Classification of topological manifolds by the Euler characteristic and the K-theory ranks of C^*-algebras

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Abstract. We consider classification of homeomorphism classes of connected sums of closed surfaces by the Euler characteristic and the K-theory ranks of C^*-algebras. We next consider classification of those of connected sums of higher dimensional, closed topological manifolds by the Euler characteristic and the K-theory ranks of C^*-algebras.

1 Introduction In this paper, first of all, we consider classification (of homeomorphism classes) of connected sums of closed surfaces such as the real two-dimensional, sphere, torus, and the real projective plane, by the Euler characteristic in the K-theory of C^*-algebras. We obtain the Euler characteristic formula for the C^*-algebras corresponding to connected sums of closed surfaces and show that the classification list obtained by Euler characteristic in K-theory for the corresponding C^*-algebras is the same as the classification list for connected sums of closed surfaces by the Euler characteristic in homology, well known (see for instance, [5] inspired or [4]). As well, we consider another classification of these two-dimensional topological manifolds by the K-theory group (free) ranks (i.e. the Betti numbers) of the corresponding C^*-algebras.

We next consider classification (of homeomorphism classes) of connected sums of higher dimensional, closed topological manifolds such as higher dimensional, spheres, tori, and real projective spaces, by the Euler characteristic in the K-theory of C^*-algebras. We obtain the Euler characteristic formulae for the C^*-algebras corresponding to connected sums of the closed topological manifolds and do the classification list for connected sums of the closed topological manifolds by the K-theory Euler characteristic for the corresponding C^*-algebras. As well, we consider another classification of these higher dimensional, closed topological manifolds by the K-theory group (free) ranks (i.e. the Betti numbers) (as well as the K-theory torsion ranks in some cases) of the corresponding C^*-algebras.

In the process, and in the end we obtain the list of K-theory groups for the C^*-algebras considered so far in this paper, and as well we show that those closed topological manifolds are classifiable (up to homeomorphism) by K-theory data for C^*-algebras (together with dimension for spaces).

As a generalization from connected sums of topological manifolds, in the final section we define and consider connected sums of C^*-algebras viewed as noncommutative connected sums and obtain their Euler characteristic formula and K-theory rank formulae.

Now recall that the Euler characteristic (in K-theory) of a C^*-algebra $\mathfrak{A}$ is (first introduced by Hiroshi Takai and) defined to be the alternative sum of the Betti numbers of the K-theory groups of $\mathfrak{A}$:

$$\chi(\mathfrak{A}) = b_0(\mathfrak{A}) - b_1(\mathfrak{A}) \equiv \mathrm{rank}_{\mathbb{Z}} K_0(\mathfrak{A}) - \mathrm{rank}_{\mathbb{Z}} K_1(\mathfrak{A}),$$

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where each $\mathrm{rank}_{\mathbb{Z}} K_j(\mathfrak{A})$ means the (free) $\mathbb{Z}$-rank of the free summmand of the abelian group $K_j(\mathfrak{A})$ of $\mathfrak{A}$ ($j = 0, 1$). We say that a C^*-algebra $\mathfrak{A}$ has Euler number n if $\chi(\mathfrak{A}) = n$, where n is an integer, or may as well be $+\infty$ or $-\infty$ (or formally $\pm\infty - \pm\infty$).

We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on a separable, infinite dimensional, Hilbert space. Denote by $M_n(\mathbb{C})$ the $n \times n$ matrix C^*-algebra over the complex field $\mathbb{C}$. We denote by $C(X)$ the C^*-algebra of all complex-valued, continuous functions on a compact Hausdorff space X. Denote by $C_0(X)$ the C^*-algebra of all $\mathbb{C}$-valued, continuous functions on a locally compact Hausdorff space X vanishing at infinity.

Recall some basic facts on the Euler characteristic for C^*-algebras, which can be found in [8] or [9] and are used in the following sections without mentioning.

• Group isomorphisms $K_0(\mathbb{C}) \cong K_0(M_n(\mathbb{C})) \cong K_0(\mathbb{K}) \cong \mathbb{Z}$, and $K_1(\mathbb{C}) \cong K_1(M_n(\mathbb{C})) \cong K_1(\mathbb{K}) \cong 0$, so that $\chi(\mathbb{C}) = \chi(M_n(\mathbb{C})) = \chi(\mathbb{K}) = 1$.

• $K_0(C_0(\mathbb{R}^{2n})) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{R}^{2n})) \cong 0$ by the Bott periodicity, so that $\chi(C_0(\mathbb{R}^{2n})) = 1$. Also, $K_0(C_0(\mathbb{R}^{2n+1})) \cong 0$ and $K_1(C_0(\mathbb{R}^{2n+1})) \cong \mathbb{Z}$, so that $\chi(C_0(\mathbb{R}^{2n+1})) = -1$. Indeed, $\chi(C_0(X \times \mathbb{R})) = \chi(SC_0(\mathbb{R})) = -\chi(C_0(X))$ for X a locally compact Hausdorff space X. Moreover, $\chi(S\mathfrak{A}) = -\chi(\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$, with the suspension $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$, because $K_j(S\mathfrak{A}) = K_{j+1}(\mathfrak{A})$ with $j + 1 \bmod 2$.

• We have $\chi(C(X)) = \chi(X)$ the Euler characteristic of X in homology (or cohomology in the several definitions) via the Euler-Poincaré formula.

• For a short exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras, we have $\chi(\mathfrak{A}) = \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I})$ if each term is finite.

• For a tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of C^*-algebras, we have $\chi(\mathfrak{A} \otimes \mathfrak{B}) = \chi(\mathfrak{A}) \cdot \chi(\mathfrak{B})$ if each term is finite and if one of the tensor product factors belongs to the bootstrap category or the UCT class, which is deduced from the Künneth formula in K-theory of C^*-algebras.

Refer to [1] or [10] for some facts on K-theory of C^*-algebras, used below.

2 Connected sums of closed surfaces

A closed surface is a compact (real) 2-dimensional topological manifold without boundary. Let M, N be closed surfaces. The connected sum $M \# N$ of M and N is defined to be the closed surface obtained by removing the 2-dimensional closed unit disks D viewed on M and N from themselves and gluing the open differences $M \setminus D$ and $N \setminus D$ by attaching the unit circle S^1 (or the 1-dimensional torus $\mathbb{T}$) to them as their boundaries.

Theorem 2.1. *Let M, N be closed surfaces and $M \# N$ be their connected sum. Then*

$$\chi(C(M \# N)) = \chi(C(M)) + \chi(C(N)) - 2.$$

Proof. By the definition of $M \# N$, we have the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D) \sqcup (N \setminus D)) \xrightarrow{i} C(M \# N) \xrightarrow{q} C(\mathbb{T}) \to 0,$$

where i is the inclusion map and q is the quotient map since $\mathbb{T}$ attached in gluing is closed in $M \# N$, and $\sqcup$ means the disjoint union, and the closed ideal is isomorphic to the direct sum $C_0(M \setminus D) \oplus C_0(N \setminus D)$. Therefore, it follows that

$$\chi(C(M \# N)) = \chi(C_0(M \setminus D)) + \chi(C_0(N \setminus D)) + \chi(C(\mathbb{T})),$$

with $\chi(C(\mathbb{T})) = 0$, which follows from that there is the short exact sequence $0 \to C_0(\mathbb{R}) \to C(\mathbb{T}) \to \mathbb{C} \to 0$ of C^*-algebras.

Moreover, we have

$$0 \to C_0(M \setminus D) \xrightarrow{i} C(M) \xrightarrow{q} C(D) \to 0,$$

so that

$$\chi(C(M)) = \chi(C_0(M \setminus D)) + \chi(C(D)),$$

with $\chi(C(D)) = 1$, which follows from that D is contractible.

The same holds for N. □

Note that there is a homeomorphism between (2 times)-successive connected sums of three closed surfaces M_1, M_2, M_3, denoted as

$$(M_1\#M_2)\#M_3 \approx M_1\#(M_2\#M_3).$$

We denote both sides by $M_1\#M_2\#M_3$ or $\#_{i=1}^3 M_i$ (called a 3-connected sum) and apply this convention for more successive connected sums of finitely many closed surfaces.

Corollary 2.2. *Let $M_1\#M_2\cdots\#M_n$ be an $(n-1)$-successive connected sum of closed surfaces $M_1, M_2, \cdots, M_n$. Then*

$$\chi(C(M_1\#M_2\cdots\#M_n)) = \sum_{k=1}^n \chi(C(M_k)) - 2(n-1).$$

Example 2.3. Let $\mathbb{T}^2$ be the 2-dimensional torus. Denote by $\#^n\mathbb{T}^2$ the $(n-1)$-successive connected sum of n-copies of $\mathbb{T}^2$, and we call it the n-connected sum of $\mathbb{T}^2$. A closed surface is said to be an orientable closed surface with genus n if it is homeomorphic to the n-connected sum $\#^n\mathbb{T}^2$, and we denote it by $T(n)$. Set $T(1) \approx \mathbb{T}^2$. Then, by Theorem 2.1,

$$\chi(C(T(n))) = \chi(C(\#^n\mathbb{T}^2)) = -2(n-1) = 2-2n,$$

since $C(T(n))$ is isomorphic to $C(\#^n\mathbb{T}^2)$ and $\chi(C(\mathbb{T}^2)) = \chi(C(\mathbb{T}\times\mathbb{T})) = \chi(C(\mathbb{T})\otimes C(\mathbb{T})) = \chi(C(\mathbb{T}))\cdot\chi(C(\mathbb{T})) = 0$.

A closed surface is said to be a closed surface with genus zero if it is homeomorphic to the 2-dimensional sphere S^2, and we denote it by $S(0)$. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(\mathbb{R}^2) \to C(S^2) \to \mathbb{C} \to 0,$$

so that

$$\chi(C(S(0))) = \chi(C(S^2)) = \chi(C_0(\mathbb{R}^2)) + \chi(\mathbb{C}) = 1+1 = 2.$$

Note that $M\#S^2$ is homeomorphic to M for any closed surface M and that

$$\chi(C(M\#S(0))) = \chi(C(M)) + \chi(C(S(0))) - 2 = \chi(C(M)).$$

Let P^2 be the real 2-dimensional projective plane, obtained (in $\mathbb{R}^4$) by gluing the boundary of the Möbius band M_b with that of the closed unit disk D of $\mathbb{R}^2$, where the Möbius band M_b is obtained from a 2-dimensional closed interval I by identifying one edge E of four edges of I with the opposite edge with one twist. Denote by $\#^nP^2$ the $(n-1)$-successive connected sum of n-copies of P^2, and we call it the n-connected sum of P^2. A closed surface is said to be a non-orientable closed surface with genus n if it is homeomorphic to the n-connected sum $\#^nP^2$, and we denote it by $P(n)$. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(M_b^\circ) \to C(P^2) \to C(D) \to 0,$$

where M_b° is the interior of M_b. Also,

$$0 \to C_0(I^\circ) \to C_0(M_b^\circ) \to C_0(E^\circ) \to 0$$

with $I^\circ \approx \mathbb{R}^2$ and $E^\circ \approx \mathbb{R}$. Therefore,

$$\begin{aligned}\chi(C(P^2)) &= \chi(C_0(M_b^\circ)) + \chi(C(D)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0(\mathbb{R}^2)) + \chi(C_0(\mathbb{R})) + 1 = 1 - 1 + 1 = 1.\end{aligned}$$

Moreover, by Theorem 2.1,

$$\chi(C(P(n))) = \chi(C(\#^n P^2)) = n \cdot 1 - 2(n-1) = 2 - n.$$

Note that the 2-connected sum $P^2\#P^2$ is homeomorphic to the Klein bottle K^2, obtained by gluing two Möbius bands M_b along their boundaries homeomorphic to S^1. Note also that $\mathbb{T}^2\#P^2 \approx \#^3P^2$. An intuitive explanation for this fact is that $\mathbb{T}^2$ viewed as a square 2-dimensional closed interval with four edges identified with opposites is transformed by cutting the interval on the diagonal to the Klein bottle K^2 viewed as a square 2-dimensional closed interval with two edges identified with opposites (with no twist) and with the other two edges identified with opposites with one twist in the connected sum $\mathbb{T}^2\#P^2$. Moreover, $(\#^m\mathbb{T}^2)\#(\#^nP^2) \approx \#^{n+2m}P^2$.

Refer to [5] or [4] about connected sums of closed surfaces.

It is well known as a remarkable fact in (low dimensional) algebraic topology that closed surfaces (or compact 2-dimensional topological manifolds without boundary) X are classified as in the list of the Table 1, which becomes the same list as obtained by our Euler numbers $\chi(C(X))$ for the C^*-algebras $C(X)$:

Table 1: Classification for closed surfaces

Euler number	Orientable	Non-orientable
2	$S^2 \approx S(0)$	No
1	No	$P^2 \approx P(1)$
0	$\mathbb{T}^2 \approx T(1)$	$P^2\#P^2 \approx P(2) \approx K^2$
-1	No	$\#^3P^2 \approx P(3) \approx \mathbb{T}^2\#P^2$
-2	$\mathbb{T}^2\#\mathbb{T}^2 \approx T(2)$	$\#^4P^2 \approx P(4)$
$3-2n$	No	$\#^{2n-1}P^2 \approx P(2n-1)$
$2-2n$	$\#^n\mathbb{T}^2 \approx T(n)$	$\#^{2n}P^2 \approx P(2n)$

We now compute K-theory groups.

Theorem 2.4. *Let M, N be closed surfaces and $M\#N$ be their connected sum. Then*

$$\begin{aligned}K_0(C(M\#N)) &\cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\ K_1(C(M\#N)) &\cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial),\end{aligned}$$

where each $[1]$ means the K_0-class of the unit 1 of $C(M)$ or $C(N)$, with $\mathbb{Z}[1] \cong \mathbb{Z}$, and $[z]$ means the K_1-class of $K_1(C(\mathbb{T}))$ corresponding to the coordinate function on $\mathbb{T}$, with $\mathbb{Z}[z] = K_1(C(\mathbb{T})) \cong \mathbb{Z}$, and $\partial\mathbb{Z}[z]$ is the image under the associated boundary map ∂, whose kernel is denoted by $\ker(\partial)$ and is isomorphic to (a subgroup of) $\mathbb{Z}$ or zero.

Remark. The image $\partial\mathbb{Z}[z]$ may not be isomorphic to $\mathbb{Z}$ in general, which depends on $\ker(\partial)$, and the image may not split into a direct summand of $K_0(C(M\#N))$ in general, while each $\mathbb{Z}[1] \cong \mathbb{Z}$ always splits in its K_0-group. Also, the K-theory groups may have torsion in general. Examples are given after the proof below or later.

Proof. The six-term exact sequence of K-theory groups follows from the short exact sequence of $C(M\#N)$ in the proof of Theorem 2.1:

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \xrightarrow{i_*} & K_0(C(M\#N)) & \xrightarrow{q_*} & K_0(C(\mathbb{T})) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(\mathbb{T})) & \xleftarrow{q_*} & K_1(C(M\#N)) & \xleftarrow{i_*} & K_1(\mathfrak{I}) \end{array}$$

with $\mathfrak{I} = C_0((M \setminus D) \sqcup (N \setminus D))$ and $K_j(C(\mathbb{T})) = \mathbb{Z}[z] \cong \mathbb{Z}$ $(j = 0, 1)$ and

$$K_j(C_0((M \setminus D) \sqcup (N \setminus D))) \cong K_j(C_0(M \setminus D)) \oplus K_j(C_0(N \setminus D))$$

$(j = 0, 1)$, where the maps i_* and q_* are induced from the maps i and q, and ∂ are boundary maps (or the index map on the left and the exponential map on the right). The map q_* on K_0 sends the K_0-class of the unit of $C(M\#N)$ to that of $C(\mathbb{T})$, and hence is onto. Thus, ∂ on the right is zero by exactness of the diagram.

Moreover, we also have the following diagram:

$$\begin{array}{ccccc} K_0(C_0(M \setminus D)) & \xrightarrow{i_*} & K_0(C(M)) & \xrightarrow{q_*} & K_0(C(D)) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(D)) & \xleftarrow{q_*} & K_1(C(M)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D)) \end{array}$$

with $K_0(C(D)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C(D)) \cong K_1(\mathbb{C}) \cong 0$ since D is contractible. The map q_* on K_0 sends the K_0-class of the unit of $C(M)$ to that of $C(D)$, and hence is onto. Thus, both of the boundary maps ∂ are zero. Therefore, the diagram implies that

$$K_0(C(M)) \cong K_0(C_0(M \setminus D)) \oplus \mathbb{Z},$$
$$K_1(C(M)) \cong K_1(C_0(M \setminus D)),$$

where the direct summand $\mathbb{Z}$ corresponds to the K_0-class $[1]$ of the unit 1 of $C(M)$. Note also that $K_1(C_0(M \setminus D)) \cong K_1(C_0(M \setminus D)^+)$, where the C^*-algebra unitization $C_0(M \setminus D)^+$ by $\mathbb{C}$ is isomorphic to $C((M \setminus D)^+)$, with the one-point compactification $(M \setminus D)^+$, which is in fact homeomorphic to M.

The same holds for N.

It then follows from the first six-term diagram above that

$$K_0(C(M\#N)) \cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\},$$
$$K_1(C(M\#N)) \cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial).$$

□

Example 2.5. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((S^2 \setminus D) \sqcup (S^2 \setminus D)) \xrightarrow{i} C(S^2\#S^2) \xrightarrow{q} C(\mathbb{T}) \to 0$$

with $S^2 \setminus D \approx \mathbb{R}^2$ and $S^2\#S^2 \approx S^2$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial \\ \mathbb{Z}[z] & \xleftarrow{q_*} & 0 & \xleftarrow{i_*} & 0 \oplus 0. \end{array}$$

Therefore, the boundary map ∂ on the left is nonzero, and as in Theorem 2.4,

$$\begin{aligned} K_0(C(S^2\#S^2)) &\cong \mathbb{Z}\oplus\{[(K_0(C(S^2))/\mathbb{Z})\oplus(K_0(C(S^2))/\mathbb{Z})]\}/\partial\mathbb{Z}[z] \\ &\cong \mathbb{Z}\oplus\{[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^2, \\ K_1(C(S^2\#S^2)) &\cong K_1(C(S^2))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong 0\oplus 0\oplus 0\cong 0. \end{aligned}$$

Note that $\mathbb{Z}\oplus\mathbb{Z}$ is torsion free, so that $\partial\mathbb{Z}[z]\cong\mathbb{Z}$ and $\ker(\partial)\cong 0$, and that the diagram above does not involve torsion, so that $[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\cong\mathbb{Z}$.

Next, there is the following short exact sequence of C^*-algebras:

$$0\to C_0((\mathbb{T}^2\setminus D)\sqcup(S^2\setminus D)) \xrightarrow{\ i\ } C(\mathbb{T}^2\#S^2) \xrightarrow{\ q\ } C(\mathbb{T})\to 0$$

with $\mathbb{T}^2\#S^2\approx\mathbb{T}^2$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^2\setminus D))\oplus\mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & \mathbb{Z}^2 & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^2\setminus D))\oplus 0. \end{array}$$

Moreover, the exact sequence $0\to C_0(\mathbb{T}^2\setminus D)\to C(\mathbb{T}^2)\to C(D)\to 0$ implies

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^2\setminus D)) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ \partial=0\uparrow & & & & \downarrow\partial=0 \\ 0 & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & K_1(C_0(\mathbb{T}^2\setminus D)) \end{array}$$

so that $K_0(C_0(\mathbb{T}^2\setminus D))\cong\mathbb{Z}$ and $K_1(C_0(\mathbb{T}^2\setminus D))\cong\mathbb{Z}^2$. Therefore, the boundary map ∂ on the left in the second six-term diagram in this example is nonzero, and as in Theorem 2.4, by the same reasoning as above,

$$\begin{aligned} K_0(C(\mathbb{T}^2\#S^2)) &\cong \mathbb{Z}\oplus\{[(K_0(C(\mathbb{T}^2))/\mathbb{Z})\oplus(K_0(C(S^2))/\mathbb{Z})]\}/\partial\mathbb{Z}[z] \\ &\cong \mathbb{Z}\oplus\{[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^2, \\ K_1(C(\mathbb{T}^2\#S^2)) &\cong K_1(C(\mathbb{T}^2))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong\mathbb{Z}^2\oplus 0\oplus 0\cong\mathbb{Z}^2. \end{aligned}$$

Now, let M be a closed surface. There is the following short exact sequence of C^*-algebras:

$$0\to C_0((M\setminus D)\sqcup(S^2\setminus D)) \xrightarrow{\ i\ } C(M\#S^2) \xrightarrow{\ q\ } C(\mathbb{T})\to 0$$

with $S^2\setminus D\approx\mathbb{R}^2$ and $M\#S^2\approx M$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0(M\setminus D))\oplus\mathbb{Z} & \xrightarrow{i_*} & K_0(C(M\#S^2)) & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & K_1(C(M\#S^2)) & \xleftarrow{i_*} & K_1(C_0(M\setminus D))\oplus 0. \end{array}$$

Moreover, $K_1(C_0(M\setminus D))\cong K_1(C_0(M\setminus D)^+)\cong K_1(C(M))$. Therefore, the map q_* on K_1 is zero, so that $\ker(\partial)=0$. Also, $K_0(C(M))\cong K_0(C_0(M\setminus D)^+)\cong K_0(C_0(M\setminus D))\oplus\mathbb{Z}$. Thus, as in Theorem 2.4,

$$\begin{aligned} K_0(C(M\#S^2)) &\cong \mathbb{Z}\oplus\{[K_0(C_0(M\setminus D))\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z}\oplus K_0(C_0(M\setminus D))\cong K_0(C(M)), \\ K_1(C(M\#S^2)) &\cong K_1(C(M))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong K_1(C(M)). \end{aligned}$$

Corollary 2.6. *The formula obtained in Theorem* 2.1 *follows from Theorem* 2.4.

Proof. Note that each quotient by $\mathbb{Z}[1]$, or by $\partial\mathbb{Z}[z]$ together with $\ker(\partial)$ as a set, in Theorem 2.4 corresponds to either one rank lowering the free ranks of those K_0-groups, or either the same or one rank raising the free ranks of those K_1-groups, respectively, where either $\partial\mathbb{Z}[z]$ or $\ker(\partial)$ has rank one, Hence,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 3 = \chi(C(M)) + \chi(C(N)) - 2.$$

□

Corollary 2.7. *Let M_i $(1 \leq i \leq n)$ be closed surfaces and $\#_{i=1}^n M_i$ be their successive connected sum. Then, inductively,*

$$\begin{aligned}
&K_0(C(\#_{i=1}^n M_i)) \\
&\cong \mathbb{Z} \oplus \{[(K_0(C(\#_{i=1}^{n-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_n))/\mathbb{Z}[1])]/\partial_{n-1}\mathbb{Z}[z]\}, \\
&\cong \mathbb{Z} \oplus \{[((\cdots(\mathbb{Z} \oplus \{[(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]\}) \\
&\quad \cdots/\mathbb{Z}[1]) \oplus (K_0(C(M_n))/\mathbb{Z}[1])]/\partial_{n-1}\mathbb{Z}[z]\}, \\
&K_1(C(\#_{i=1}^n M_i)) \cong [\oplus_{i=1}^n K_1(C(M_i))] \oplus [\oplus_{i=1}^{n-1} \ker(\partial_i)],
\end{aligned}$$

where each $[1]$ *means the K_0-class of the unit* 1 *of* $C(M_i)$*, and* $[z] \in K_1(C(\mathbb{T}))$ *the generating class, and each $\partial_i = \partial$ is the boundary map in each step in induction.*

Corollary 2.8. *That Corollary* 2.2 *follows from this Corollary* 2.7.

Proof. Note that each quotient by $\mathbb{Z}[1]$, or by $\partial_i\mathbb{Z}[z]$ together with $\ker(\partial_i)$ as a set, in Corollary 2.7 corresponds to either one rank lowering the free ranks of those K_0-groups, or either the same or one rank raising the free ranks of those K_1-groups, respectively, where either $\partial_i\mathbb{Z}[z]$ or $\ker(\partial_i)$ has rank one. Hence, inductively,

$$\begin{aligned}
\chi(C(\#_{i=1}^n M_i)) &= \chi(C(\#_{i=1}^{n-1} M_i)) + \chi(C(M_i)) - 2 \\
&= \cdots = \sum_{i=1}^n \chi(C(M_i)) - 2(n-1).
\end{aligned}$$

□

Example 2.9. Since $K_j(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ $(j = 0, 1)$, it is obtained by Theorem 2.4 that

$$\begin{aligned}
K_0(C(\mathbb{T}^2\#\mathbb{T}^2)) &\cong \mathbb{Z} \oplus \{[(\mathbb{Z}^2/\mathbb{Z}) \oplus (\mathbb{Z}^2/\mathbb{Z})]\}/\partial\mathbb{Z} \\
&\cong \mathbb{Z} \oplus \{[\mathbb{Z} \oplus \mathbb{Z}]/\partial\mathbb{Z}\} \cong \mathbb{Z}^2, \\
K_1(C(\mathbb{T}^2\#\mathbb{T}^2)) &\cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \ker(\partial) \cong \mathbb{Z}^4.
\end{aligned}$$

Hence, $\chi(C(\mathbb{T}^2\#\mathbb{T}^2)) = (3-1) - 4 = -2$.

Moreover, it is obtained by Corollary 2.7 that

$$\begin{aligned}
K_0(C(\#^n\mathbb{T}^2)) &\cong \mathbb{Z} \oplus \{[(\cdots([\mathbb{Z} \oplus \mathbb{Z}]/\partial_1\mathbb{Z})\cdots) \oplus \mathbb{Z}]/\partial_{n-1}\mathbb{Z}\}, \\
K_1(C(\#^n\mathbb{T}^2)) &\cong [\oplus_{i=1}^n \mathbb{Z}^2] \oplus [\oplus_{i=1}^{n-1} \ker(\partial_i)] \cong \mathbb{Z}^{2n}.
\end{aligned}$$

Hence, $\chi(C(\#^n\mathbb{T}^2)) = (1 + n - (n-1)) - 2n = 2 - 2n$. Indeed, it is deduced that $K_0(C(\#^n\mathbb{T}^2)) \cong \mathbb{Z}^2$ by repeating the reasoning as before.

As for the real 2-dimensional, projective plane P^2, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C(P^2)$ in Example 2.3, becomes

$$\begin{array}{ccccc} K_0(C_0(M_b^\circ)) & \longrightarrow & K_0(C(P^2)) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle \partial=0}\uparrow & & & & \downarrow{\scriptstyle \partial=0} \\ 0 & \longleftarrow & K_1(C(P^2)) & \longleftarrow & K_1(C_0(M_b^\circ)) \end{array}$$

so that

$$\begin{aligned} K_0(C(P^2)) &\cong \mathbb{Z} \oplus K_0(C_0(M_b^\circ)), \\ K_1(C(P^2)) &\cong K_1(C_0(M_b^\circ)). \end{aligned}$$

Indeed, P^2 is viewed as the one-point compactification of the interior M_b° of the Möbius band M_b. Moreover, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C_0(M_b^\circ)$ in Example 2.3, becomes

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(C_0(M_b^\circ)) & \longrightarrow & 0 \\ {\scriptstyle \partial}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ \mathbb{Z} & \longleftarrow & K_1(C_0(M_b^\circ)) & \longleftarrow & 0. \end{array}$$

Furthermore, it follows from the diagram that $K_1(C_0(M_b^\circ))$ viewed as a subgroup of $\mathbb{Z}$ without torsion is isomorphic to $\mathbb{Z}$ or zero. On the other hand, there is the quotient map from $C_0(M_b^\circ)^+ \cong C((M_b^\circ)^+)$ to $C_0(\mathbb{R})^+ \cong C(\mathbb{T})$, which induces the induced map q_* on K_1-groups to be zero. In fact, $(M_b^\circ)^+$ is homeomorphic to the Moore space of order two, so that $K_0(C((M_b^\circ)^+)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $K_1(C((M_b^\circ)^+)) \cong 0$ (see [7] and [6, 12.3]). Hence it follows that $K_0(C_0(M_b^\circ)) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $K_1(C_0(M_b^\circ)) \cong 0$. Therefore, we do have

$$K_0(C(P^2)) \cong \mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad K_1(C(P^2)) \cong 0.$$

Hence $\chi(C(P^2)) = 1$. (Note that these results are compatible with those in homology theory for P^2 in the sense that the Euler characteristic obtained in homology theory for P^2 coincides with our Euler number for $C(P^2)$. See for instance, [3].)

It is obtained by Theorem 2.4 that

$$\begin{aligned} K_0(C(P^2\#P^2)) &\cong \mathbb{Z} \oplus \{[(([\mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\mathbb{Z}) \oplus ([\mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\mathbb{Z})]/\partial_1\mathbb{Z}\} \\ &\cong \mathbb{Z} \oplus \{[(\mathbb{Z}/\partial\mathbb{Z}) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_1\mathbb{Z}\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z} \oplus \mathbb{Z}_2^2, \\ K_1(C(P^2\#P^2)) &\cong 0 \oplus 0 \oplus \ker(\partial_1) \cong \mathbb{Z} \end{aligned}$$

with in fact $\mathbb{Z}/\partial\mathbb{Z} \cong \mathbb{Z}_2$, so that $\partial_1\mathbb{Z} \cong 0$ and $\ker(\partial_1) \cong \mathbb{Z}$. Hence $\chi(C(P^2\#P^2)) = 1-1 = 0$. Indeed, it follows from [6, 12.3] that the image $\partial_1\mathbb{Z}$ is zero.

Moreover, it is obtained by Corollary 2.7 that

$$\begin{aligned} K_0(C(\#^n P^2)) &\cong \mathbb{Z} \oplus \{[(\cdots([(\mathbb{Z}/\partial\mathbb{Z}) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_1\mathbb{Z})\cdots) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_{n-1}\mathbb{Z}\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_2^n, \\ K_1(C(\#^n P^2)) &\cong [\oplus_{i=1}^n 0] \oplus [\oplus_{i=1}^{n-1}\ker(\partial_i)] \cong \mathbb{Z}^{n-1}. \end{aligned}$$

Hence $\chi(C(\#^n P^2)) = 1-(n-1) = 2-n$. In fact, since $\mathbb{Z}/\partial\mathbb{Z} \cong \mathbb{Z}_2$, we have the image $\partial_1\mathbb{Z}$ isomorphic to 0, and inductively, the image $\partial_{n-1}\mathbb{Z}$ isomorphic to 0, so that each $\ker(\partial_i)$ is isomorphic to $\mathbb{Z}$.

Table 2: Classification for closed surfaces

K_0 rank	Orientable	Non-orientable
2	$S^2 \approx S(0)$ $\mathbb{T}^2 \approx T(1)$ $\#^n\mathbb{T}^2 \approx T(n)$	No
1	No	$P^2 \approx P(1)$ $P^2\#P^2 \approx P(2) \approx K^2$ $\#^n P^2 \approx P(n)$

It follows from the Table 2 that:

Corollary 2.10. *The rank of K_0-groups for C^*-algebras can not classify homeomorphism classes of orientable closed surfaces, and as well, the rank of K_0-groups for C^*-algebras does not classify homeomorphism classes of non-orientable closed surfaces.*

But, the rank of K_0-groups for C^*-algebras does distinguish the class of homeomorphism classes of orientable, closed surfaces from the class of those of non-orientable, closed surfaces.

Table 3: Classification for closed surfaces

K_1 rank	Orientable	Non-orientable
$2n$	$\#^n\mathbb{T}^2 \approx T(n)$	$\#^{2n+1}P^2 \approx P(2n+1)$
$2n-1$	No	$\#^{2n}P^2 \approx P(2n)$
4	$\mathbb{T}^2\#\mathbb{T}^2 \approx T(2)$	$\#^5P^2 \approx P(5)$
3	No	$\#^4P^2 \approx P(4)$
2	$\mathbb{T}^2 \approx T(1)$	$\#^3P^2 \approx P(3) \approx \mathbb{T}^2\#P^2$
1	No	$P^2\#P^2 \approx P(2) \approx K^2$
0	$S^2 \approx S(0)$	$P^2 \approx P(1)$

It follows from the Table 3 that:

Corollary 2.11. *The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of orientable, closed surfaces and as well, those of non-orientable, closed surfaces.*

3 Connected sums of closed topological manifolds A closed topological manifold is a compact real n-dimensional topological manifold without boundary ($n \geq 1$). Let M, N be n-dimensional closed topological manifolds. The connected sum $M\#N$ of M and N is defined to be the closed topological manifold obtained by removing the n-dimensional closed unit balls D^n (of $\mathbb{R}^n$) viewed on M and N from themselves and gluing the open differences $M \setminus D^n$ and $N \setminus D^n$ by attaching the $(n-1)$-dimensional sphere S^{n-1} (or the boundary ∂D^n of D^n) to them as their boundaries ($n \geq 2$), where when $n = 1$, $D^1 = [-1, 1]$ the closed interval and $S^0 = \partial D^1 = \{-1, 1\}$ the set of two points.

Theorem 3.1. *Let M, N be n-dimensional closed topological manifolds and $M\#N$ be their connected sum ($n \geq 1$). If n is even, then*

$$\chi(C(M\#N)) = \chi(C(M)) + \chi(C(N)) - 2,$$

and if n is odd, then

$$\chi(C(M\#N)) = \chi(C(M)) + \chi(C(N)).$$

Proof. By the definition of $M\#N$, we have the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D^n) \sqcup (N \setminus D^n)) \to C(M\#N) \to C(S^{n-1}) \to 0,$$

where $\sqcup$ means the disjoint union, and the closed ideal is isomorphic to the direct sum $C_0(M \setminus D^n) \oplus C_0(N \setminus D^n)$. Therefore, it follows that

$$\chi(C(M\#D)) = \chi(C_0(M \setminus D^n)) + \chi(C_0(N \setminus D^n)) + \chi(C(S^{n-1})).$$

Moreover, we have

$$0 \to C_0(M \setminus D^n) \to C(M) \to C(D^n) \to 0,$$

so that

$$\chi(C(M)) = \chi(C_0(M \setminus D^n)) + \chi(C(D^n)),$$

with $\chi(C(D^n)) = 1$ since D^n is contractible to a point, so that $C(D^n)$ is contractible to $\mathbb{C}$, and the Euler characteristic is stable under homotopy equivalence in C^*-algebras.

The same holds for N. Also, we have

$$0 \to C_0(\mathbb{R}^{n-1}) \to C(S^{n-1}) \to \mathbb{C} \to 0$$

since S^{n-1} is viewed as the one-point compactification of $\mathbb{R}^{n-1}$, so that

$$\chi(C(S^{n-1})) = \begin{cases} -1 + 1 = 0 & \text{if } n \text{ is even,} \\ 1 + 1 = 2 & \text{if } n \text{ is odd} \end{cases}$$

($n \geq 2$), where note that $C(S^0) \cong \mathbb{C}^2$, and hence the equation above for $n = 1$ also holds. □

Corollary 3.2. *Let $M_1\#M_2 \cdots \#M_l$ be an $(l-1)$-successive connected sum of closed n-dimensional topological manifolds $M_1, M_2, \cdots, M_l$. If n is even, then*

$$\chi(C(M_1\#M_2 \cdots \#M_l)) = \sum_{k=1}^{l} \chi(C(M_k)) - 2(l-1),$$

and if n is odd, then

$$\chi(C(M_1\#M_2 \cdots \#M_l)) = \sum_{k=1}^{l} \chi(C(M_k)).$$

As an interest, we consider higher dimensional analogues of closed surfaces.

Example 3.3. Let $\mathbb{T}^n$ be the n-dimensional torus. Then $\chi(C(\mathbb{T}^n)) = 0$ since $\chi(C(\mathbb{T}^n)) = \chi(C(\Pi^n \mathbb{T})) = \chi(\otimes^n C(\mathbb{T})) = \chi(C(\mathbb{T}))^n = 0^n$, with $\Pi^n \mathbb{T} = \mathbb{T} \times \cdots \times \mathbb{T}$. Therefore, we have

$$\chi(C(\#^l \mathbb{T}^n)) = \begin{cases} -2(l-1) = 2 - 2l & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Let S^n be the n-dimensional sphere. Then

$$\chi(C(\#^l S^n)) = \begin{cases} 2l - 2(l-1) = 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Note that for M a closed n-dimensional topological manifold,

$$\chi(C(M\#S^n)) = \begin{cases} \chi(C(M)) + \chi(C(S^n)) - 2 = \chi(C(M)) & \text{if } n \text{ is even,} \\ \chi(C(M)) + \chi(C(S^n)) = \chi(C(M)) & \text{if } n \text{ is odd.} \end{cases}$$

Indeed, $M\#S^n \approx M$. Thus, in particular, $\#^l S^n \approx S^n$.

Let P^n be the real n-dimensional projective space, which is a quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ (or of S^n), where two points x, y in $\mathbb{R}^{n+1}\setminus\{0\}$ are equivalent if there is $t \in \mathbb{R}$ such that $x = ty$. Now we view P^n as obtained by gluing the boundary of the n-dimensional Möbius band M_b^n with that of the closed unit ball D^n of $\mathbb{R}^n$, where the n-dimensional Möbius band M_b^n defined by us is obtained from the product space $I = [0,1] \times [\{-\infty\} \cup (P^{n-1})^- \cup \{\infty\}]$ by identifying one edge $E \approx [0,1]$ at $-\infty$ with the opposite edge at ∞ with one twist, so that $(P^{n-1})^- \cup \{\pm\infty\} \approx P^{n-1}$ with $+\infty = -\infty$ identified, where $(P^{n-1})^-$ means our uncompactification of P^{n-1} by removing one point. One can check that this should be true, as follows. We have the decomposition $S^n = S^n_+ \cup S^{n-1} \cup S^n_-$ as a disjoint union, where $S^n_+ \cup S^n_- = S^n \setminus S^{n-1}$ with the north and south poles contained in S^n_+ and S^n_- respectively. Then S^n_+ is homeomorphic to the interior of D^n and is identified with S^n_- in P^n, and P^n is obtained by gluing the boundary P^{n-1} of the n-dimensional Möbius band M_b^n with that of D^n mapped in P_n by the quotient map from S^n to P^n. Refer also [3, Section 2.8] for the cell decomposition for P^n as $P^n = P^{n-1} \sqcup \mathbb{R}^n$ a disjoint union.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M_b^n)^\circ) \to C(P^n) \to C(D^n) \to 0,$$

where $(M_b^n)^\circ$ is the interior of M_b^n. Also, we have

$$0 \to C_0(I^\circ) \to C_0((M_b^n)^\circ) \to C_0(E^\circ) \to 0.$$

Therefore, we get

$$\begin{aligned} \chi(C(P^n)) &= \chi(C_0((M_b^n)^\circ)) + \chi(C(D^n)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0((P^{n-1})^- \times \mathbb{R})) + \chi(C_0(\mathbb{R})) + 1 \\ &= -\chi(C_0((P^{n-1})^-)) - 1 + 1 \\ &= -[\chi(C(P^{n-1})) - 1] \\ &= -\chi(C(P^{n-1})) + 1, \end{aligned}$$

$(n \geq 1)$, where we have $0 \to C_0((P^{n-1})^-) \to C(P^{n-1}) \to \mathbb{C} \to 0$ split, and $P^1 \approx S^1 = \mathbb{T}$, and P^0 is identified with the quotient $\{-1 = +1\}$ of S^0. The equation obtained above is converted to

$$a_n - \frac{1}{2} = -\left(a_{n-1} - \frac{1}{2}\right)$$

$(n \geq 1)$ with $a_n = \chi(C(P^n))$ for $n \geq 1$, so that

$$\begin{aligned} \chi(C(P^n)) &= \frac{1}{2}(1 + (-1)^n) \quad (n \geq 1) \\ &= \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(Note that this result is compatible with that in homology theory for P^n in the sense that the Euler characteristic in homology theory for P^n coincides with our Euler number for $C(P^n)$. Indeed,

$$H_0(P^{2n}) \cong \mathbb{Z}, \quad H_{2k}(P^{2n}) \cong 0, \quad \text{and} \quad H_{2k-1}(P^{2n}) \cong \mathbb{Z}/2\mathbb{Z} \quad (k = 1, \cdots, n),$$

so that $\chi(P^{2n}) = 1$. Also,

$$\begin{aligned} &H_0(P^{2n+1}) \cong \mathbb{Z}, \quad H_{2k}(P^{2n+1}) \cong 0, \quad H_{2n+1}(P^{2n+1}) \cong \mathbb{Z}, \\ &\text{and} \quad H_{2k-1}(P^{2n+1}) \cong \mathbb{Z}/2\mathbb{Z} \quad (k = 1, \cdots, n), \end{aligned}$$

so that $\chi(P^{2n+1}) = 0$ (see [3, Section 3.7]).)

Moreover, if n is even, then

$$\chi(C(\#^l P^n)) = l - 2(l-1) = 2 - l,$$

and if n is odd, then

$$\chi(C(\#^l P^n)) = 0.$$

Note that $P^n \# P^n$ is homeomorhic to the closed topological manifold obtained by glueing two n-dimensional Möbius bands M_b^n along with their boundaries homeomorphic to P^{n-1}, which we may call it the n-dimensional Klein bottle, and denote it by K^n. Note also that $\mathbb{T}^2 \# P^2 \approx \#^3 P^2$, and as well, we may have $\mathbb{T}^n \# P^n \approx \#^3 P^n$ $(n \geq 3)$ (as our consequence). (Our intuitive explanation for this is that $\mathbb{T}^n$ viewed as a square n-dimensional closed interval with edges identified with opposites (with no twist) is transformed by cutting the interval on the diagonal to the Klein bottle K^n viewed as a square n-dimensional closed interval with edges identified with opposites with half twisted and with half no twisted alternatively in the connected sum $\mathbb{T}^n \# P^n$.) Indeed, if n is even,

$$\begin{aligned} \chi(C(\mathbb{T}^n \# P^n)) &= \chi(C(\mathbb{T}^n)) + \chi(C(P^n)) - 2 = 0 + 1 - 2 = -1, \\ \chi(C(\#^3 P^n)) &= 2 - 3 = -1, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} \chi(C(\mathbb{T}^n \# P^n)) &= \chi(C(\mathbb{T}^n)) + \chi(C(P^n)) \\ &= 0 + 0 = 0, \\ \chi(C(\#^3 P^n)) &= 0. \end{aligned}$$

Table 4: Classification for closed topological manifolds with dimension even

Euler number	Orientable	Non-orientable
2	$S^{2n} \approx \#^l S^{2n}$, S^0	No
1	No	P^{2n}, P^0
0	$\mathbb{T}^{2n}$	$P^{2n} \# P^{2n} = K^{2n}$
-1	No	$\#^3 P^{2n}$
-2	$\mathbb{T}^{2n} \# \mathbb{T}^{2n}$	$\#^4 P^{2n}$
$3 - 2l$	No	$\#^{2l-1} P^{2n}$
$2 - 2l$	$\#^l \mathbb{T}^{2n}$	$\#^{2l} P^{2n}$

Note that $\#^3 P^{2n} \approx \mathbb{T}^{2n} \# P^{2n}$ by the reason given above. It follows from the Table 4 that

Corollary 3.4. *Let n be a natural number. The Euler characteristic in K-theory of C^*-algebras classifies even $2n$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the even $2n$-dimensional torus $\mathbb{T}^{2n}$ and the even $2n$-dimensional sphere S^{2n}, and also does even $2n$-dimensional non-orientable topological manifolds in the homeomorphism classes of connected sums of the even $2n$-dimensional projective space P^{2n}.*

Table 5: Classification for closed topological manifolds with dimension odd

Euler number	Orientable	Orientable
0	$\mathbb{T}^{2n+1}$, $\#^l\mathbb{T}^{2n+1}$ $S^{2n+1} \approx \#^l S^{2n+1}$	P^{2n+1} $\#^l P^{2n+1}$

Note that P^n is orientable if n is odd but not if n is even (see [3]). We also have $P^{2n+1}\#P^{2n+1} = K^{2n+1}$ and $\mathbb{T}^{2n+1}\#P^{2n+1} \approx \#^3P^{2n+1}$. It follows from the Table 5 that

Corollary 3.5. *Let n be a natural number. The Euler characteristic in K-theory of C^*-algebras can not classify odd $(2n+1)$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the odd $(2n+1)$-dimensional torus $\mathbb{T}^{2n+1}$ and the odd $(2n+1)$-dimensional sphere S^{2n+1}, and does not classify odd $(2n+1)$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the odd $(2n+1)$-dimensional projective space P^{2n+1}.*

We now consider another decomposition for a sort of substitute of P^n in our sense. As well, one may use the K-theory of its corresponding (different) C^*-algebras to classify homeomorphism classes of connected sums of P^n.

Example 3.6. As a contrast to P^n, and as a sort of substitute of P^n, we may define P_n to be a closed topological manifold obtained by gluing the boundary of the n-dimensional Möbius band $M_{b,n}$ (the same name as before, but with different fibers) with that of the closed unit ball D^n of $\mathbb{R}^n$, where the n-dimensional Möbius band $M_{b,n}$ defined by us is obtained from the product space $I = [0,1] \times [\{-\infty\} \cup \mathbb{R}^{n-1} \cup \{\infty\}]$ by identifying one edge $E \approx [0,1]$ at $-\infty$ with the opposite edge at ∞ with one twist, so that $\mathbb{R}^{n-1} \cup \{\pm\infty\} \approx S^{n-1}$ with $+\infty = -\infty$ identified. We have the decomposition $S^n = S^n_+ \cup S^{n-1} \cup S^n_-$ as a disjoint union, where $S^n_+ \cup S^n_- = S^n \setminus S^{n-1}$ with the north and south poles contained in S^n_+ and S^n_- respectively. Then S^n_+ is homeomorphic to the interior of D^n and is identified with S^n_- in P_n, and P_n is obtained by gluing the boundary S^{n-1} of the n-dimensional Möbius band $M_{b,n}$ with that of D^n.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M_{b,n})^\circ) \to C(P_n) \to C(D^n) \to 0,$$

where $(M_{b,n})^\circ$ is the interior of $M_{b,n}$. Also,

$$0 \to C_0(I^\circ) \to C_0((M_{b,n})^\circ) \to C_0(E^\circ) \to 0.$$

Therefore, we do obtain

$$\begin{aligned}\chi(C(P_n)) &= \chi(C_0((M_{b,n})^\circ)) + \chi(C(D^n)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0(\mathbb{R}^n)) + \chi(C_0(\mathbb{R})) + 1 \\ &= \begin{cases} 1 - 1 + 1 = 1 & \text{if } n \text{ is even,} \\ -1 - 1 + 1 = -1 & \text{if } n \text{ is odd.} \end{cases}\end{aligned}$$

(Note that these results are not compatible with those in homology theory for P^n, so that the decomposition for P_n should be not applied to P^n, but one can use this result for a more better classification for connected sums of P_n as compared with that given above, when n is odd.) Indeed, moreover,

$$\chi(C(\#^l P_n)) = \begin{cases} l - 2(l-1) = 2 - l & \text{if } n \text{ is even,} \\ -l \quad (\neq 0) & \text{if } n \text{ is odd.} \end{cases}$$

(As a question, there must be a suitable topological reason for the last inequality. The reason may be that P_n is less twisted than P^n.)

Note that $P_n \# P_n$ is homeomorhic to the closed topological manifold obtained by glueing two n-dimensional Möbius bands $M_{b,n}$ along with their boundaries homeomorphic to S^{n-1}, which we may call it the n-dimensional Klein bottle (the same name as before), and denote it by K_n (slightly different). Note also that we may have that $\mathbb{T}^n \# P_n \approx \#^3 P_n$ if n is even, but not if n is odd. Indeed, if n is even, then

$$\begin{aligned}\chi(C(\mathbb{T}^n \# P_n)) &= 0 + 1 - 2 = -1, \\ \chi(C(\#^3 P_n)) &= 2 - 3 = -1,\end{aligned}$$

but if n is odd, then

$$\begin{aligned}\chi(C(\mathbb{T}^n \# P_n)) &= 0 + (-1) = -1, \\ \chi(C(\#^3 P_n)) &= -3 \neq -1.\end{aligned}$$

In the following we compute K-theory groups.

Theorem 3.7. *Let M, N be n-dimensional closed topological manifolds and $M \# N$ be their connected sum. If n is even, then*

$$\begin{aligned}K_0(C(M\#N)) &\cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\ K_1(C(M\#N)) &\cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial),\end{aligned}$$

and if n is odd, then

$$\begin{aligned}K_0(C(M\#N)) &\cong \mathbb{Z} \oplus [(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])], \\ K_1(C(M\#N)) &\cong [K_1(C(M)) \oplus K_1(C(N))]/\partial\mathbb{Z}[p],\end{aligned}$$

where each $[1]$ means the K_0-class of the unit 1 (of $C(M)$, $C(N)$, and $C(S^{n-1})$), and $[z]$ means the generating K_1-class of $K_1(C(S^{n-1})) \cong \mathbb{Z}$ when n is even, and $[p]$ means the non-trivial K_0-class of $K_0(C(S^{n-1})) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ when n is odd, with the image $\partial\mathbb{Z}[p] \cong \mathbb{Z}$.

Remark. See the Remark for Theorem 2.4 for more details on notes, also applied to the image $\partial\mathbb{Z}[p]$.

Proof. The six-term exact sequence of K-theory groups follows from the short exact sequence of $C(M\#N)$ in the proof of Theorem 3.1:

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \xrightarrow{i_*} & K_0(C(M\#N)) & \xrightarrow{q_*} & K_0(C(S^{n-1})) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(S^{n-1})) & \xleftarrow{q_*} & K_1(C(M\#N)) & \xleftarrow{i_*} & K_1(\mathfrak{I}) \end{array}$$

where $\mathfrak{I} = C_0((M \setminus D^n) \sqcup (N \setminus D^n))$, and if $n = 2k$ even and if $n = 2k+1$ odd, then respectively,

$$K_j(C(S^{2k-1})) \cong \begin{cases} \mathbb{Z}[1] & j = 0, \\ \mathbb{Z}[z] & j = 1, \end{cases} \quad K_j(C(S^{2k})) \cong \begin{cases} \mathbb{Z}[1] \oplus \mathbb{Z}[p] & j = 0, \\ 0 & j = 1, \end{cases}$$

since $0 \to C_0(\mathbb{R}^{n-1}) \to C(S^{n-1}) \to \mathbb{C} \to 0$ is a split, short exact sequence of C^*-algebras, and also

$$K_j(C_0((M \setminus D^n) \sqcup (N \setminus D^n))) \cong K_j(C_0(M \setminus D^n)) \oplus K_j(C_0(N \setminus D^n))$$

$(j = 0, 1)$, and where the maps i_* and q_* are induced from the maps i and q, and ∂ are boundary maps (or the index map on the left and the exponential map on the right). The map q_* on K_0 sends the K_0-class of the unit of $C(M\#N)$ to that of $C(S^{n-1})$, and hence is onto if n is even. Thus, ∂ on the right is zero if n is even. If n is odd, one can see in general that the kernel $\ker(\partial)$ contains $\mathbb{Z}[1]$.

Moreover, we also have the following diagram:

$$\begin{array}{ccccc} K_0(C_0(M \setminus D^n)) & \xrightarrow{i_*} & K_0(C(M)) & \xrightarrow{q_*} & K_0(C(D^n)) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(D^n)) & \xleftarrow{q_*} & K_1(C(M)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D^n)) \end{array}$$

with $K_0(C(D^n)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C(D^n)) \cong K_1(\mathbb{C}) \cong 0$ since D^n is contractible. The map q_* on K_0 sends the K_0-class of the unit of $C(M)$ to that of $C(D^n)$, and hence is onto. Thus, both of the boundary maps ∂ are zero. Therefore, the diagram implies that

$$K_0(C(M)) \cong K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z},$$
$$K_1(C(M)) \cong K_1(C_0(M \setminus D^n)),$$

where the direct summand $\mathbb{Z}$ corresponds to the K_0-class [1] of the unit 1 of $C(M)$. Note also that $K_1(C_0(M\setminus D^n)) \cong K_1(C_0(M\setminus D^n)^+)$, where the unitization $C_0(M\setminus D^n)^+$ is isomorphic to $C((M \setminus D^n)^+)$, with the one-point compactification $(M \setminus D^n)^+$ homeomorphic to M.

The same holds for N.

Furthermore, the map q_* on K_1 in the first diagram in this proof is zero when n is odd, so that the kernel $\ker(\partial) = 0$.

It then follows consequently that if n is even, then

$$K_0(C(M\#N)) \cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\},$$
$$K_1(C(M\#N)) \cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial),$$

and also, if n is odd, then

$$\begin{aligned} K_0(C(M\#N)) &\cong \mathbb{Z} \oplus [(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])], \\ K_1(C(M\#N)) &\cong [K_1(C(M)) \oplus K_1(C(N))]/\partial\mathbb{Z}[p], \end{aligned}$$

which follows from exactness of the six-term diagram of K-theory groups above in the first of the proof, where $K_0(C(S^{2k})) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ with $n = 2k+1$, for which $\partial[1] = [0]$ the zero class, but $\partial[p] \neq [0]$. This follows by considering $M\#N \cong (M\#N)\#S^n$ and by the several cases and the general case in Example 3.8 below. We indeed have the following commutative diagram:

$$\begin{array}{ccccc} C(M\#N) & \xrightarrow{q} & C(S^{n-1}) & \longrightarrow & 0 \\ f\downarrow & & \downarrow g & & \| \\ C((M\#N)\#S^n) & \xrightarrow{q} & C(S^{n-1}) & \longrightarrow & 0 \end{array}$$

with f the isomorphism induced from the homeomorphism and g the isomorphism to make the diagram commutative, so that the following diagram commutes

$$\begin{array}{ccc} K_0(C(M\#N)) & \xrightarrow{q_*} & K_0(C(S^{n-1})) \\ f_*\downarrow & & \downarrow g_* \\ K_0(C((M\#N)\#S^n)) & \xrightarrow{q_*} & K_0(C(S^{n-1})) \end{array}$$

with f_* and g_* isomorphisms induced from f and g respectively. □

Example 3.8. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((S^n \setminus D^n) \sqcup (S^n \setminus D^n)) \xrightarrow{i} C(S^n\#S^n) \xrightarrow{q} C(S^{n-1}) \to 0$$

with $S^n \setminus D^n \approx \mathbb{R}^n$ and $S^n\#S^n \approx S^n$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow \partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & 0 & \xleftarrow{i_*} & 0 \oplus 0 \end{array}$$

and if n is odd,

$$\begin{array}{ccccc} 0 \oplus 0 & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{q_*} & \mathbb{Z}[1] \oplus \mathbb{Z}[p] \\ \partial=0\uparrow & & & & \downarrow \partial \\ 0 & \xleftarrow{q_*} & \mathbb{Z} & \xleftarrow{i_*} & \mathbb{Z} \oplus \mathbb{Z}. \end{array}$$

Therefore, if n is even, then

$$\begin{aligned} K_0(C(S^n\#S^n)) &\cong \mathbb{Z} \oplus \{[(K_0(C(S^n))/\mathbb{Z}) \oplus (K_0(C(S^n))/\mathbb{Z})]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z} \oplus \{[\mathbb{Z} \oplus \mathbb{Z}]/\partial\mathbb{Z}[z]\} \cong \mathbb{Z}^2, \\ K_1(C(S^n\#S^n)) &\cong K_1(C(S^n)) \oplus K_1(C(S^n)) \oplus \ker(\partial) \cong 0. \end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(S^n\#S^n)) &\cong \mathbb{Z}\oplus[(K_0(C(S^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})]\oplus(\ker(\partial)/\mathbb{Z}[1])\\
&\cong \mathbb{Z}\oplus[0\oplus 0]\oplus 0\cong\mathbb{Z},\\
K_1(C(S^n\#S^n)) &\cong [K_1(C(S^n)\oplus K_1(C(S^n))]/\partial\mathbb{Z}[p]\\
&\cong [\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[p]\cong\mathbb{Z}.
\end{aligned}$$

Note that the diagram above when n is odd involves no torsion.

Next, there is the following short exact sequence of C^*-algebras:

$$0\to C_0((\mathbb{T}^n\setminus D^n)\sqcup(S^n\setminus D^n))\xrightarrow{\ i\ } C(\mathbb{T}^n\#S^n)\xrightarrow{\ q\ } C(S^{n-1})\to 0$$

with $\mathbb{T}^n\#S^n\approx\mathbb{T}^n$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{T}^n\setminus D^n))\oplus\mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^{2^{n-1}} & \xrightarrow{q_*} & \mathbb{Z}\\
\partial\uparrow & & & & \downarrow\partial=0\\
\mathbb{Z}[z] & \xleftarrow{q_*} & \mathbb{Z}^{2^{n-1}} & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^n\setminus D^n))\oplus 0
\end{array}$$

and if n is odd,

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{T}^n\setminus D^n))\oplus 0 & \xrightarrow{i_*} & \mathbb{Z}^{2^{n-1}} & \xrightarrow{q_*} & \mathbb{Z}[1]\oplus\mathbb{Z}[p]\\
\partial=0\uparrow & & & & \downarrow\partial\\
0 & \xleftarrow{q_*} & \mathbb{Z}^{2^{n-1}} & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^n\setminus D^n))\oplus\mathbb{Z}.
\end{array}$$

Moreover, the exact sequence $0\to C_0(\mathbb{T}^n\setminus D^n)\to C(\mathbb{T}^n)\to C(D^n)\to 0$ implies

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{T}^n\setminus D^n)) & \longrightarrow & \mathbb{Z}^{2^{n-1}} & \longrightarrow & \mathbb{Z}\\
\partial=0\uparrow & & & & \downarrow\partial=0\\
0 & \longleftarrow & \mathbb{Z}^{2^{n-1}} & \longleftarrow & K_1(C_0(\mathbb{T}^n\setminus D^n))
\end{array}$$

so that $K_0(C_0(\mathbb{T}^n\setminus D^n))\cong\mathbb{Z}^{2^{n-1}-1}$ and $K_1(C_0(\mathbb{T}^n\setminus D^n))\cong\mathbb{Z}^{2^{n-1}}$. Therefore, the boundary map ∂ on the left in the first diagram in this case is nonzero when n is even, and if n is even, then

$$\begin{aligned}
K_0(C(\mathbb{T}^n\#S^n)) &\cong \mathbb{Z}\oplus\{[(K_0(C(\mathbb{T}^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})]\}/\partial\mathbb{Z}[z]\\
&\cong \mathbb{Z}\oplus\{[\mathbb{Z}^{2^{n-1}-1}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^{2^{n-1}},\\
K_1(C(\mathbb{T}^n\#S^n)) &\cong K_1(C(\mathbb{T}^n))\oplus K_1(C(S^n))\oplus\ker(\partial)\cong\mathbb{Z}^{2^{n-1}}\oplus 0\oplus 0,
\end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(\mathbb{T}^n\#S^n)) &\cong \mathbb{Z}\oplus[(K_0(C(\mathbb{T}^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})]\\
&\cong \mathbb{Z}\oplus[\mathbb{Z}^{2^{n-1}-1}\oplus 0]\cong\mathbb{Z}^{2^{n-1}},\\
K_1(C(\mathbb{T}^n\#S^n)) &\cong [K_1(C(\mathbb{T}^n))\oplus K_1(C(S^n))]/\partial\mathbb{Z}[p]\\
&\cong [\mathbb{Z}^{2^{n-1}}\oplus\mathbb{Z}]/\partial\mathbb{Z}[p]\cong\mathbb{Z}^{2^{n-1}}.
\end{aligned}$$

Note that $\ker(\partial) = \mathbb{Z}[1]$ follows from that $\partial\mathbb{Z}[p] \cong \mathbb{Z}$ by the diagram.

Let M be an n-dimensional closed topological manifold. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D^n) \sqcup (S^n \setminus D^n)) \xrightarrow{\ i\ } C(M\#S^n) \xrightarrow{\ q\ } C(S^{n-1}) \to 0$$

with $S^n \setminus D^n \approx \mathbb{R}^n$ and $M\#S^n \approx M$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{array}{ccccc} K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z} & \xrightarrow{i_*} & K_0(C(M\#S^n)) & \xrightarrow{q_*} & \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & K_1(C(M\#S^n)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D^n)) \oplus 0 \end{array}$$

and if n is odd,

$$\begin{array}{ccccc} K_0(C_0(M \setminus D^n)) \oplus 0 & \xrightarrow{i_*} & K_0(C(M\#S^n)) & \xrightarrow{q_*} & \mathbb{Z}[1] \oplus \mathbb{Z}[p] \\ \partial=0 \uparrow & & & & \downarrow \partial \\ 0 & \xleftarrow{q_*} & K_1(C(M\#S^n)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D^n)) \oplus \mathbb{Z}. \end{array}$$

Moreover, $K_1(C_0(M \setminus D^n)) \cong K_1(C_0(M \setminus D^n)^+) \cong K_1(C(M))$. Therefore, the map q_* on K_1 is zero, when n is even, so that $\ker(\partial) = 0$. Also, $K_0(C(M)) \cong K_0(C_0(M \setminus D)^+) \cong K_0(C_0(M \setminus D)) \oplus \mathbb{Z}$. Thus, if n is even, then

$$\begin{aligned} K_0(C(M\#S^n)) &\cong \mathbb{Z} \oplus \{[K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z}]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z} \oplus K_0(C_0(M \setminus D^n)) \oplus 0 \cong K_0(C(M)), \\ K_1(C(M\#S^n)) &\cong K_1(C(M)) \oplus K_1(C(S^n)) \oplus \ker(\partial) \cong K_1(C(M)), \end{aligned}$$

and if n is odd, then

$$\begin{aligned} K_0(C(M\#S^n)) &\cong \mathbb{Z} \oplus [K_0(C_0(M \setminus D^n)) \oplus 0] \\ &\cong \mathbb{Z} \oplus K_0(C_0(M \setminus D^n)) \cong K_0(C(M)), \\ K_1(C(M\#S^n)) &\cong [K_1(C_0(M \setminus D^n)) \oplus \mathbb{Z}]/\partial\mathbb{Z}[p] \cong K_1(C(M)). \end{aligned}$$

Note that $\ker(\partial) = \mathbb{Z}[1]$ follows from that $\partial\mathbb{Z}[p] \cong \mathbb{Z}$ by the diagram.

Corollary 3.9. *The formula obtained in Theorem* 3.1 *follows from Theorem* 3.7.

Proof. Note that each quotient by $\mathbb{Z}[1]$, $\partial\mathbb{Z}[z]$ together with $\ker(\partial)$, or $\partial\mathbb{Z}[p]$ in Theorem 3.7, respectively, corresponds to one rank lowering or one rank raising of the free ranks of those K_0-groups or K_1-groups, respectively. Hence, if n is even,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 3 = \chi(C(M)) + \chi(C(N)) - 2,$$

and if n is odd,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 2 - (-1) = \chi(C(M)) + \chi(C(N)).$$

□

Corollary 3.10. *Let M_i $(1 \leq i \leq l)$ be n-dimensional closed topological manifolds and $\#_{i=1}^l M_i$ be their connected sum. If n is even, then inductively,*

$$\begin{aligned}
&K_0(C(\#_{i=1}^l M_i)) \\
&\cong \mathbb{Z} \oplus \{[(K_0(C(\#_{i=1}^{l-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\} \\
&\cong \mathbb{Z} \oplus \{[((\cdots(\mathbb{Z} \oplus \{[(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]\}) \\
&\quad \cdots)/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\}, \\
&K_1(C(\#_{i=1}^l M_i)) \cong [\oplus_{i=1}^l K_1(C(M_i))] \oplus [\oplus_{i=1}^{l-1} \ker(\partial_i)],
\end{aligned}$$

and if n is odd, then inductively,

$$\begin{aligned}
&K_0(C(\#_{i=1}^l M_i)) \\
&\cong \mathbb{Z} \oplus [(K_0(C(\#_{i=1}^{l-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z} \oplus [((\cdots(\mathbb{Z} \oplus [(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]) \\
&\quad \cdots)/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])], \\
&K_1(C(\#_{i=1}^l M_i)) \cong [K_1(C(\#_{i=1}^{l-1} M_i)) \oplus K_1(C(M_l))]/\partial_{l-1}\mathbb{Z}[p] \\
&\cong [((\cdots([K_1(C(M_1)) \oplus K_1(C(M_2))]/\partial_1\mathbb{Z}[p]) \\
&\quad \cdots)/\partial_{l-2}\mathbb{Z}[p]) \oplus K_1(C(M_l))]/\partial_{l-1}\mathbb{Z}[p],
\end{aligned}$$

where each $[1]$ *means the* K_0*-class of the unit* 1 *of* $C(M_i)$*, and* $[z] \in K_1(C(S^{n-1})) \cong \mathbb{Z}$ *when n is even, and* $[p] \in K_1(C(S^{n-1})) = \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ *when n is odd, and each* $\partial_i = \partial$ *is the boundary map in each step in induction.*

Corollary 3.11. *That Corollary* 3.2 *follows from this Corollary* 3.10*.*

Proof. Note that each quotient by $\mathbb{Z}[1]$, $\partial_i\mathbb{Z}[z]$ together with $\ker(\partial_i)$, or $\partial_i\mathbb{Z}[p]$ in Corollary 3.10, repsectively, corresponds to one rank lowering or one rank raising of the free ranks of those either K_0-groups or K_1-groups, respectively. Hence, if n is even, then inductively,

$$\begin{aligned}
\chi(C(\#_{i=1}^l M_i)) &= 1 + \chi(C(\#_{i=1}^{l-1} M_i)) + \chi(C(M_l)) - 3 \\
&= \cdots = \sum_{i=1}^{l} \chi(C(M_i)) - 2(l-1),
\end{aligned}$$

and if n is odd, then inductively,

$$\begin{aligned}
\chi(C(\#_{i=1}^l M_i)) &= 1 + \chi(C(\#_{i=1}^{l-1} M_i)) + \chi(C(M_l)) - 2 + 1 \\
&= \cdots = \sum_{i=1}^{l} \chi(C(M_i)).
\end{aligned}$$

□

Example 3.12. Since $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ $(j = 0, 1)$, it is obtained by Theorem 3.7 that if n is even, then

$$\begin{aligned}
K_0(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong \mathbb{Z} \oplus \{[(\mathbb{Z}^{2^{n-1}}/\mathbb{Z}) \oplus (\mathbb{Z}^{2^{n-1}}/\mathbb{Z})]/\partial\mathbb{Z}[z]\} \\
&\cong \mathbb{Z} \oplus \{[\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial\mathbb{Z}[z]\} \cong \mathbb{Z}^{2^n-2}, \\
K_1(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong \mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}} \oplus \ker(\partial) \cong \mathbb{Z}^{2^n},
\end{aligned}$$

with $\partial\mathbb{Z}[z] \cong \mathbb{Z}$ since a finitely many times direct sum of $\mathbb{Z}$ is torsion free, so that $\ker(\partial) = 0$, and as well, the K_0-group above can not involve torsion because its quotient by q_* is $\mathbb{Z}$, and if n is odd, then

$$\begin{aligned} K_0(C(\mathbb{T}^n\#\mathbb{T}^n)) &\cong \mathbb{Z} \oplus [(\mathbb{Z}^{2^{n-1}}/\mathbb{Z}) \oplus (\mathbb{Z}^{2^{n-1}}/\mathbb{Z})] \\ &\cong \mathbb{Z} \oplus [\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}] \cong \mathbb{Z}^{2^n-1}, \\ K_1(C(\mathbb{T}^n\#\mathbb{T}^n)) &\cong [\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}}]/\partial\mathbb{Z}[p] \cong \mathbb{Z}^{2^n-1}. \end{aligned}$$

Hence,

$$\chi(C(\mathbb{T}^n\#\mathbb{T}^n)) = \begin{cases} (2^n-2) - 2^n = -2 & \text{if } n \text{ is even,} \\ (2^n-1) - (2^n-1) = 0 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, it is obtained by Corollary 3.10 that if n is even, then

$$\begin{aligned} K_0(C(\#^l\mathbb{T}^n)) &\cong \mathbb{Z} \oplus \{[(\cdots([\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial_1\mathbb{Z}[z])\cdots) \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial_{l-1}\mathbb{Z}[z]\} \\ &\cong \mathbb{Z}^{1+l(2^{n-1}-1)-(l-1)} \cong \mathbb{Z}^{2+l(2^{n-1}-2)}, \\ K_1(C(\#^l\mathbb{T}^n)) &\cong [\oplus_{i=1}^l \mathbb{Z}^{2^{n-1}}] \oplus [\oplus_{i=1}^{l-1}\ker(\partial_i)] \cong \mathbb{Z}^{l2^{n-1}}, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} K_0(C(\#^l\mathbb{T}^n)) &\cong \mathbb{Z} \oplus [(\cdots[\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]\cdots) \oplus \mathbb{Z}^{2^{n-1}-1}] \\ &\cong \mathbb{Z}^{l2^{n-1}-l+1}, \\ K_1(C(\#^l\mathbb{T}^n)) &\cong [(\cdots([\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}}]/\partial_1\mathbb{Z}[p])\cdots) \oplus \mathbb{Z}^{2^{n-1}}]/\partial_{l-1}\mathbb{Z}[p] \cong \mathbb{Z}^{l2^{n-1}-(l-1)}. \end{aligned}$$

Hence,

$$\chi(C(\#^l\mathbb{T}^n)) = \begin{cases} [2 + l(2^{n-1}-2)] - l2^{n-1} = 2 - 2l & \text{if } n \text{ is even,} \\ [l2^{n-1} - l + 1] - (l2^{n-1} - (l-1)) = 0 & \text{if } n \text{ is odd.} \end{cases}$$

As for the real n-dimensional projective space P^n, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C(P^n)$ in Example 3.3, becomes

$$\begin{array}{ccccc} K_0(C_0((M_b^n)^\circ)) & \longrightarrow & K_0(C(P^n)) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle\partial=0}\uparrow & & & & \downarrow{\scriptstyle\partial=0} \\ 0 & \longleftarrow & K_1(C(P^n)) & \longleftarrow & K_1(C_0((M_b^n)^\circ)) \end{array}$$

so that

$$\begin{aligned} K_0(C(P^n)) &\cong \mathbb{Z} \oplus K_0(C_0((M_b^n)^\circ)), \\ K_1(C(P^n)) &\cong K_1(C_0((M_b^n)^\circ)). \end{aligned}$$

Indeed, P^n is viewed as the one-point compactification of the interior $(M_b^n)^\circ$ of the n-dimensional Möbius band M_b^n. Moreover, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C_0((M_b^n)^\circ)$ in Example 3.3, becomes

$$\begin{array}{ccccc} K_1(C_0((P^{n-1})^-)) & \longrightarrow & K_0(C_0((M_b^n)^\circ)) & \longrightarrow & 0 \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ \mathbb{Z} & \longleftarrow & K_1(C_0((M_b^n)^\circ)) & \longleftarrow & K_0(C_0((P^{n-1})^-)) \end{array}$$

and

$$K_1(C_0((P^{n-1})^-)) \cong K_1(C_0((P^{n-1})^-)^+) \cong K_1(C(P^{n-1})) \quad \text{and}$$
$$K_0(C_0((P^{n-1})^-)) \cong K_0(C(P^{n-1}))/\mathbb{Z}[1].$$

We now determine the K-theory groups of $C(P^n)$ inductively by the diagram above, as follows. Since $K_0(C(P^2)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $K_1(C(P^2)) \cong 0$,

$$K_0(C(P^3)) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z},$$
$$K_1(C(P^3)) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

so that it follows that

$$K_0(C(P^4)) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$
$$K_1(C(P^4)) \cong 0$$

and hence we obtain inductively that for $k \geq 1$,

$$\begin{cases} K_0(C(P^{2k})) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \\ K_1(C(P^{2k})) \cong 0, \end{cases}$$

and

$$\begin{cases} K_0(C(P^{2k+1})) \cong \mathbb{Z}, \\ K_1(C(P^{2k+1})) \cong \mathbb{Z} \oplus \mathbb{Z}_2. \end{cases}$$

(Note that this result is compatible with that in homology mentioned in Example 3.3. In fact, the compatiblity does imply it.) It is deduced that

$$\chi(C(P^n)) = 1 - \chi(C(P^{n-1})) \quad (n \geq 1)$$

with $\chi(C(P^2)) = 1$ and $\chi(C(P^1)) = \chi(C(S^1)) = 0$ and $C(P^0) \cong \mathbb{C}$. Moreover, the equation obtained above is converted to

$$a_n - \frac{1}{2} = -\left(a_{n-1} - \frac{1}{2}\right)$$

$(n \geq 1)$ with $a_n = \chi(C(P^n))$, so that we obtain the same formula for $\chi(C(P^n))$ as in Example 3.3. Let

$$\alpha_n = \mathrm{rank}_{\mathbb{Z}} K_0(C(P^n)) \text{ and } \beta_n = \mathrm{rank}_{\mathbb{Z}} K_1(C(P^n)).$$

If n is even $(n \geq 2)$, then

$$\alpha_n = 1 \quad \text{and} \quad \beta_n = 0,$$

and if n is odd $(n \geq 3)$, then

$$\alpha_n = 1 \quad \text{and} \quad \beta_n = 1.$$

It is obtained by Theorem 3.7 that if n is even, then

$$\begin{aligned} K_0(C(P^n \# P^n)) &\cong \mathbb{Z} \oplus \{[(K_0(C(P^n))/\mathbb{Z}[1]) \oplus (K_0(C(P^n))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z} \oplus \{[\mathbb{Z}_2 \oplus \mathbb{Z}_2]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z} \oplus [\oplus^2 \mathbb{Z}_2], \\ K_1(C(P^n \# P^n)) &\cong K_1(C(P^n)) \oplus K_1(C(P^n)) \oplus \ker(\partial) \\ &\cong 0 \oplus 0 \oplus \mathbb{Z} \cong \mathbb{Z}, \end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(P^n\#P^n)) &\cong \mathbb{Z}\oplus[(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z}\oplus[0\oplus 0]\oplus(\ker(\partial)/\mathbb{Z}[1])\cong\mathbb{Z}, \\
K_1(C(P^n\#P^n)) &\cong [K_1(C(P^n))\oplus K_1(C(P^n))]/\partial\mathbb{Z}[p] \\
&\cong [(\mathbb{Z}\oplus\mathbb{Z}_2)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial\mathbb{Z}[p] \\
&\cong [\mathbb{Z}^2\oplus\mathbb{Z}_2^2]/\partial\mathbb{Z}[p]\cong\mathbb{Z}\oplus\mathbb{Z}_2^2.
\end{aligned}$$

Hence, if n is even, then $\chi(C(P^n\#P^n))=1-1=0$, and if n is odd, then $\chi(C(P^n\#P^n))=1-(2-1)=0$.

Moreover, it is obtained by Corollary 3.10 that if n is even, then

$$\begin{aligned}
K_0(C(\#^lP^n)) &\cong \mathbb{Z}\oplus\{[((\cdots\mathbb{Z}\oplus([(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]) \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\} \\
&\cong \mathbb{Z}\oplus\{[((\cdots(\mathbb{Z}\oplus([\mathbb{Z}_2\oplus\mathbb{Z}_2]/\partial_1\mathbb{Z}[z]) \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus\mathbb{Z}_2]/\partial_{l-1}\mathbb{Z}[z]\}\cong\mathbb{Z}\oplus(\mathbb{Z}_2)^l, \\
K_1(C(\#^lP^n)) &\cong [\oplus_{i=1}^l K_1(C(P^n))]\oplus[\oplus_{i=1}^{l-1}\ker(\partial_i)] \\
&\cong 0\oplus[\oplus_{i=1}^{l-1}\ker(\partial_i)]\cong\mathbb{Z}^{l-1},
\end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(\#^lP^n)) &\cong \mathbb{Z}\oplus[((\cdots(\mathbb{Z}\oplus[(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z}\oplus[(\cdots(0\oplus[0\oplus 0])\cdots)\oplus 0]\cong\mathbb{Z}, \\
K_1(C(\#^lP^n)) &\cong [(\cdots[K_1(C(P^n))\oplus K_1(C(P^n))]/\partial_1\mathbb{Z}[p] \\
&\qquad\cdots)\oplus K_1(C(P^n))]/\partial_{l-1}\mathbb{Z}[p] \\
&\cong [(\cdots[(\mathbb{Z}\oplus\mathbb{Z}_2)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial_1\mathbb{Z}[p] \\
&\qquad\cdots)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial_{l-1}\mathbb{Z}[p]\cong\mathbb{Z}\oplus\mathbb{Z}_2^l.
\end{aligned}$$

Hence, if n is even, then $\chi(C(\#^lP^n))=1-(l-1)=2-l$, and if n is odd, then $\chi(C(\#^lP^n))=1-(l-(l-1))=0$.

Table 6: Classification for closed topological manifolds with dimension even

K_0 rank	Orientable	Non-orientable
$2+l(2^{2n-1}-2)$	$\#^l\mathbb{T}^{2n}$	No
$2^{2n}-2$	$\mathbb{T}^{2n}\#\mathbb{T}^{2n}$	No
2^{2n-1}	$\mathbb{T}^{2n}$	No
8	$\mathbb{T}^4\#\mathbb{T}^4$	No
2	$S^{2n}\approx\#^lS^{2n}$, S^0 $\#^n\mathbb{T}^2\approx T(n)$	No
1	No	P^0, P^{2n}, $\#^nP^2\approx P(n)$ $P^{2n}\#P^{2n}$, $\#^lP^{2n}$

Note that $S^0=\{-1,1\}$ and P^0 is the one point set as the quotient of S^0.

It follows from the Table 6 that:

Corollary 3.13. *Let n be a natural number with $n \geq 1$. The rank of K_0-groups for C^*-algebras can not classify homeomorphism classes of connected sums of the 2-dimensional, orientable closed topological manifolds $\mathbb{T}^2$ and S^2.*

But, if $n \geq 2$, it does classify homeomorphism classes of connected sums of the $2n$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n}$ and S^{2n}.

And the rank of K_0-groups can not classify homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n} $(n \geq 1)$.

Table 7: Classification for closed topological manifolds with dimension even

K_1 rank	Orientable	Non-orientable
$l2^{2n-1}$	$\#^l\mathbb{T}^{2n}$	$\#^{l2^{2n-1}+1}P^2$, $\#^{l2^{2n-1}+1}P^{2k}$
$l2n$	$\#^{ln}\mathbb{T}^2$	$\#^{l2n+1}P^2$, $\#^{l2n+1}P^{2k}$
2^{2n}	$\mathbb{T}^{2n}\#\mathbb{T}^{2n}$	$\#^{2^{2n}+1}P^2$, $\#^{2^{2n}+1}P^{2k}$
2^{2n-1}	$\mathbb{T}^{2n}$	$\#^{2^{2n-1}+1}P^2$, $\#^{2^{2n-1}+1}P^{2k}$
$2l$	$\#^l\mathbb{T}^2 \approx T(n)$	$\#^{2l+1}P^2$, $\#^{2l+1}P^{2n}$
$2l-1$	No	$\#^{2l}P^2$, $\#^{2l}P^{2n}$
4	$\mathbb{T}^2\#\mathbb{T}^2 \approx T(2)$	$\#^5P^2$, $\#^5P^{2n}$
3	No	$\#^4P^2$, $\#^4P^{2n}$
2	$\mathbb{T}^2 \approx T(1)$	$\#^3P^2$, $\#^3P^{2n}$
1	No	$P^2\#P^2 \approx P(2) \approx K^2$ $P^{2n}\#P^{2n}$
0	$S^{2n} \approx \#^l S^{2n}$, S^0	P^0, $P^2 \approx P(1)$, P^{2n}

It follows from the Table 7 that:

Corollary 3.14. *Let n be a natural number with $n \geq 1$. The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of connected sums of the $2n$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n}$ and S^{2n}, and does classify homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n}.*

Table 8: Classification for closed topological manifolds with dimension odd

K_0 rank	Orientable	Orientable
$1 + l(2^{2n} - 1)$	$\#^l\mathbb{T}^{2n+1}$	No
$2^{2n+1} - 1$	$\mathbb{T}^{2n+1}\#\mathbb{T}^{2n+1}$	No
2^{2n}	$\mathbb{T}^{2n+1}$	No
4	$\mathbb{T}^3$	No
1	$S^1 = \mathbb{T}$, $S^{2n+1} \approx \#^l S^{2n+1}$	$S^1 \approx P^1$, P^{2n+1}, $\#^l P^{2n+1}$

It follows from the Table 8 that:

Corollary 3.15. *Let n be a natural number with $n \geq 1$. The rank of K_0-groups for C^*-algebras does classify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n+1}$ and S^{2n+1}, but the rank of K_0-groups does not classify homeomorphism classes of connected sums of $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.*

Table 9: Classification for closed topological manifolds with dimension odd

K_1 rank	Orientable	Orientable
$1 + l(2^{2n} - 1)$	$\#^l \mathbb{T}^{2n+1}$	No
$2^{2n+1} - 1$	$\mathbb{T}^{2n+1} \# \mathbb{T}^{2n+1}$	No
2^{2n}	$\mathbb{T}^{2n+1}$	No
4	$\mathbb{T}^3$	No
1	$S^1 = \mathbb{T}$, $S^{2n+1} \approx \#^l S^{2n+1}$	$S^1 \approx P^1$, P^{2n+1}, $\#^l P^{2n+1}$

Note that the list of ranks and items in the Table 8 is exactly the same as that in the Table 9. It follows from the Table 9 that:

Corollary 3.16. *Let n be a natural number with $n \geq 1$. The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n+1}$ and S^{2n+1}, but does not classify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.*

As in the examples considered so far, consequently, one can say that

Corollary 3.17. *The free ranks of K-theory groups K_0 or K_1 of C^*-algebras are more classifiable or the same level invariants for closed topological manifolds than or as the Euler characteristic of C^*-algebras, respectively.*

But the Euler characteristic of C^*-algebras are more easily computable and more beautiful numerically than the K-theory group ranks of C^*-algebras.

Remark. It follows from our K-theory group formulae obtained so far that the K-theory groups written as quotients in some examples and cases may have torsion in general (but may not in some corresponding cases). But without knowing its information, we could determine the K-theory group ranks and the Euler characteristic for C^*-algebras. As a question, it should be of interest to understand more about the K-theory torsion (or torsion freeness), which in fact could be known from more about the boundary maps.

Table 10: The K-theory groups for the C^*-algebras of topological manifolds

C^*-algebra	K_0-group	K_1-group
$C(S^{2n})$	$\mathbb{Z}^2$	0
$C(S^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$
$C(\#^l \mathbb{T}^{2n})$	$\mathbb{Z}^{2+l(2^{2n-1}-2)}$	$\mathbb{Z}^{l 2^{2n-1}}$
$C(\#^l \mathbb{T}^{2n+1})$	$\mathbb{Z}^{1+l(2^{2n}-1)}$	$\mathbb{Z}^{1+l(2^{2n}-1)}$
$C(P^{2n})$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0
$C(P^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$C(\#^l P^{2n})$	$\mathbb{Z} \oplus \mathbb{Z}_2^l$	$\mathbb{Z}^{l-1}$
$C(\#^l P^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^l$

It follows from the Table 10 that

Corollary 3.18. *The torsion rank $t_0(\mathfrak{A})$ of K_0-groups of C^*-algebras $\mathfrak{A}$ with respect to $\mathbb{Z}_2$ (or any other torsion groups in general) classifies homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n}.*

The torsion rank $t_1(\mathfrak{A})$ of K_1-groups of C^*-algebras $\mathfrak{A}$ with respect to $\mathbb{Z}_2$ (or any other torsion groups in general) classifies homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.

As well, the torsion freeness for both K_0 and K_1-groups of C^*-algebras distinguish homeomorphism classes of connected sums of spheres S^n and tori $\mathbb{T}^n$ from those of connected sums of projective spaces P^n, and becomes a more better invariant than orientaion for manifolds in this case.

Added before the last minute, as a summary we obtain, with $\emptyset$ to mean empty,

Table 11: Do or not classify the closed topological manifolds

Manifolds	χ	b_0	b_1	t_0	t_1
Orientable closed surfaces S^2, $\#^l\mathbb{T}^2$	Yes	No	Yes	$\emptyset$	$\emptyset$
Non-orientable closed surfaces $\#^l P^2$	Yes	No	Yes	Yes	$\emptyset$
Even $2n(\geq 4)$ dimensional, orientable closed manifolds S^{2n}, $\#^l\mathbb{T}^{2n}$	Yes	Yes	Yes	$\emptyset$	$\emptyset$
Even $2n(\geq 4)$ dimensional, non-orientable closed manifolds $\#^l P^{2n}$	Yes	No	Yes	Yes	$\emptyset$
Odd $2n+1(\geq 3)$ dimensional, orientable closed manifolds S^{2n+1}, $\#^l\mathbb{T}^{2n+1}$	No	Yes	Yes	$\emptyset$	$\emptyset$
Odd $2n+1(\geq 3)$ dimensional, orientable closed manifolds $\#^l P^{2n+1}$	No	No	No	$\emptyset$	Yes

The last table shows that

Corollary 3.19. *All the closed topological manifolds X as in the list are classifiable (up to homeomorphism) by using K-theory data such as either the Euler charactersistic $\chi(C(X))$, the Betti numbers b_j of $K_j(C(X))$ $(j=0,1)$, or the torsion ranks t_j of $K_j(C(X))$ $(j=0,1)$, together with dimension of X (not K-theoretic) and torsion freeness for both $K_0(C(X))$ and $K_1(C(X))$ (or orientaion of X in part).*

Remark. Now comes out a natural question (to be considered), whether one can know that the converse of that corollary holds or not. Namely, determine the (suitable) class of closed topological manifolds, which are classifiable by those data as complete invariants.

Furthermore, a moment of thought implies that, as a class to answer the question,

Theorem 3.20. *For X and Y two closed topological manifolds as in the list above, X is homeomophic to Y if and only if*

$$K_0(C(X)) \oplus K_1(C(X)) \cong K_0(C(Y)) \oplus K_1(C(Y))$$

and $\dim X = \dim Y$.

Remark. Moreover, the (covering) dimension for spaces can be replaced with the real rank for C^*-algebras. Indeed, for X a compact Hausdorff space, $\dim X = \mathrm{RR}(C(X))$ (see [2]).

Therefore,

Corollary 3.21. *K-theory groups and real rank for C^*-algebras are complete invariants for those closed topological manifolds X (up to homeomorphism).*

4 Noncommutative connected sums As a generalization of connected sums of closed topological manifolds to C^*-algebras, we define a connected sum $\mathfrak{A}\#\mathfrak{B}$ of two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a unital C^*-algebra $\mathfrak{D}$ as a common quotient, having a quotient $\mathfrak{E}$, to be the following extension of $\mathfrak{E}$ by the direct sum $\mathfrak{I} \oplus \mathfrak{K}$:

$$0 \to \mathfrak{I} \oplus \mathfrak{K} \xrightarrow{i} \mathfrak{A}\#\mathfrak{B} \xrightarrow{q} \mathfrak{E} \to 0,$$

where we have

$$0 \to \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{D} \to 0,$$
$$0 \to \mathfrak{K} \xrightarrow{i} \mathfrak{B} \xrightarrow{q} \mathfrak{D} \to 0,$$

and $\mathfrak{E}$ is a quotient of $\mathfrak{D}$, where each i is the inclusion map and each q is the quotient map.

As a note, in the definition we may replace the closed ideals $\mathfrak{I}$, $\mathfrak{K}$, and $\mathfrak{I} \oplus \mathfrak{K}$ with $\mathfrak{I} \otimes \mathbb{K}$, $\mathfrak{K} \otimes \mathbb{K}$, and $[\mathfrak{I} \oplus \mathfrak{K}] \otimes \mathbb{K}$, respectively, if necessary as in the extension theory of C^*-algebras. Also, the connected sum $\mathfrak{A}\#\mathfrak{B}$ defined may not be unique, which depends on the extension theory of C^*-algebras and can be unique as an equivalence class in the theory, so that $\mathfrak{A}\#\mathfrak{B}$ is one representative of the connected sums defined above. Also, C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{D}$ in the definition may not be unital. Note that the unital case of C^*-algebras corresponds to the compact case of spaces, as in this paper, and the non-unital case does to the non-compact case, not dealt with here.

Theorem 4.1. *Let $\mathfrak{A}\#\mathfrak{B}$ be the connected sum of two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$. Then*

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{A}) + \chi(\mathfrak{B}) - 2 \cdot \chi(\mathfrak{D}) + \chi(\mathfrak{E}).$$

Proof. It follows from the definition of $\mathfrak{A}\#\mathfrak{B}$ above that

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{I}) + \chi(\mathfrak{K}) + \chi(\mathfrak{E}),$$

and also that

$$\chi(\mathfrak{A}) = \chi(\mathfrak{I}) + \chi(\mathfrak{D}), \quad \text{and} \quad \chi(\mathfrak{B}) = \chi(\mathfrak{K}) + \chi(\mathfrak{D}).$$

Therefore, we obtain

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{A}) + \chi(\mathfrak{B}) - 2 \cdot \chi(\mathfrak{D}) + \chi(\mathfrak{E}).$$

□

Example 4.2. Let X, Y be compact Hausdorff spaces and $C(X), C(Y)$ be the C^*-algebras of all continuous, complex-valued functions on X, Y respectively. Assume that there is a closed subset D of X which is identified with a closed subset of Y. Let $E = \partial D$ be the boundary of D, which is closed in D. Then one can define the connected sum $C(X)\#C(Y)$ in our sense to be the following extension:

$$0 \to C_0(X \setminus D) \oplus C_0(Y \setminus D) \to C(X)\#C(Y) \to C(E) \to 0.$$

It follows that

$$\chi(C(X)\#C(Y)) = \chi(C(X)) + \chi(C(Y)) - 2 \cdot \chi(C(D)) + \chi(C(\partial D)).$$

Compare with those formulae in Theorems 2.1 and 3.1, contained in this formula and in that of Theorem 4.1.

To define the 2-successive connected sum of three unital C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ wiht a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$, we assume that there are short exact sequences of C^*-algebras:

$$0 \to \mathfrak{I}_j \to \mathfrak{A}_j \to \mathfrak{D} \to 0$$

($j = 1, 2, 3$) and

$$0 \to \mathfrak{I}_{j,k} \to \mathfrak{A}_j \# \mathfrak{A}_k \to \mathfrak{D} \to 0$$

($1 \leq j, k \leq 3$ and $j \neq k$). We then define the 2-successive connected sum $(\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l$ to be the following extension:

$$0 \to \mathfrak{I}_{j,k} \oplus \mathfrak{I}_l \to (\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l \to \mathfrak{E} \to 0,$$

which may not be unique. Also, the associativity for the connected sum may not holds, i.e., $(\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l \not\cong \mathfrak{A}_j \# (\mathfrak{A}_k \# \mathfrak{A}_l)$ in general. (Checking this should be another task to be continued elsewhere.) Anyhow, we can define the $(n-1)$-successive connected sum of unital C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$ to be inductively as

$$\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n = (\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \# \mathfrak{A}_3 \cdots) \# \mathfrak{A}_n$$

in this order, where we need to assume that there are short exact sequences of C^*-algebras:

$$0 \to \mathfrak{I}_j \to \mathfrak{A}_j \to \mathfrak{D} \to 0$$

for $1 \leq j \leq n$ and

$$0 \to \mathfrak{I}_{1,2,\cdots,k} \to (\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \cdots) \# \mathfrak{A}_k \to \mathfrak{D} \to 0$$

($k = 2, \cdots, n-1$), so that one can define the following extensions:

$$0 \to \mathfrak{I}_{1,2,\cdots,k} \oplus \mathfrak{I}_{k+1} \to ((\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \cdots) \# \mathfrak{A}_k) \# \mathfrak{A}_{k+1} \to \mathfrak{E} \to 0$$

for $1 \leq k \leq n-1$. We omit to write this assumption in what follows.

Corollary 4.3. *Let $\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n$ be the $(n-1)$-successive connected sum of unital C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ by a common quotient $\mathfrak{D}$ with a quotient $\mathfrak{E}$. Then*

$$\chi(\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i) + (n-1)[\chi(\mathfrak{E}) - 2 \cdot \chi(\mathfrak{D})].$$

Proposition 4.4. *Let $\mathfrak{A}_1 \# \mathfrak{A}_2$ be the connected sum of two unital C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with a common quotient $\mathfrak{D} = \mathfrak{A}_1/\mathfrak{I}_1 = \mathfrak{A}_2/\mathfrak{I}_2$ with a quotient $\mathfrak{E}$. Then*

$$\begin{array}{ccccc} K_0(\mathfrak{I}_1) \oplus K_0(\mathfrak{I}_2) & \xrightarrow{i_*} & K_0(\mathfrak{A}_1 \# \mathfrak{A}_2) & \xrightarrow{q_*} & K_0(\mathfrak{E}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{E}) & \xleftarrow{q_*} & K_1(\mathfrak{A}_1 \# \mathfrak{A}_2) & \xleftarrow{i_*} & K_1(\mathfrak{I}_1) \oplus K_1(\mathfrak{I}_2) \end{array}$$

and

$$\begin{array}{ccccc} K_0(\mathfrak{I}_j) & \xrightarrow{i_*} & K_0(\mathfrak{A}_j) & \xrightarrow{q_*} & K_0(\mathfrak{D}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{D}) & \xleftarrow{q_*} & K_1(\mathfrak{A}_j) & \xleftarrow{i_*} & K_1(\mathfrak{I}_j) \end{array}$$

$(j=1,2)$, from which $K_l(\mathfrak{I}_j)$ $(j,l=0,1)$ are computable in terms of the K-theory groups of given C^*-algebras, so that

$$\begin{array}{ccccc} 0 \to [K_l(\mathfrak{I}_1 \oplus \mathfrak{I}_2)]/\partial K_{l+1}(\mathfrak{E}) & \longrightarrow & K_l(\mathfrak{A}_1 \# \mathfrak{A}_2) & \longrightarrow & q_*(K_l(\mathfrak{A}_1 \# \mathfrak{A}_2)) \to 0 \\ \| & & & & \| \\ \mathrm{coker}(\partial) & & & & \ker(\partial) \end{array}$$

$(l=0,1)$ and

$$\begin{array}{ccccc} 0 \to K_l(\mathfrak{I}_j)/\partial K_{l+1}(\mathfrak{D}) & \longrightarrow & K_l(\mathfrak{A}_j) & \longrightarrow & q_*(K_l(\mathfrak{A}_j)) \to 0 \\ \| & & & & \| \\ \mathrm{coker}(\partial) & & & & \ker(\partial) \end{array}$$

$(l=0,1)$, where $l+1$ (mod 2). It follows that the K-theory groups $K_l(\mathfrak{A}_1 \# \mathfrak{A}_2)$ as well as $K_l(\mathfrak{A}_j)$ are determined by the cokernels $\mathrm{coker}(\partial)$ *and the kernels* $\ker(\partial)$ *of the boundary maps ∂ (up and down arrows) in the left and right sides (that are index and exponential maps, respectively).*

Proposition 4.5. *Let $\#_{j=1}^n \mathfrak{A}_j$ be the $(n-1)$-successive connected sum of unital C^*-algebras $\mathfrak{A}_j$ $(1 \le j \le n)$ by a common quotient $\mathfrak{D}$ with a quotient $\mathfrak{E}$. Then inductively,*

$$\begin{array}{ccccc} K_0(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n) & \xrightarrow{i_*} & K_0((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) & \xrightarrow{q_*} & K_0(\mathfrak{E}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{E}) & \xleftarrow{q_*} & K_1((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) & \xleftarrow{i_*} & K_1(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n) \end{array}$$

and

$$\begin{array}{ccccc} K_0(\mathfrak{I}_j) & \xrightarrow{i_*} & K_0(\mathfrak{A}_j) & \xrightarrow{q_*} & K_0(\mathfrak{D}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{D}) & \xleftarrow{q_*} & K_1(\mathfrak{A}_j) & \xleftarrow{i_*} & K_1(\mathfrak{I}_j) \end{array}$$

$(1 \le j \le n)$, in particular, when $j=n$, and

$$\begin{array}{ccccc} K_0(\mathfrak{I}_{1,\cdots,n-1}) & \xrightarrow{i_*} & K_0(\#_{j=1}^{n-1}\mathfrak{A}_j) & \xrightarrow{q_*} & K_0(\mathfrak{D}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{D}) & \xleftarrow{q_*} & K_1(\#_{j=1}^{n-1}\mathfrak{A}_j) & \xleftarrow{i_*} & K_1(\mathfrak{I}_{1,\cdots,n-1}) \end{array}$$

with $\#_{j=1}^{n-1}\mathfrak{A}_j = (\#_{j=1}^{n-2}\mathfrak{A}_j)\#\mathfrak{A}_{n-1}$ as the next step, from which $K_l(\mathfrak{I}_{1,\cdots,n-1})$ $(l=0,1)$ are computed inductively in terms of the K-theory groups of given C^*-algebras, so that

$$\begin{array}{c} 0 \to K_l(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n)/\partial K_{l+1}(\mathfrak{E}) = \mathrm{coker}(\partial) \\ \downarrow \\ K_l((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) \\ \downarrow \\ q_*(K_l((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n)) = \ker(\partial) \to 0 \end{array}$$

$(l = 0, 1)$ *and*

$$\begin{array}{ccccc} 0 \to K_l(\mathfrak{I}_j)/\partial K_{l+1}(\mathfrak{D}) & \longrightarrow & K_l(\mathfrak{A}_j) & \longrightarrow & q_*(K_l(\mathfrak{A}_j)) \to 0 \\ \| & & & & \| \\ \mathrm{coker}(\partial) & & & & \ker(\partial) \end{array}$$

$(l = 0, 1)$, *where* $l + 1$ $(mod\ 2)$, *in particular, when* $j = n$, *and*

$$\begin{array}{c} 0 \to K_l(\mathfrak{I}_{1,\cdots,n-1})/\partial K_{l+1}(\mathfrak{D}) = \mathrm{coker}(\partial) \\ \downarrow \\ K_l(\#_{j=1}^{n-1}\mathfrak{A}_j) \\ \downarrow \\ q_*(K_l(\#_{j=1}^{n-1}\mathfrak{A}_j)) = \ker(\partial) \to 0 \end{array}$$

$(l = 0, 1)$ *with* $\#_{j=1}^{n-1}\mathfrak{A}_j = (\#_{j=1}^{n-2}\mathfrak{A}_j)\#\mathfrak{A}_{n-1}$, *for which its K-theory groups are computed similarly as the case of* $\#_{j=1}^{n}\mathfrak{A}_j$ *above. It follows that the K-theory groups* $K_l(\#_{j=1}^{n}\mathfrak{A}_j)$, $K_l(\#_{j=1}^{n-1}\mathfrak{A}_j)$, $\cdots$, *as well as* $K_l(\mathfrak{A}_j)$ *are determined inductively by the cokernels* coker(∂) *and the kernels* ker(∂) *of the boundary maps* ∂ (*up and down arrows*) *in the left and right sides* (*that are index and exponential maps, respectively*).

Corollary 4.6. *The K-theory groups of successive connected sums of C^*-algebras in our sense is computable inductively if the six-term diagrams in the proposition are computable inductively in the sense that the cokernels and the kernels of the boundary maps associated with the diagrams can be determined.*

Example 4.7. Principal examples in the commutative case should be those in Sections 2 and 3 and that of Example 4.2. Principal examples in the even noncommutative case should be from the tenor product C^*-algebras of the commutative C^*-algebras in the commutative case, respectively tensored with noncommutative C^*-algebras such as $M_n(\mathbb{C})$, $\mathbb{K}$ and any other C^*-algebras with their K-theory groups computable.

In the case of $M_n(\mathbb{C})$ and $\mathbb{K}$, the noncommutative connected sums have the same Euler characteristic and the same K-theory as the commutative connected sums without tensoring with $M_n(\mathbb{C})$ and $\mathbb{K}$ by the stability of K-theory groups.

Some more complicated examples can be given by replacing the tensor product C^*-algebras viewed as the trivial bundle C^*-algebras, with more general bundle C^*-algebras (or continuous fields of C^*-algebras) (or with crossed product C^*-algebras with suitable actions, viewed as skewed tensor product C^*-algebras) but their base spaces should have the topological strucure of connected sums of spaces involved.

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