be spaces and T a subset of $X \times Y$. For $x \in X$, let $T^m(x) = \{y \in Y : (x, y) \in T\}$. $T^m(x)$ may not be non-empty for every $x \in X$. For T^m to be a multifunction, $T^m(x)$ should be non-empty for every $x \in X$. For this we may consider only those $x \in X$ such that $(x, y) \in T$ for some $y \in Y$. Let $X_T = p_1(T) = \{x \in X : (x, y) \in T$ for some $y \in Y\}$. Then T^m is a multifunction from X_T to Y and $T \subset X_T \times Y$. In order that concepts concerning a multifunction make sense for T^m, we need to consider X_T in place of X. For $y \in Y$, let $T_y = \{x \in X : (x, y) \in T\}$. Let $Y_T = p_2(T) = \{y \in Y : (x, y) \in T$ for some $x \in X\}$. Note that $T \subset X_T \times Y_T$.

Lemma 2.3 *Let X, Y be two spaces, and let T be a subset of $X \times Y$.*
(a) *If T is closed in $X_T \times Y$, then for every compact subset A of $X_T, T^m(A)$ is a closed subset of Y.*
(b) *If T is closed in $X_T \times Y$, then $T^{m\cap}(B)$ is closed in X_T for every compact subset B of Y.*

Now we note that every multifunction is of the form T^m. Let F be a multifunction from X to Y. Let $T_F = F\text{-}mgrX = \{(x, y) : x \in X, y \in F(x)\}$. Let $x \in X$. Since $F(x) \neq \emptyset$, $(T_F)^m$ is a multifunction from X to Y.

Remark 2.4 (a) $F = (T_F)^m$.
(b) $p_2(T_F) = F(X)$.
Proof. (a) Let $x \in X$. For $y \in Y, y \in (T_F)^m(x)$ iff $(x, y) \in T_F$, i.e iff $y \in F(x)$.
(b) Since $T_F = F\text{-}mgrX$, by Remark 2.1(ii), $p_2(T_F) = F(X)$.

We note the following before the next observation.
Let F be a multifunction from X to Y. For $F(X) \subset Z \subset Y$, F is a multifunction from X to Z, and $F\text{-}mgrX(Y) = F\text{-}mgrX(Z)$.

Lemma 2.5 *For spaces X and Y, with X connected, let F be a multifunction from X to Y with connected values. Then the multigraph of F is connected if one of the following conditions hold:*
(i) *F is a connectivity multifunction.*
(ii) *F is lower semicontinuous.*
(iii) *F is upper semicontinuous with compact values.*
(iv) *$F^{\cap}(\{y\})$ is open in X for $y \in Y$.*
(v) *F is a closed compact multifunction.*

Proof. (i) Since F is a connectivity multifunction and X is connected, F has connected multigraph.
(ii) and (iii). By Theorem 3.2 of [1], multigraph of F is connected.
(iv) By Remark 2.1(i), (iv)$\Rightarrow$(ii).
(v) Let $Z = cl_Y(F(X))$, F be a compact multifunction form X to Z. Since $T_F = F$-$mgrX, T_F$ is closed. Now by Lemma 2.3(b) and Remark 2.4(a), F is upper semicontinuous. By (a) of Lemma 2.3, F has compact values. Now by (iii), multigraph of F is connected.

3 Connected spaces with endpoints and Multifunctions

Let X be a set with a total order $<$ on it. For $x \in X$, let $L(x) = \{y \in X : y < x\}, U(x) = \{y \in X : x < y\}$ [6]. Let $L = \{(s,t) \in X \times X : t < s\}$ and $U = \{(s,t) \in X \times X : s < t\}$. Then it can be seen that $L = \bigcup\{\{s\} \times L(s) : s \in X\} = \bigcup\{U(s) \times \{s\} : s \in X\}$ and $U = \bigcup\{\{s\} \times U(s) : s \in X\} = \bigcup\{L(s) \times \{s\} : s \in X\}$.

We denote the cardinality of a set X by $|X|$.

Lemma 3.1 *Let X be a COTS such that X is T_0 whenever $|X| = 2$. Then $U \cup \Delta(O)$ and $L \cup \Delta(O)$ are open in $X \times X$.*

Proof. Case (i): $|X| = 2$, i.e., X has only two points. Since X is a connected non-indiscrete space, it follows that $X = \{s,t\}$, with a Sierpinski topology, say $\{\emptyset, \{t\}, X\}$ and $s < t$. Then $U \cup \Delta(O) = X \times \{t\}$, which is open in $X \times X$. That $L \cup \Delta(O)$ is open is proved similarly.

Case (ii): $|X| > 2$, i.e., X has at least three points. Let $(s,t) \in U \cup \Delta(O)$. Then X is $T_{1/2}$ by Proposition 2.9 of [6]. Now if $\{s\}$ and $\{t\}$ are open in X, then $\{(s,t)\} = \{s\} \times \{t\}$ is open in $X \times X$. If $\{s\}$ is open and $\{t\}$ is closed, using Theorem 2.7 and Lemma 2.8 of [6], $\{s\} \times (U(s))^{+s}$ is open in $X \times X$ and $(s,t) \in \{s\} \times (U(s))^{+s} \subset U \cup \Delta(O)$. If $\{s\}$ is closed and $\{t\}$ is open, using Theorem 2.7 and Lemma 2.8 of [6], $(L(t))^{+t} \times \{t\}$ is open in $X \times X$ and $(s,t) \in (L(t))^{+t} \times \{t\} \in U \cup \Delta(O)$. In the case when $\{s\}$ and $\{t\}$ are closed, there is some point y of X such that $s < y < t$ by Lemma 2.8(b) and (c) of [6]. Since $\{y\}$ is either open or closed in X, by Theorem 2.7 and Lemma 2.8 of [6], either $(U(y))^{+y}$ and $(L(y))^{+y}$ or $U(y)$ and $L(y)$ are open in X. So either $(L(y))^{+y} \times (U(y))^{+y}$ or $L(y) \times U(y)$ is open in $X \times X$ and $(s,t) \in L(y) \times U(y) \subset (L(y))^{+y} \times (U(y))^{+y} \subset U \cup \Delta(O)$. Thus $U \cup \Delta(O)$ is open in $X \times X$. Since, in a COTS there are two total orders and each of these orders is the reverse of the other, $L \cup \Delta(O)$ is open in $X \times X$.

Theorem 3.2 *For two multifunctions F, G from a space X to a connected space Y with endpoints such that Y is T_0 whenever $|Y| = 2$, one of which is onto, if either $(F \times G)(X)$ is connected or $F \times G$ has a connected multigraph, then there exists some $w \in X$ such that $F(w) \cap G(w) \neq \emptyset$.*

Proof. In view of Remark 2.4(b), we prove the result by contradiction under the assumption that $(F \times G)(X)$ is connected. Suppose not; then $F(w) \cap G(w) = \emptyset$ for every $w \in X$. By the given condition Y is a space with endpoints, so $Y = S[a,b]$. Let $H = F \times G$. Since, by Theorem 3.2 of [3], Y is a COTS with end points a and b (with $a < b$), $H(X) \subset L \cup U$ in $Y \times Y$. So $(L \cup \Delta(O)) \cap H(X) = L \cap H(X)$ and $(U \cup \Delta(O)) \cap H(X) = U \cap H(X)$. Using Lemma 3.1, $L \cup \Delta(O)$ and $U \cup \Delta(O)$ are open in $Y \times Y$. By given condition, either $F(X) = Y$ or $G(X) = Y$. First assume that $F(X) = Y$. Then we pick $x_a, x_b \in X$ such that $a \in F(x_a)$ and $b \in F(x_b)$. Let $y_a \in G(x_a)$ and $y_b \in G(x_b)$. Since $F(x_a) \cap G(x_a) = \emptyset$, so $a < y_a$. Similarly $y_b < b$. This implies that $(a, y_a) \in U \cap H(X)$ and $(b, y_b) \in L \cap H(X)$. Thus we get a separation of $H(X)$ as $L \cap H(X)$ and $U \cap H(X)$ are disjoint non-empty open subsets of $H(X)$. This gives a contradiction as $H(X)$ is connected by Remark 2.4(b). Thus $F(X) \neq Y$. Similarly we have $G(X) \neq Y$. This leads to again a contradiction to the given condition. The proof is complete.

Theorem 1 of [8] gives a sort of fixed point theorem for a multifunction on a connected LOTS with two end points. Every connected LOTS with end points is a connected space with endpoints, but the converse need not be true. The following theorem and corollary are about a connected space with endpoints; so they strengthen Theorems 1 and 2 of [8] respectively.

Theorem 3.3 *Let X be a connected space with endpoints such that X is T_0 whenever $|X| = 2$. Let F be a multifunction on X with connected multigraph. Then there exists some $w \in X$ such that $w \in F(w)$.*

Proof. The theorem follows by taking $X = Y$ and $G(x) = \{x\}$ for $x \in X$ in Theorem 3.2.

Corollary 3.4 *Let X be a connected space with endpoints such that X is T_0 whenever $|X| = 2$. Let F be a multifunction on X with connected values. Then there exists some $w \in X$ such that $w \in F(w)$, if one of the following conditions hold:*
(i) *F is a connectivity multifunction.*
(ii) *F is lower semicontinuous.*
(iii) *F is upper semicontinuous with compact values.*
(iv) *$F^{\cap}(y)$ is open in X for $y \in X$.*
(v) *F is a closed compact multifunction.*

Proof. The result follows by Lemma 2.5 and Theorem 3.3.

The following two theorems respectively strengthen Theorems 2.1 and 2.2 of [9] because here $[0, 1]$ is replaced by a connected space with endpoints (with no separation axioms assumed).

Theorem 3.5 *Let X be a connected space and Y be a space admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$. Let F, G be two multifunctions from X to Y, with connected values and one of which is onto. Assume that both F and G are either upper semicontinuous with compact values, or lower semicontinuous. Then there exists some $w \in X$ such that $F(w) \cap G(w) \neq \emptyset$.*

Proof. By the given condition we have a connected space Z with endpoints, say a and b and a one-one, onto and continuous function $h : Y \to Z$. Let $H = (h^p \circ F) \times (h^p \circ G)$. By Lemmas 2.2 and 2.5, multigraph of H is connected. Now by Theorem 3.2, there exists some $w \in X$ such that $h(F(w)) \cap h(G(w)) \neq \emptyset$. This implies that $F(w) \cap G(w) \neq \emptyset$ as h is one-one.

Below we have some results in which we assume a subset of a product space of two spaces to be connected. It may be added that Theorem 2.5 of [9] is handy to know the connectedness of a given set in a product space.

Theorem 3.6 *Let X, Y be two spaces, with Y admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$, and let T be a connected subset of $X \times Y$. Let Φ be a multifunction from X to Y, with connected values. Assume that Φ is either upper semicontinuous with compact values, or lower semicontinuous.*
(i) *If $Y_T = Y$ or $\Phi(X_T) = Y$, then $T \cap (\Phi\text{-}mgrX) \neq \emptyset$.*
(ii) *If $X_T = X$ and Φ is onto, then $T \cap (\Phi\text{-}mgrX) \neq \emptyset$.*

Proof. (i) $F = (i_Y)^2 (= i_Y \circ p_2)$ and $G = \Phi^1 (= \Phi \circ p_1)$ are multifunctions from T to Y. So using the given condition, F and G are either upper semicontinuous with compact values, or lower semicontinuous. Also F and G have connected values and so by the given condition, one of F and G is onto. Now by applying Theorem 3.5 to F and G, the result follows.

(ii) It follows from the assumption of (ii) that the hypothesis $\Phi(X_T) = Y$ of (i) is satisfied.

The following particular case of theorem 3.6 is about fixed point of a multifunction.

Corollary 3.7 *Let X be a space admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$. Let Φ be a multifunction from X to X, with connected values. Assume that Φ is either upper semicontinuous with compact values, or lower semicontinuous. If Δ is a connected set of $X \times X$, then there exists some $x_0 \in X$ such that $x_0 \in \Phi(x_0)$.*

Proof. Since $X_\Delta = X$, the result follows by taking $Y = X$ and $T = \Delta$ in Theorem 3.6.

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