SOME RESULTS ON BN_1-ALGEBRAS

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Abstract. BN_1-algebras have been introduced by C. B. Kim and H. S. Kim. Here we give an equivalent definition of BN_1-algebras and show that every BN_1-algebra is a loop. Moreover we prove that an algebra is BN_1-algebra if and only if it is a commutative BG-algebra. We also prove that the class of associative BN_1-algebras coincides with the class of Coxeter algebras. Finally we indicate the interrelationships between BN_1-algebras and several algebras.

1 Introduction In 1966, K. Iséki introduced in [3] the concept of BCI-algebras as algebras connected with some logics. Next, in 1983, Q. P. Hu and X. Li ([1]) defined BCH-algebras which are a generalization of BCI-algebras. Several years later, Y. B. Jun, E. H. Roh and H. S. Kim ([4]) introduced a wide class of abstract algebras called BH-algebras. Recently, C. B. Kim and H. S. Kim introduced in [7] the notion of a BN_1-algebra. They defined a *BN_1-algebra* as an algebra $(A; *, 0)$ of type $(2, 0)$ (i.e., a nonempty set A with a binary operation $*$ and a constant 0) satisfying the following axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(BN) $(x * y) * z = (0 * z) * (y * x)$,
(BN_1) $x = (x * y) * y$.

Every Boolean group (that is, Abelian group all of whose elements have order 2) is a BN_1-algebra. The class of all BN_1-algebras is a proper subclass of the class of BN-algebras defined in [7]. A. Walendziak introduced in [12] BF-algebras which are a generalization of BN-algebras and B-algebras ([10]). C. B. Kim and H. S. Kim defined in [6] BM-algebras and proved that every BM-algebra is a B-algebra. They also introduced BG-algebras ([5]) as a generalization of B-algebras.

We will denote by **BCI** (resp., $\mathbf{BCH/BH/B/BM/BG/BF/BN/BN_1}$) the class of all BCI-algebras (resp., BCH/BH/B/BM/BG/BF/BN/BN_1-algebras). The interrelationships between some classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \to \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

In this paper we study BN_1-algebras. We give another axiomatization of BN_1-algebras and prove that every BN_1-algebra is a loop. Moreover we show that the concept of a BN_1-algebra is equivalent to the concept of a commutative BG-algebra. We also show that the class of associative BN_1-algebras coincides with the class of Coxeter algebras. Finally we consider the relationships between BN_1-algebras and several algebras.

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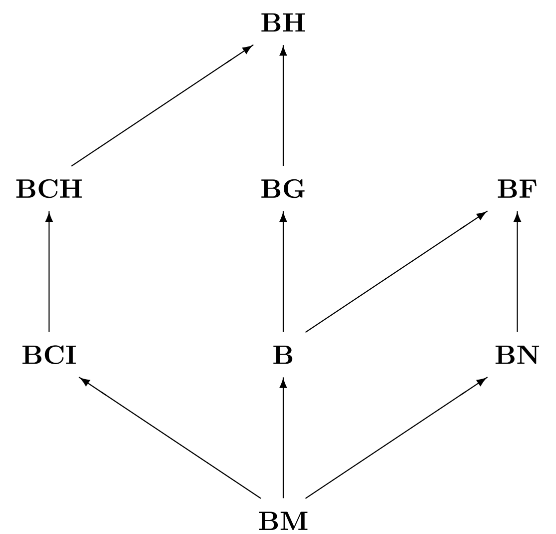

Figure 1

2 Preliminaries Throughout this paper $\mathcal{A}$ will denote an algebra $(A; *, 0)$ of type $(2, 0)$.

An algebra $\mathcal{A}$ is said to be a *BH-algebra* ([4]) if it satisfies (B1), (B2) and the following axiom:

(BH) $\quad x * y = y * x = 0 \Longrightarrow x = y.$

A BH-algebra $\mathcal{A}$ with the condition

(BCH) $\quad (x * y) * z = (x * z) * y$

(for all $x, y, z \in A$) is called a *BCH-algebra.* In [1], it is proved that $\mathcal{A}$ is a BCH-algebra if and only if it satisfies (B1), (BH), and (BCH).

A BH-algebra $\mathcal{A}$ satisfying the identity

(BCI) $\quad ((x * y) * (x * z)) * (z * y) = 0$

is called a *BCI-algebra.* Recall that according to the H. S. Li's axiom system ([9]), an algebra $\mathcal{A}$ is a BCI-algebra if and only if it satisfies (B2), (BH), and (BCI).

Remark 2.1. We know that every BCI-algebra is a BCH-algebra and every BCH-algebra is a BH-algebra.

Let an algebra $\mathcal{A}$ satisfy identities (B1) and (B2). We say that $\mathcal{A}$ is a *B-algebra* (resp., *BF/BG/BN-algebra*) if $\mathcal{A}$ satisfies axiom (B) (resp., (BF)/(BG)/(BN)), where:

(B) $\quad (x * y) * z = x * [z * (0 * y)],$
(BF) $\quad 0 * (x * y) = y * x,$
(BG) $\quad x = (x * y) * (0 * y),$
(BN) $\quad (x * y) * z = (0 * z) * (y * x),$

An algebra $\mathcal{A}$ is called a *BM-algebra* ([6]) if it satisfies (B2) and the following axiom:

(BM) $\quad (x * y) * (x * z) = z * y.$

Remark 2.2. From Theorem 2.6 of [6] it follows that every BM-algebra is a B-algebra. By Theorem 2.2 and Proposition 2.8 of [5], every B-algebra is a BG-algebra and every BG-algebra is a BH-algebra. It is easy to see that (BM) implies (BCI). Therefore the class of BM-algebras is a subclass of the class of BCI-algebras.

An algebra $\mathcal{A}$ is said to be 0-*commutative* (resp., *commutative*) if $x * (0 * y) = y * (0 * x)$ (resp., $x * y = y * x$) for any $x, y \in A$.

Remark 2.3. In [6], it is proved that $\mathcal{A}$ is a BM-algebra if and only if it is a 0-commutative B-algebra. C. B. Kim and H. S. Kim ([7]) showed that an algebra is a BN-algebra if and only if it is a 0-commutative BF-algebra (therefore, every BN-algebra is a BF-algebra). By Corollary 2.12 of [7], every BM-algebra is a BN-algebra.

Proposition 2.4. ([7]) *If* $(A; *, 0)$ *is a BN-algebra, then*

(a) $0 * (0 * x) = x$,
(b) $y * x = (0 * x) * (0 * y)$

for all $x, y \in A$.

H. S. Kim, Y. H. Kim and J. Neggers introduced the concepts of Coxeter algebras and pre-Coxeter algebras. A *Coxeter algebra* ([8]) is an algebra $\mathcal{A}$ satisfying identities (B1), (B2) and

(As) $x * (y * z) = (x * y) * z$.

It is known that a Coxeter algebra is a special type of abelian groups (see [8]). In [7], it is proved that $\mathcal{A}$ is a Coxeter algebra if and only if it is a BN-algebra satisfying the following axiom:

(D) $(x * y) * z = x * (z * y)$.

Proposition 2.5. ([6]) *Every Coxeter algebra is a BM-algebra.*

Proposition 2.6. ([6]) *If* $\mathcal{A}$ *is a BM-algebra satisfying the condition*

(B2') $0 * x = x$,

then it is a Coxeter algebra.

A commutative BH-algebra is called a *pre-Coxeter algebra* (shortly, *PC-algebra*). The class of all Coxeter algebras (resp., pre-Coxeter algebras) we denote by **CA** (resp., **PC**). Every Coxeter algebra is a PC-algebra and there is a PC-algebra which is not a Coxeter algebra (see [8]). Consequently, **CA** is a proper subclass of **PC**. Every BM-algebra satisfying the condition (B2') is a PC-algebra (see Theorem 3.7 of [6]). In general, a PC-algebra need not be a BM-algebra (see Example 3.8 of [6]).

From Proposition 2.5 and Remark 2.3 we obtain

$$\mathbf{CA} \subset \mathbf{BM} \subset \mathbf{BN} \subset \mathbf{BF}. \tag{1}$$

Let $\mathcal{A}$ be a PC-algebra. Observe that $\mathcal{A}$ is a BN-algebra. Indeed, $(x * y) * z = z * (y * x) = (0 * z) * (y * x)$ for all $x, y, z \in A$. Therefore, $\mathcal{A}$ satisfies (BN) and consequently, $\mathcal{A}$ is a BN-algebra. Thus

$$\mathbf{PC} \subset \mathbf{BN}. \tag{2}$$

3. On BN$_1$-algebras By definition, $\mathcal{A} = (A; *, 0)$ is a BN$_1$-algebra if and only if it is a BN-algebra satisfying (BN$_1$).

Example 3.1. Let $A = \{0, 1\}$ and $*$ be defined by the following table:

$*$	0	1
0	0	1
1	1	0

Then $(A; *, 0)$ is a BN_1-algebra.

Example 3.2. Let $A = \{0, 1, 2, 3\}$ and define the binary operation "$*$" on A by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(A; *, 0)$ is a BN_1-algebra (In fact, A is the Klein 4-group.)

Example 3.3. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $*$ be defined by the following table:

$*$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	6	7	4	5	9	8
2	2	3	0	1	8	6	5	9	4	7
3	3	2	1	0	7	8	9	4	5	6
4	4	6	8	7	0	9	1	3	2	5
5	5	7	6	8	9	0	2	1	3	4
6	6	4	5	9	1	2	0	8	7	3
7	7	5	9	4	3	1	8	0	6	2
8	8	9	4	5	2	3	7	6	0	1
9	9	8	7	6	5	4	3	2	1	0

It is easy to check that $(A; *, 0)$ is a BN_1-algebra.

Proposition 3.4. *If* $(A; *, 0)$ *is a* BN_1*-algebra, then*

(P1) $0 * x = x$,
(P2) $x = (x * y) * (0 * y)$,
(P3) $x * y = y * x$,
(P4) $x = y * (y * x)$.
(P5) $x * y = 0 \Longrightarrow x = y$,
(P6) $x * y = y \Longrightarrow x = 0$,
(P7) $x * y = x \Longrightarrow y = 0$,
(P8) $x * y = x * z \Longrightarrow y = z$,

for all $x, y, z \in A$.

Proof. Let $x, y, z \in A$.

(P1) Applying (BN_1) and (B1) we have $x = (x * x) * x = 0 * x$, that is, (P1) holds.

(P2) By (BN_1) and (P1).

(P3) From (P1) and Theorem 2.4 (b) we obtain

$$x * y = (0 * y) * (0 * x) = y * x.$$

(P4) Clear.

(P5) By Corollary 3.10 of [7].

(P6) Let $x * y = y$. Using (BN_1) and (B1) we get $x = (x * y) * y = y * y = 0$. Therefore (P6) is satisfied.

(P7) The proof is similar to the proof of (P6).

(P8) Let $x * y = x * z$. Hence $x * (x * y) = x * (x * z)$. By (P4), $y = z$. Thus (P8) holds. □

Proposition 3.5. *Every BN_1-algebra has the unique solution property.*

Proof. Let $\mathcal{A}$ be a BN_1-algebra and $a, b \in A$. It is easy to see that the equations $x * b = a$ and $b * x = a$ have solutions given by $x = a * b$ and $x = b * a$, respectively. (P8) implies that in each case, such x is unique. □

Theorem 3.6. *Every BN_1-algebra is a loop.*

Proof. Let $\mathcal{A}$ be a BN_1-algebra. Since $x * 0 = 0 * x = x$ for each $x \in A$ and $\mathcal{A}$ has the unique solution property, we conclude that $\mathcal{A}$ is a loop. □

Remark 3.7. There is a loop which is not a BN_1-algebra. Let $A = \{0, 1, 2, 3, 4\}$ and define the binary operation "$*$" on A by the following table:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Then $(A; *, 0)$ is a loop but it is not a BN_1-algebra, since $(1 * 2) * 2 = 3 * 2 = 4 \neq 1$.

Theorem 3.8. *An algebra $\mathcal{A}$ is a BN_1-algebra if and only if it satisfies the following axioms:*

(B1) $x * x = 0$,

(C) $x * y = y * x$,

(BN_1) $(x * y) * y = x$.

Proof. Let $\mathcal{A}$ be a BN_1-algebra. By definition and property (P3), $\mathcal{A}$ satisfies (B1), (BN_1) and (C).

Conversely, suppose that the above identities hold in $\mathcal{A}$. From (BN_1) and (B1) we have $x = (x * x) * x = 0 * x$ for all $x \in A$, that is, (B2') is satisfied. Using commutativity of $*$ we get (B2). Observe that (BN) also holds in $\mathcal{A}$. Let $x, y, z \in A$. Applying (C) and (B2') we obtain

$$(x * y) * z = z * (y * x) = (0 * z) * (y * x).$$

Thus $\mathcal{A}$ is a BN-algebra and finally, $\mathcal{A}$ is a BN_1-algebra. □

Theorem 3.9. *An algebra $\mathcal{A}$ is a BN_1-algebra if and only if it is a commutative BG-algebra.*

Proof. Let $\mathcal{A}$ be a BN_1-algebra. By (P2), $\mathcal{A}$ satisfies (BG). From property (P3) we see that the operation $*$ is commutative.

Conversely, if $\mathcal{A}$ is a commutative BG-algebra, then $\mathcal{A}$ satisfies (B1), (C) and (BN_1). From Theorem 3.8 it follows that $\mathcal{A}$ is a BN_1-algebra. □

It is easy to see that every Coxeter algebra is a BN_1-algebra, that is,

$$\mathbf{CA} \subset \mathbf{BN}_1. \tag{3}$$

Proposition 3.10. *If $\mathcal{A}$ is a BN_1-algebra, then it is a PC-algebra.*

Proof. From (P5) it follows that $\mathcal{A}$ satisfies the condition (BH). Since the operation $*$ is commutative, we see that $\mathcal{A}$ is a commutative BH-algebra, that is, $\mathcal{A}$ is a PC-algebra. □

The converse of Proposition 3.10 does not hold in general. The PC-algebra $(A; *, 0)$ given in Example 4.7 of [8] is not a BN_1-algebra, since $(2 * 1) * 1 = 3 \neq 2$.

Remark 3.11. Let $A = \{0, 1, 2\}$ and $*$ be defined by the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(A; *, 0)$ is a BM-algebra (see [6]) but it is not a PC-algebra. Consequently, $\mathbf{BM} \not\subseteq \mathbf{PC}$. Hence $\mathbf{BM} \not\subseteq \mathbf{BN}_1$.

Remark 3.12. The BN_1-algebra given in Example 3.3 is not a BM-algebra, since $(1 * 3) * (1 * 4) = 2 * 6 = 5 \neq 7 = 4 * 3$. Therefore, $\mathbf{BN}_1 \not\subseteq \mathbf{BM}$ and hence $\mathbf{PC} \not\subseteq \mathbf{BM}$.

From (1)–(3), Proposition 3.10, and Remarks 3.11 and 3.12 we obtain the following interrelationships between some of the class of algebras mentioned above.

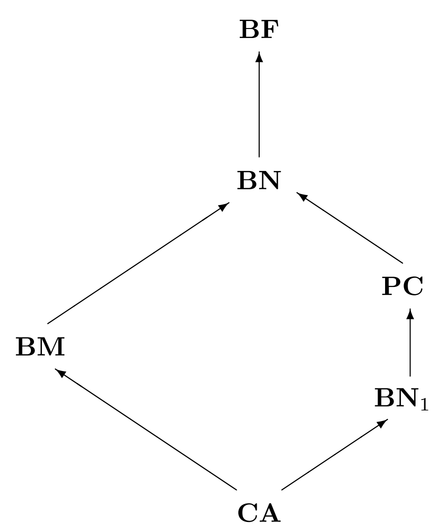

Figure 2

Lemma 3.13. *If an algebra $\mathcal{A}$ satisfies the commutative law* (C) *and* (B2'), *then condition* (As) *implies condition* (B).

Proof. Using associativity, commutativity and (B2') we obtain

$$(x * y) * z = x * (y * z) = x * (z * y) = x * (z * (0 * y))$$

for all $x, y, z \in A$, that is, (B) holds.

Lemma 3.14. *If $\mathcal{A}$ satisfies* (B2'), *then condition* (B) *implies condition* (D).

Proof. Let $x, y, z \in A$. We have

$$(x * y) * z = x * (z * (0 * y)) = x * (z * y),$$

i.e., (D) is true in $\mathcal{A}$. □

Lemma 3.15. *Let $\mathcal{A}$ satisfy* (C). *Then* (D) *implies* (BCH).

Proof. Let $x, y, z \in A$. Applying commutativity of $*$ and (D) we get

$$(x * y) * z = (y * x) * z = y * (z * x) = (x * z) * y.$$

Thus (BCH) is valid in $\mathcal{A}$. □

Lemma 3.16. *Let $\mathcal{A}$ satisfy* (C) *and* (BN_1). *Then* (BCH) *implies* (BM).

Proof. Let $x, y, z \in A$. Using (BCH), (C) and (BN_1) we have

$$(x * y) * (x * z) = (x * (x * z)) * y = ((z * x) * x)) * y = z * y,$$

i.e., (BM) holds in $\mathcal{A}$. □

Lemma 3.17. *Let* (B1) *hold in $\mathcal{A}$. Then condition* (BM) *implies condition* (BCI).

Proof. By (BM) and (B1),

$$((x * y) * (x * z)) * (z * y) = (z * y) * (z * y) = 0$$

for all $x, y, z \in A$. This means that $\mathcal{A}$ satisfies (BCI). □

Lemma 3.18. *In BN_1-algebras,* (BCI) *implies* (As).

Proof. Let $\mathcal{A}$ be a BN_1-algebra satisfying (BCI) and $x, y, z \in A$. Then $((x * y) * (x * z)) * (z * y) = 0$. By (P5), $(x * y) * (x * z) = z * y$. Therefore $\mathcal{A}$ is a BM-algebra. From (P1) we see that (B2') holds in $\mathcal{A}$. Applying Proposition 2.6 we get (As). □

From Lemmas 3.13 – 3.18 we have the following result.

Theorem 3.19. *In a BN_1-algebra, the conditions* (As), (B), (D), (BCH), (BM), *and* (BCI) *are all equivalent.*

Corollary 3.20. *An algebra $\mathcal{A} = (A; *, 0)$ is a Coxeter algebra if and only if it is a BN_1-algebra with the associative law for $*$.*

Remark 3.21. From Theorem 3.19 it follows that

$$\mathbf{B} \cap \mathbf{BN}_1 = \mathbf{BCH} \cap \mathbf{BN}_1 = \mathbf{BM} \cap \mathbf{BN}_1 = \mathbf{BCI} \cap \mathbf{BN}_1 = \mathbf{CA}.$$

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