WEIGHTED VARIABLE MODULATION SPACES

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Abstract. The aim of this paper is to develop a theory of weighted modulation spaces with variable exponent. All we assume on the exponent is that the essential infimum of the exponent is positive. We shall show that the auxiliary parameter can be removed assuming, in addition, that the weight belongs to the variable Muckenhoupt class and that the exponents satisfy the log-Hölder condition and the log-decay condition. Under these assumptions, we prove the molecular decomposition theorem and boundedness of pseudo-differential operators with symbol S^0_{00}.

1 Introduction The aim of this paper is two-fold: one is to develop a theory for modulation spaces with variable exponents; the other is to establish that the results carry over to the weighted setting to a large extent by introducing an auxiliary parameter $a > 0$.

Let us start by recalling the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ proposed by Nakano in 1951 [10, 11], where Nakano actually worked on $[0, 1]$; see [8] for a detailed account. Let $L^0(\mathbb{R}^n)$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$. Let also $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function throughout this paper, which is sometimes referred to as an exponent. Define the Lebesgue space $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent by:

$$L^{p(\cdot)}(R^n) \equiv \left\{ f \in L^0(R^n) : \int_{R^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

Equip $L^{p(\cdot)}(\mathbb{R}^n)$ with the norm given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}$$

for $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Now we move on to the weighted setting. By a "weight function", we mean a measurable function w defined on $\mathbb{R}^n$ such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Let $p(\cdot)$ be an exponent such that

$$0 < p_- \equiv \operatorname*{essinf}_{x \in \mathbb{R}^n} p(x) \le p_+ \equiv \operatorname*{esssup}_{x \in \mathbb{R}^n} p(x) < \infty$$

and w be a weight function. One defines the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ by

$$L^{p(\cdot)}(w) \equiv \left\{ f \in L^0(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x)\, dx < \infty \right\},$$

as a linear space, and the norm is given by:

$$\|f\|_{L^{p(\cdot)}(w)} \equiv \inf \left\{ \lambda > 0 \ : \ \int_{\mathbb{R}^n} \left(\frac{|f(x)| w(x)^{1/p(x)}}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

We define the weighted vector-valued Lebesgue space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ with variable exponents based on the above definition.

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Definition 1.1. Let $p(\cdot)$ and $q(\cdot)$ be exponents satisfying

$$0 < p_- \le p_+ < \infty, \quad 0 < q_- \le q_+ < \infty, \tag{1.1}$$

and let w be a weight. One defines the weighted vector-valued function space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ by

$$\ell^{q(\cdot)}(L^{p(\cdot)}(w)) \equiv \left\{ \{f_m\}_{m\in\mathbb{Z}^n} \subset L^0(\mathbb{R}^n) \,:\, \sum_{m\in\mathbb{Z}^n} \left\| |f_m|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty \right\}$$

as a linear space, and the norm is given by:

$$\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} = \inf \left\{ \lambda > 0 \,:\, \sum_{m\in\mathbb{Z}^n} \left\| \left| \frac{f_m}{\lambda} \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le 1 \right\}$$

for $\{f_m\}_{m\in\mathbb{Z}^n} \subset L^0(\mathbb{R}^n)$.

Now we define the weighted modulation space $M_{p(\cdot),q(\cdot),a}(w)$ with variable exponents by using the following standard operators in time frequency analysis:

- For a measurable function f on $\mathbb{R}^n$ and $m, l \in \mathbb{Z}^n$, define $M_m f$ and $T_l f$ by $M_m f(x) \equiv \exp(im \cdot x) f(x)$ and $T_l f(x) \equiv f(x - l)$, respectively.
- Define the Fourier transform and its inverse by

$$\mathcal{F}f(\xi) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) \exp(-ix\cdot\xi)\, dx,\ \mathcal{F}^{-1}f(x) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi) \exp(ix\cdot\xi)\, d\xi.$$

- Let $Q(r) = \{x \in \mathbb{R}^n \,:\, \max\{|x_1|, |x_2|, \ldots, |x_n|\} \le r\}$.

With these definitions in mind, we present the definition of the weighted modulation space $M_{p(\cdot),q(\cdot),a}(w)$ with variable exponents.

Definition 1.2. Suppose that $p(\cdot)$ and $q(\cdot)$ satisfies (1.1) and $a > 0$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\chi_{Q(1/4)} \le \mathcal{F}\phi \le \chi_{Q(2)} \tag{1.2}$$

and

$$\sum_{m\in\mathbb{Z}^n} T_m[\mathcal{F}\phi](x) > 0 \tag{1.3}$$

for all $x \in \mathbb{R}^n$. Then the space $M_{p(\cdot),q(\cdot),a}(w)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{M_{p(\cdot),q(\cdot),a}(w)} \equiv \|\{(M_m\phi * f)_a\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

is finite, where

$$(M_m\phi * f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1 + |x - y|)^a}. \tag{1.4}$$

The next result justifies the notation $M_{p(\cdot),q(\cdot),a}(w)$.

Theorem 1.1. *Let $p(\cdot), q(\cdot) : \mathbb{R}^n \to (0,\infty)$ be variable exponents satisfying* (1.1). *Then the definition of the set $M_{p(\cdot),q(\cdot),a}(w)$ is independent of the choice of ϕ; different choices of admissible functions yield equivalent norms.*

With the new parameter a, our assumption on w can be minimized in order that $M_{p(\cdot),q(\cdot),a}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. More quantitatively, we have the following assertion:

Theorem 1.2. *Let w be a weight such that*

$$\int_{[0,1]^n} w(x)\,dx < \infty. \tag{1.5}$$

Let $f \in M_{p(\cdot),q(\cdot),a}(w)$. Then there exists $C > 0$ such that

$$\sup_{m\in\mathbb{Z}^n} \|(1+|\cdot|)^{-a}(M_m\phi * f)_a\|_{L^\infty} \le C\|f\|_{M_{p(\cdot),q(\cdot),a}(w)}.$$

In particular, $M_{p(\cdot),q(\cdot),a}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.2 shows that (1.5) is sufficient to guarantee that our new space $M_{p(\cdot),q(\cdot),a}(w)$ is a subset of $\mathcal{S}'(\mathbb{R}^n)$.

We impose on $p(\cdot)$ the log-Hölder continuity condition:

$$|p(x)-p(y)| \le \frac{c_{\log}(p)}{\log(\mathrm{e}+|x-y|^{-1})} \quad \text{for} \quad x,y\in\mathbb{R}^n, \tag{1.6}$$

and the log decay condition;

$$|p(x)-p_\infty| \le \frac{c^*}{\log(e+|x|)} \quad \text{for} \quad x\in\mathbb{R}^n, \tag{1.7}$$

where p_∞ is a real number, $c_{\log}(p)$ and c^* are positive constants independent of x and y. We say that $p(\cdot)$ satisfies the globally log-Hölder condition if $p(\cdot)$ satisfies both (1.6) and (1.7).

We also consider the sequence space $m_{p(\cdot),q(\cdot),a}(w)$ to prove the molecular decomposition theorem.

Definition 1.3. Let $p(\cdot)$ and $q(\cdot)$ be exponents satisfying (1.1) and $a > 0$. One defines a space $m_{p(\cdot),q(\cdot),a}(w)$ as the set of all complex sequences $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$ such that

$$\sum_{m\in\mathbb{Z}^n} \left\| \left| \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{(1+|y|)^a} \right) \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

For such a sequence λ, define the quasi-norm by

$$\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)} \equiv \inf\left\{ T>0 : \sum_{m\in\mathbb{Z}^n} \left\| \left| \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{T(1+|y|)^a} \right) \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le 1 \right\}.$$

We consider the following condition on weights:

$$(1+|\cdot|)^{-a} \in L^{p(\cdot)}(w) < \infty. \tag{1.8}$$

As the following theorem shows, (1.8) is a natural and minimal condition.

Theorem 1.3. *Let $w : \mathbb{R}^n \to [0,\infty)$ be a measurable function. Assume that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1) *and $a > 0$. Then assumption* (1.8) *is necessary and sufficient for $m_{p(\cdot),q(\cdot),a}(w)$ to contain an element other than* 0.

We may ask ourselves whether the parameter a is essential. If the weight is good enough, then we can show that a is not essential as long as $a \gg 0$. We invoke the following definition from [2, 3, 4].

For a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, a measurable function w is said to be an $A_{p(\cdot)}$ weight if $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$ and

$$\sup_Q \left(\frac{1}{|Q|} \|w^{1/p(\cdot)}\chi_Q\|_{L^{p(\cdot)}} \|w^{-1/p(\cdot)}\chi_Q\|_{L^{p'(\cdot)}} \right) < \infty \tag{1.9}$$

holds, where the supremum is taken over all open cubes $Q \subset \mathbb{R}^n$ whose sides are parallel to the coordinate axes and $p'(x)$ is the conjugate exponent of $p(x)$, that is, $1/p(x)+1/p'(x) = 1$.

In the above definition, when $a \gg 0$ and $w \in A_{p(\cdot)}$, the space $m_{p(\cdot),q(\cdot),a}(w)$ does not depend on a, as the following theorem shows.

Theorem 1.4. *Assume that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1), *$p_- > 1$, $w \in A_{p(\cdot)}$ and $a \gg 0$. Assume, in addition, that $p(\cdot)$ and $q(\cdot)$ are globally* log*-Hölder continuous. Then $\lambda \in m_{p(\cdot),q(\cdot),a}(w)$ if and only if*

$$\left\| \left\{ \sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \chi_{l+[0,1)^n}(\cdot) \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} < \infty.$$

We also consider the molecular decomposition. For $x \in \mathbb{R}^n$, we write $\langle x \rangle \equiv \sqrt{1+|x|^2}$. Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K-function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an (m,l)-molecule if it satisfies $|\partial^\alpha(e^{-im\cdot x}\tau(x))| \le \langle x-l \rangle^{-N}$, $x \in \mathbb{R}^n$ for $|\alpha| \le K$. Set

$$\mathcal{M} \equiv \{M = \{\mathrm{mol}_{ml}\}_{m,l \in \mathbb{Z}^n} \subset C^K \ : \ \mathrm{mol}_{ml} \text{ is an } (m,l)\text{-molecule for every } m,l \in \mathbb{Z}^n\}.$$

We shall develop a theory of decomposition based on the above definition.

Theorem 1.5. *Let $a \gg N+n$. Assume, in addition, that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1).

(i) *Let $\phi, \kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfy*

$$\chi_{Q(1/4)} \le \mathcal{F}\phi \le \chi_{Q(2)}, \quad \sum_{l \in \mathbb{Z}^n} T_l[\mathcal{F}\phi] \equiv 1 \tag{1.10}$$

and

$$0 \le \kappa \le \chi_{Q(2)}, \quad \sum_{l \in \mathbb{Z}^n} T_l \kappa \equiv 1. \tag{1.11}$$

The decomposition, called Gabor decomposition, holds for $M_{p(\cdot),q(\cdot),a}(w)$. More precisely, we have $\{T_l M_m[\mathcal{F}^{-1}\kappa]\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}$ and the mapping

$$f \in M_{p(\cdot),q(\cdot),a}(w) \mapsto \lambda = \{M_m\phi * f(l)\}_{m,l \in \mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)$$

is bounded. Furthermore, any $f \in M_{p(\cdot),q(\cdot),a}(w)$ admits the following Gabor decomposition

$$f = \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot T_l M_m[\mathcal{F}^{-1}\kappa],$$

$$\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} = \{M_m\phi * f(l)\}_{m,l \in \mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w). \tag{1.12}$$

(ii) *Suppose we are given* $M = \{\mathrm{mol}_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}$ *and* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)$. *Then*

$$f \equiv \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \mathrm{mol}_{ml} \tag{1.13}$$

converges unconditionally in $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore,* f *belongs to* $M_{p(\cdot),q(\cdot),a}(w)$ *and satisfies the quasi-norm estimate* $\|f\|_{M_{p(\cdot),q(\cdot),a}(w)} \le C\,\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)}$. *In particular, the convergence of* (1.13) *takes place in* $M_{p(\cdot),q(\cdot),a}(w)$.

Corollary 1.6. *Under assumption* (1.8), $\mathcal{S}(\mathbb{R}^n) \subset M_{p(\cdot),q(\cdot),a}(w)$.

As an application, we shall show that the pseudo-differential operator with symbol S^0_{00} is bounded on $M_{p(\cdot),q(\cdot),a}(w)$. Recall that $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is an S^0_{00}-symbol if

$$\partial_x^\beta \partial_\xi^\alpha a \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$$

for all multi-indices α and β. The pseudo-differential operator $a(X,D)$ is defined by

$$a(X,D)f(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x,\xi)\mathcal{F}f(\xi) e^{ix\cdot\xi}\,d\xi.$$

In [9, Lemma 3.2], the authors showed that the set $\mathcal{M}$ is preserved by $a(X,D)$. Thus, we have the following result, which is a direct corollary of Theorem 1.5:

Theorem 1.7. *Let* $a \in S^0_{00}$. *Then* $a(X,D)$ *is a bounded linear operator on* $M_{p(\cdot),q(\cdot),a}(w)$.

Remark 1.1. When $p(\cdot) \equiv q(\cdot) \equiv 2$, $a > n/2$ and $w \equiv 1$, we have $M_{p(\cdot),q(\cdot),a}(w) = L^2(\mathbb{R}^n)$. It may be interesting to note that Sjöstrand proved this result when $M_{p(\cdot),q(\cdot),a}(w) = L^2(\mathbb{R}^n)$ by using the so-called T^*T-method, while our method is beyond the reach of this method employed in [12].

We organize the remaining part of this paper as follows: The proofs of Theorem 1.1 through Theorem 1.4 can be found in Section 2. In Section 3, we shall develop a theory of decomposition and we prove Theorem 1.5.

2 Fundamental structure of $M_{p(\cdot),q(\cdot),a}(w)$

2.1 Proof of Theorem 1.1 Let ϕ, ψ be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) and (1.3). Let us choose a smooth function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\mathcal{F}\Phi(\xi) \sum_{m \in \{-2,-1,0,1,2\}^n} \mathcal{F}\psi(\xi - m) = \mathcal{F}\phi(\xi).$$

Then we have

$$\phi = (2\pi)^{n/2}\Phi * \sum_{m \in \{-2,-1,0,1,2\}^n} M_m\psi$$

and hence

$$M_l\phi = (2\pi)^{n/2} M_l\Phi * \sum_{m \in \{-2,-1,0,1,2\}^n} M_{l+m}\psi,$$

which implies

$$\begin{aligned}
&|M_l\phi * f(x-y)| \\
&\le C \sum_{m\in\{-2,-1,0,1,2\}^n} \int_{\mathbb{R}^n} |\Phi(z)| \cdot |M_{l+m}\psi * f(x-y-z)|\,dz \\
&\le C \left(\int_{\mathbb{R}^n} |\Phi(z)|(1+|z+y|)^a\,dz\right) \sum_{m\in\{-2,-1,0,1,2\}^n} \sup_{w\in\mathbb{R}^n} \frac{|M_{l+m}\psi * f(x-w)|}{(1+|w|)^a} \\
&\le C(1+|y|)^a \sum_{m\in\{-2,-1,0,1,2\}^n} \sup_{w\in\mathbb{R}^n} \frac{|M_{l+m}\psi * f(x-w)|}{(1+|w|)^a}
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. This in turn implies

$$(M_l\phi * f)_a(x) \le C \sum_{m\in\{-2,-1,0,1,2\}^n} (M_{l+m}\psi * f)_a(x).$$

Due to symmetry, we see that different choices of admissible functions yield equivalent norms.

2.2 Proof of Theorem 1.2 Let $m \in \mathbb{Z}^n$ be fixed and take $x \in m + [0,1]^n$. Then we have

$$\begin{aligned}
(1+|x|)^{-a}(M_m\phi * f)_a(x) &= \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1+|x-y|)^a(1+|x|)^a} \\
&\le C \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1+|y|)^a} \\
&\le C \inf_{z\in[0,1]^n} \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1+|y-z|)^a}.
\end{aligned}$$

If we use (1.5), then we obtain

$$(1+|x|)^{-a}(M_m\phi * f)_a(x) \le C \left\| \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1+|y-\cdot|)^a} \right\|_{L^{p(\cdot)}(w)} = C\|(M_m\phi * f)_a\|_{L^{p(\cdot)}(w)}.$$

This then yields

$$(1+|x|)^{-a}(M_m\phi * f)_a(x) \le C\|f\|_{M_{p(\cdot),q(\cdot),a}(w)}.$$

Thus, the proof is complete.

2.3 Proof of Theorem 1.3 We justify the condition (1.8); we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)\setminus\{0\}$. Then there exist $m_0, l_0 \in \mathbb{Z}^n$ such that $\lambda_{m_0 l_0} \neq 0$. Set

$$\rho_{ml} \equiv \begin{cases} \lambda_{m_0 l_0} & (m,l) = (m_0, l_0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\rho \equiv \{\rho_{ml}\}_{m,l\in\mathbb{Z}^n}$ belongs to $m_{p(\cdot),q(\cdot),a}(w) \setminus \{0\}$. This implies

$$0 < \|\rho\|_{m_{p(\cdot),q(\cdot),a}(w)} = \left\| \sup_{y\in\mathbb{R}^n} \left(|\lambda_{m_0 l_0}| \frac{\chi_{l_0+[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

Hence, we have

$$\left\| \sup_{y\in\mathbb{R}^n} \left(\frac{\chi_{[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le (1+|l_0|)^{aq_+} \left\| \sup_{y\in\mathbb{R}^n} \left(\frac{\chi_{l_0+[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

Therefore, we obtain

$$\begin{aligned} &\left\| \left| \sup_{y\in\mathbb{R}^n} \left(\frac{\chi_{[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right) \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \\ &\le \max\left\{ \left\| \sup_{y\in\mathbb{R}^n} \left(\frac{\chi_{[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right) \right\|_{L^{p(\cdot)}(w)}^{q_-}, \left\| \sup_{y\in\mathbb{R}^n} \left(\frac{\chi_{[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right) \right\|_{L^{p(\cdot)}(w)}^{q_+} \right\} < \infty \end{aligned}$$

and hence (1.8). □

2.4 Proof of Theorem 1.4 Firstly, we prove the following lemma to prove Lemma 2.2.

Lemma 2.1. *Let $p(\cdot)$ satisfy the* log*-Hölder conditions. Define $\eta_a(x) \equiv \dfrac{1}{(1+|x|)^a}$ for $x \in \mathbb{R}^n$. If $a > c_{\log}(p)$, then there exists a constant $C > 0$ such that*

$$b^{p(x)}\eta_{2a}(x-y) \le C b^{p(y)}\eta_a(x-y) \tag{2.1}$$

holds for any $1 \le b < \infty$ and $x, y \in \mathbb{R}^n$.

Proof. We use a similar argument to the proof of [5, Lemma 6.1]. We may assume $|x-y| \ge b$ due to the log Hölder continuity of $p(\cdot)$. We fix the smallest natural number $k \ge 2$ such that $|x-y| \le b^{-1+k}$. Then, for such x, y and k, $1+|x-y| \sim b^k$ holds and we have

$$\frac{\eta_{2a}(x-y)}{\eta_a(x-y)} \le c(1+b^k)^{-a} \le cb^{-ka}. \tag{2.2}$$

Furthermore, by the Hölder continuity of $p(\cdot)$ and $a > c_{\log}(p)$, we see that

$$b^{p(y)-p(x)} \ge b^{-c_{\log}(p)/\log(e+|x-y|^{-1})} \ge b^{-c_{\log}(p)} \ge b^{-(k-1)a}. \tag{2.3}$$

Hence, the desired inequality (2.1) holds thanks to (2.2) and (2.3) as well as the fact that $a > c_{\log}(p)$. □

We need the following auxiliary estimate akin to the one in [1].

Lemma 2.2. *Let $p(\cdot), q(\cdot)$ satisfy the log-Hölder condition as well as the log decay condition. Let $1 \le p_- \le p_+ < \infty$ and let also $w \in A_{p(\cdot)}$. Let $a > 2\max\{n, c_{\log}(q)\}$. Set $\eta_a(x) \equiv (1+|x|)^{-a}$. Then, for any $\{f_m\}_{m\in\mathbb{Z}^n} \in \ell^{q(\cdot)}(L^{p(\cdot)}(w))$,*

$$\|\{\eta_a * f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \le C\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

Proof. We follow the idea in the work by Almeida and Hästö, which is listed above. Without loss of generality, we can assume $\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} = 1$. Then it is easy to see that $\|f_m\|_{L^{p(\cdot)}(w)} \le 1$ hold for any $m \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}^n$ be fixed and $\delta = \||f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}$. By the argument of [1, Proof of Lemma 4.7] with Lemma 2.1 which takes the place of [1, Lemma 4.3], we have $\|\delta^{-1/q(\cdot)}(\eta_a * f_m)\|_{L^{p(\cdot)}(w)} \le C\|\eta_{a/2} * [\delta^{-1/q(\cdot)} f_m]\|_{L^{p(\cdot)}(w)}$.

Denote by M the Hardy-Littlewood maximal operator; for a measurable function f define

$$Mf(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{[-r,r]^n} |f(x-y)|\, dy \quad (x \in \mathbb{R}^n).$$

Since $a > 2n$, we have $\|\eta_{a/2}\|_{L^1} < \infty$ and hence

$$|\eta_{a/2} * F(x)| \le CMF(x)$$

for all positive measurable functions F. Since $w \in A_{p(\cdot)}$, we have

$$\|\eta_{a/2} * F\|_{L^{p(\cdot)}(w)} \le C\|F\|_{L^{p(\cdot)}(w)}. \tag{2.4}$$

Note that

$$\|\delta^{-1/q(\cdot)}(\eta_a * f_m)\|_{L^{p(\cdot)}(w)} \le C\|\eta_{a/2} * [\delta^{-1/q(\cdot)} f_m]\|_{L^{p(\cdot)}(w)} \le C\|\delta^{-1/q(\cdot)} f_m\|_{L^{p(\cdot)}(w)},$$

where for the second inequality we used (2.4). Note that

$$\begin{aligned} &\min\{\|h^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_-}, \|h^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_+}\} \\ &\le \|h\|_{L^{p(\cdot)}} \le \max\{\|h^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_-}, \|h^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_+}\} \end{aligned}$$

for any non-negative measurable function h. Therefore,

$$\begin{aligned} &\min\{\delta^{-1/q_-}\||\eta_a * f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_-}, \delta^{-1/q_+}\||\eta_a * f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_+}\} \\ &\le \max\{\delta^{-1/q_-}\||f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_-}, \delta^{-1/q_+}\||f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_+}\} = 1. \end{aligned}$$

This implies that either

$$\delta^{-1/q_-}\||\eta_a * f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_-} \le 1$$

or

$$\delta^{-1/q_+}\||\eta_a * f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}^{1/q_+} \le 1.$$

Hence,

$$\|\,|\eta_a * f_m|^{q(\cdot)}\,\|_{L^{p(\cdot)/q(\cdot)}(w)} \le C\|\,|f_m|^{q(\cdot)}\,\|_{L^{p(\cdot)/q(\cdot)}(w)}.$$

□

Now we prove Theorem 1.4. Let $\eta_a(x) \equiv (1+|x|)^{-a}$ as before. Then we have

$$\begin{aligned} \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{Z}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a} \right) &\le C \sum_{k\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-k)}{(1+|k|)^a} \right) \\ &\le C \int_{\mathbb{R}^n} \left(\sum_{l\in\mathbb{Z}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-z)}{(1+|z|)^a} \right) dz \\ &= C\eta_a * \left[\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}|\chi_{l+[0,1)^n} \right](x). \end{aligned}$$

Taking the $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$-norm, we obtain

$$\left\| \left\{ \sup_{y \in \mathbb{R}^n} \left(\sum_{l \in \mathbb{Z}^n} \frac{|\lambda_{ml}| \chi_{l+[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right) \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$
$$\leq C \left\| \left\{ \eta_a * \left[\sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \chi_{l+[0,1)^n} \right] \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

Hence, we invoke Lemma 2.2 to see that

$$\left\| \left\{ \sup_{y \in \mathbb{R}^n} \left(\sum_{l \in \mathbb{Z}^n} \frac{|\lambda_{ml}| \chi_{l+[0,1)^n}(\cdot - y)}{(1+|y|)^a} \right) \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$
$$\leq C \left\| \left\{ \sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \chi_{l+[0,1)^n} \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))},$$

as was to be shown.

3 Molecular decomposition Assumption (1.5) is also appropriate to develop a theory for the decomposition of weighted modulation spaces with variable exponent.

The following well-known lemma is used to prove Theorem 1.5. For example, we refer for the proof to the paper [6, Lemma 2.1] due to M. Frazier and B. Jawerth, who took full advantage of this equality in [7, Lemma 2.1].

Lemma 3.1. [6, Lemma 2.1], [7, Lemma 2.1] *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ with frequency support contained in $Q(2)$;*

$$\mathrm{supp}(\mathcal{F}f) \subset Q(2). \tag{3.1}$$

Assume, in addition, that $\kappa \in \mathcal{S}(\mathbb{R}^n)$ is supported on $Q(2)$ and that

$$\sum_{l \in \mathbb{Z}^n} T_l \kappa \equiv 1.$$

Then we have

$$f = (2\pi)^{-\frac{n}{2}} \sum_{l \in \mathbb{Z}^n} f(l) \cdot T_l[\mathcal{F}^{-1}\kappa]. \tag{3.2}$$

Remark 3.1. In the original version of [6, Lemma 2.1], Frazier and Jawerth did not consider condition (3.1). Instead, they decomposed f according the size of frequency support; see [6, (2.5)]. Apart from the mollification done in [6, (2.7)], their key idea of the proof is to expand a function into Fourier series; see [6, (2.8)]. This technique will be used to prove Lemma 3.1. Despite the fact that Frazier and Jawerth dealt with Besov spaces and Triebel-Lizorkin spaces in [6, 7] and that we deal with (weighted) modulation spaces, we can say that Lemma 3.1 is essentially due to Frazier and Jawerth because of the important contribution to the theory of decompositions obtained in [6, 7].

Proof of Theorem 1.5. Define $M_{p(\cdot),q(\cdot),a}(w)$ according to Definition 1.2 by using ϕ satisfying (1.10).

(i) Let $f \in M_{p(\cdot),q(\cdot),a}(w)$. Then by using (1.10) and (1.11) we expand f according to Lemma 3.1 :

$$f = \sum_{m \in Z^n} M_m \phi * f = (2\pi)^{-\frac{n}{2}} \sum_{m \in Z^n} \left(\sum_{l \in Z^n} M_m \phi * f(l) \cdot T_l M_m [\mathcal{F}^{-1}\kappa] \right).$$

Thus, if we set $\lambda_{ml} \equiv (2\pi)^{-\frac{n}{2}} M_m \phi * f(l)$, then we obtain a decomposition of f as follows:

$$f = \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \cdot T_l M_m [\mathcal{F}^{-1}\kappa] \right).$$

Let us check that this decomposition fulfills the desired property in Theorem 1.5(i).

Let $x, y \in \mathbb{R}^n$. Denote by $l_{x,y}$ an element in $\mathbb{Z}^n$ such that $x-y \in l_{x,y}+[0,1)^n$. Observe that

$$\begin{aligned}
(2\pi)^{\frac{n}{2}} \sup_{y \in \mathbb{R}^n} \left(\sum_{l \in \mathbb{R}^n} \frac{|\lambda_{ml}| \chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a} \right) &= \sup_{y \in \mathbb{R}^n} \left(\sum_{l \in \mathbb{R}^n} \frac{|M_m \phi * f(l)| \chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a} \right) \\
&= \sup_{y \in \mathbb{R}^n} \frac{|M_m \phi * f(l_{x,y})|}{(1+|y|)^a} \\
&\le \sup_{y \in \mathbb{R}^n} \frac{|M_m \phi * f(l_{x,y})|}{(1+|x-l_{x,y}|)^a} (1+|y-x+l_{x,y}|)^a \\
&\le 2^a \sup_{y \in \mathbb{R}^n} \frac{|M_m \phi * f(l_{x,y})|}{(1+|x-l_{x,y}|)^a} \\
&= 2^a (M_m \phi * f)_a(x).
\end{aligned}$$

Therefore, we obtain

$$\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)} \le C \|f\|_{M_{p(\cdot),q(\cdot),a}(w)},$$

as was to be shown.

(ii) Let $m' \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
|M_{m'} \phi * f(x)| &\le \sum_{m,l \in \mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'} \phi * \mathrm{mol}_{ml}(x)| \\
&= \sum_{m,l \in \mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'} \phi * [M_m [M_{-m} \mathrm{mol}_{ml}]](x)| \\
&= \sum_{m,l \in \mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'-m} \phi * [M_{-m} \mathrm{mol}_{ml}](x)|.
\end{aligned}$$

Note that

$$M_{m'-m} \phi * [M_{-m} \mathrm{mol}_{ml}](x) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(m'-m)y} \phi(y) (M_{-m} \mathrm{mol}_{ml})(x-y)\, dy$$

satisfies

$$|M_{m'-m} \phi * [M_{-m} \mathrm{mol}_{ml}](x)| \le C \langle m'-m \rangle^{-N} \langle x-l \rangle^{-N}.$$

Thus, it follows that

$$\frac{|M_{m'-m} \phi * [M_{-m} \mathrm{mol}_{ml}](y)|}{(1+|x-y|)^N} \le C \langle m'-m \rangle^{-N} \langle x-l \rangle^{-N}$$

for all $y \in \mathbb{R}^n$. Consequently,

$$\begin{aligned}
&\sup_{y \in \mathbb{R}^n} \frac{|M_{m'}\phi * f(y)|}{(1+|x-y|)^N} \\
&\le C \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \langle m'-m \rangle^{-N} \langle x-l \rangle^{-N} \right) \\
&= C \sum_{m \in \mathbb{Z}^n} \langle m'-m \rangle^{-N} \left(\sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \langle x-l \rangle^{-a} \langle x-l \rangle^{-N+a} \right) \\
&\le C \sum_{m \in \mathbb{Z}^n} \langle m'-m \rangle^{-N} \left(\sum_{l \in \mathbb{Z}^n} \sup_{z \in \mathbb{R}^n} \left(\sum_{l_1 \in \mathbb{Z}^n} \frac{|\lambda_{ml_1}| \chi_{l_1+[0,1)^n}(x-z)}{(1+|z|)^a} \right) \langle x-l \rangle^{-N+a} \right) \\
&\le C \sum_{m \in \mathbb{Z}^n} \langle m'-m \rangle^{-N} \left(\sup_{z \in \mathbb{R}^n} \left(\sum_{l_1 \in \mathbb{Z}^n} \frac{|\lambda_{ml_1}| \chi_{l_1+[0,1)^n}(x-z)}{(1+|z|)^a} \right) \right)
\end{aligned}$$

as long as $N > a+n$. Thus, $f \in M_{p(\cdot),q(\cdot),a}(w)$.

□

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