REPRODUCING PROPERTY FOR INTERPOLATIONAL PATH OF OPERATOR MEANS

Jun Ichi Fujii

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Abstract. We show that the solution of the 2-variable Karcher equation for the derivative solidarity coincides with the original interporational path of operator means, where the derivative solidarity for an interpolational path of operator means $A\,\mathrm{m}_t B$ is defined as $A\,\mathbf{s}_{\mathrm{m}} B = \frac{\partial A\,\mathrm{m}_t B}{\partial t}\big|_{t=0}$.

Let m be *an operator mean* in the sense of Kubo-Ando [7] which is defined by a positive operator monotone function f_{m} on the half interval $(0,\infty)$ with $f_{\mathrm{m}}(1)=1$;

$$A\,\mathrm{m}\,B = A^{\frac{1}{2}} f_{\mathrm{m}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

for positive invertible operators A and B on a Hilbert space. Thus the operator mean can be constructed by a numerical function $f_{\mathrm{m}}(x) = 1\,\mathrm{m}\,x$ which is called the representing function of m. For a symmetric operator mean m, i.e., $A\,\mathrm{m}\,B = B\,\mathrm{m}\,A$, the initial conditions

$$A\,\mathrm{m}_0 B = A, \quad A\,\mathrm{m}_{\frac{1}{2}} B = A\,\mathrm{m}\,B, \quad A\,\mathrm{m}_1 B = B$$

and the following inductive relation

$$A\,\mathrm{m}_{\frac{2k+1}{2^{n+1}}} B = (A\,\mathrm{m}_{\frac{k}{2^n}} B)\,\mathrm{m}\,(A\,\mathrm{m}_{\frac{k+1}{2^n}} B) = (A\,\mathrm{m}_{\frac{k+1}{2^n}} B)\,\mathrm{m}\,(A\,\mathrm{m}_{\frac{k}{2^n}} B) \tag{2}$$

for nonnegative numbers n and k with $2k+1 < 2^{n+1}$ determine the continuous path $A\,\mathrm{m}_t B$ from A to B of operator means. In particular,

$$A\,\mathrm{m}_{\frac{1}{2^n}} B = A\,\mathrm{m}\,(A\,\mathrm{m}_{\frac{1}{2^{n-1}}} B) = A(A\,\mathrm{m}\,(A\,\mathrm{m}_{\frac{1}{2^{n-2}}} B)) = \cdots = \overbrace{A(A(\cdots(A}^{n\text{ times}}\,\mathrm{m}\,B\overbrace{)\cdots))}^{n\text{ times}}. \tag{3}$$

Then, if the limit

$$A\,\mathbf{s}_{\mathrm{m}} B = \lim_{n\to\infty} 2^n (A\,\mathrm{m}_{\frac{1}{2^n}} B - A)$$

exists, it defines the *solidarity* whose representing function $F_{\mathbf{s}}(x) = 1\,\mathbf{s}\,x$ is a strictly increasing operator monotone function. The solidarity $\mathbf{s}$ in ([4]) is defined as a binary operation $A\,\mathbf{s}\,B$ for positive (invertible) operators A and B by

$$A\,\mathbf{s}\,B = A^{\frac{1}{2}} F\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

for some operator monotone function F on $(0,\infty)$. It has also typical properties of operator means except the monotonicity on the left-term. In particular, note that the *transformer equality*

$$T(A\,\mathbf{s}\,B)T^* = (TAT^*)\,\mathbf{s}\,(TBT^*)$$

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holds for invertible operators T.

If this path m_t is differentiable for t, then

$$A\,\mathbf{s}_{\mathrm{m}}B=\lim_{t\to 0}\frac{A\,\mathrm{m}_t B-A}{t}=\frac{\partial A\,\mathrm{m}_t B}{\partial t}\Big|_{t=0}.$$

So it is called the *derivative solidarity* for m. Its representing function $F_{\mathbf{s}}$ satisfies $F_{\mathbf{s}}(1)=0$ and $F'_{\mathbf{s}}(1)=1$ ([6]).

If a path satisfies

$$(A\,\mathrm{m}_r B)\,\mathrm{m}_t(A\,\mathrm{m}_s B)=A\,\mathrm{m}_{(1-t)r+ts}\,B$$

for all weights $r,\ s,\ t\in[0,1]$, then we call it an *interporational path* and also call the original mean an *interpolational* one as in [5, 6]. In the preceding paper [3], we showed that m_t is interpolational if and only if it satisfies the *mixing property*:

$$(a\,\mathrm{m}\,b)\,\mathrm{m}\,(c\,\mathrm{m}\,d)=(a\,\mathrm{m}\,c)\,\mathrm{m}\,(b\,\mathrm{m}\,d)$$

for all positive numbers a,b,c and d. This shows that the *logarithmic operator mean*

$$A\,\mathbf{L}\,B=A^{\frac12}\ell\left(A^{-\frac12}BA^{-\frac12}\right)A^{\frac12}$$

for the function $\ell(x)=(x-1)/\log x$ is not interpolational. We also showed in [6] that every interpolational path is convex by the maximality of the arithmetic mean in the symmetric operator means $\mathrm{m}=\mathrm{m}_{1/2}$;

$$A\,\mathrm{m}_{\frac{t+s}{2}}B=(A\,\mathrm{m}_t B)\,\mathrm{m}\,(A\,\mathrm{m}_s B)\le\frac{A\,\mathrm{m}_t B+A\,\mathrm{m}_s B}{2}.$$

Moreover it is differentiable and hence has always the derivative solidarity. This construction is similar to Uhlmann's one [9] that defines the relative entropy from interpolations.

For $r\in[-1,1]$, the following parametrized operator means $\#_t^{(r)}$, which are also called the *quasi-arithmetic* ones (cf. [2]),

$$A\#_t^{(r)}B=A^{\frac12}\left((1-t)I+t\left(A^{-\frac12}BA^{-\frac12}\right)^r\right)^{\frac1r}A^{\frac12}$$

are interpolational. The path $\#_t=\#_t^{(0)}=\lim_{\varepsilon\downarrow 0}\#_t^{(\varepsilon)}$;

$$A\#_t B=A^{\frac12}\left(A^{-\frac12}BA^{-\frac12}\right)^t A^{\frac12}$$

is that of the geometric operator mean and it is also the geodesic of the Finsler manifold of the positive invertible operators by Corach-Porta-Recht [1].

In [5, 6], we considered a map $\mathrm{m}\mapsto\mathbf{s}_{\mathrm{m}}$ from the interporational means m to the solidarities, say *Uhlmann's transform* by the above reason, but we could not discuss the inverse map then. In this paper, we will show that the solution X of the (2-variable) Karcher equation

$$(1-t)\,(X\,\mathbf{s}_{\mathrm{m}}A)+t\,(X\,\mathbf{s}_{\mathrm{m}}B)=0 \tag{4}$$

is the original path $A\,\mathrm{m}_t B$ as M.Pálfia suggested as we see later. This Karcher equation is equivalent to

$$(1-t)F\left(X^{-\frac12}AX^{-\frac12}\right)+tF\left(X^{-\frac12}BX^{-\frac12}\right)=0 \tag{4'}$$

for the representing function $F_{\mathbf{s}}(x)=1\,\mathbf{s}_{\mathrm{m}}x$.

To see this, we make some preparations. By the interpolationality of m_t, its representing function f_t has the following property:

Lemma 1. $f_s(f_t(x)) = f_{st}(x)$.

Proof. By the interpolationality, we have

$$f_s(f_t(x)) = 1\,\mathrm{m}_s(1\,\mathrm{m}_t\,x) = (1\,\mathrm{m}_0\,x)\,\mathrm{m}_s(1\,\mathrm{m}_t\,x) = 1\,\mathrm{m}_{0(1-s)+ts}\,x = f_{st}(x). \qquad \square$$

Since m is symmetric and m_t is homogeneous, we have:

Lemma 2. $B\,\mathrm{m}_{1-t}A = A\,\mathrm{m}_t B$ *and* $xf_{1-t}\left(\frac{1}{x}\right) = f_t(x)$.

Proof. The former follows from the construction (2). So we have

$$xf_{1-t}\left(\frac{1}{x}\right) = x\left(1\,\mathrm{m}_{1-t}\left(\frac{1}{x}\right)\right) = x\,\mathrm{m}_{1-t}\,1 = 1\,\mathrm{m}_t\,x = f_t(x). \qquad \square$$

Consider the derivative function $F_t(x) = \frac{\partial f_t(x)}{\partial t}$ (where $F_{\mathbf{s}} = F_0$). In [6], we showed $F_0(f_t(x)) = tF_t(x)$. Moreover we have:

Lemma 3. $F_s(f_t(x)) = tF_{ts}(x)$ *and* $F_{1-t}\left(\frac{1}{x}\right) = -\frac{1}{x}F_t(x)$.

Proof. By the definition of F_s and Lemma 1, we obtain

$$F_s(f_t(x)) = \lim_{r\to s}\frac{f_r(f_t(x)) - f_s(f_t(x))}{r-s} = t\lim_{r\to s}\frac{f_{tr}(x) - f_{ts}(x))}{tr-ts} = tF_{ts}(x).$$

Also the formula

$$-\frac{1}{x}F_t(x) = \lim_{s\to t}\frac{f_s(x)/x - f_t(x)/x}{-(s-t)} = \lim_{s\to t}\frac{f_{1-s}(1/x) - f_{1-t}(1/x)}{(1-s)-(1-t)} = F_{1-t}\left(\frac{1}{x}\right)$$

follows from the property $f_{1-t}(1/x) = \frac{f_t(x)}{x}$ in Lemma 2. $\square$

Let $\mathbf{s}_t$ be the solidarity defined by the derivative F_t. Then the above Lemma shows the formulae for tangent vectors and the transpose relation:

Theorem 4. *For the above solidarity* $\mathbf{s}_t$ *for an interpolational path* m_t,

$$A\,\mathbf{s}_s(A\,\mathrm{m}_t B) = tA\,\mathbf{s}_{ts}\,B \quad \text{and} \quad -A\,\mathbf{s}_t B = B\,\mathbf{s}_{1-t}\,A$$

for parameters $s, t \in [0,1]$.

Proof. Lemma 2 shows the first formula by

$$\begin{aligned} A\,\mathbf{s}_s(A\,\mathrm{m}_t B) &= A^{\frac{1}{2}}F_s\left(f_t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}} \\ &= tA^{\frac{1}{2}}F_{st}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} = tA\,\mathbf{s}_{ts}\,B. \end{aligned}$$

Also, since the transformer equality and Lemma 2 imply

$$X\,\mathbf{s}_t I = X(I\,\mathbf{s}_t X^{-1}) = XF_t(X^{-1}) = -F_{1-t}(X),$$

we have

$$-A\,\mathbf{s}_t B = B^{\frac{1}{2}}\left[\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)\mathbf{s}_t I\right]B^{\frac{1}{2}} = -B^{\frac{1}{2}}F_{1-t}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)B^{\frac{1}{2}} = B\,\mathbf{s}_{1-t}\,A. \qquad \square$$

Now we show the reproducing property:

Theorem 5. *Let F be the representing function for a derivative solidarity $\mathbf{s}_{\mathrm{m}}$ for an interpolational mean* m, *the solution of the Karcher equation (4) is the interpolational path* $A\,\mathrm{m}_t B$.

Proof. The homogeneity for operator means shows that we have only to show that the solution y of the numerical equation

$$(4'') \qquad (1-t)F\left(\frac{1}{y}\right)+tF\left(\frac{x}{y}\right)=0$$

is given by $y=f_t(x)=1\,\mathrm{m}_t x$. In fact, by the above lemma, we have

$$\begin{aligned}(1-t)F_0\left(\frac{1}{f_t(x)}\right)&+tF_0\left(\frac{x}{f_t(x)}\right)\\ &=-\frac{1-t}{f_t(x)}F_1(f_t(x))-\frac{t}{f_{1-t}(1/x)}F_1(f_{1-t}(1/x))\\ &=-\frac{(1-t)t}{f_t(x)}F_t(x)-\frac{t(1-t)}{f_{1-t}(1/x)}F_{1-t}(1/x)\\ &=-\frac{t(1-t)}{f_t(x)}F_t(x)+\frac{t(1-t)}{(1/x)f_t(x)}\frac{1}{x}F_t(x)=0.\end{aligned}$$

Thus we obtain $y=f_t(x)$. □

In a RIMS Workshop held at November 6–8, 2013 in Kyoto, M.Pálfia [8] posed an interesting problem when the solution of the Karcher equation (4) is a path of operator means for an operator monotone function F with $F(1)=0$ and $F'(1)=1$. So we observe the above theorem from this constructive viewpoint: In (4″), it follows from the monotonicity of F that

$$\frac{x}{y}=F^{-1}\left(-\frac{1-t}{t}F\left(\frac{1}{y}\right)\right).$$

So, putting

$$g_t(y)=yF^{-1}\left(-\frac{1-t}{t}F\left(\frac{1}{y}\right)\right),$$

we have $x=g_t(y)$. Thus if we find an interpolational path f_t with $y=f_t(g_t(y))=f_t(x)$, the solution X coincides with $A\,\mathrm{m}_t B$ for the corresponding path $1\,\mathrm{m}_t x=f_t(x)$. But we notice that g_t is not monotone:

Remark. Let m_t be an interpolational path defined by a function

$$f_t(x)=\left(1-t+t\sqrt{x}\right)^2.$$

Then it must be a solution of (4′). The derivative solidarity is determined by $F(x)=2(\sqrt{x}-1)$. Since

$$F^{-1}(z)=\left(\frac{z}{2}+1\right)^2,$$

the above function is

$$g_t(y)=y\left(1-\frac{1-t}{t}\left(\sqrt{\frac{1}{y}}-1\right)\right)^2=\frac{\left(\sqrt{y}-(1-t)\right)^2}{t^2},$$

which is convex with the minimum 0 at $y=(1-t)^2$. It is not monotone on $(0,\infty)$, but in the region $y>(1-t)^2$, the function $g_t(y)$ is monotone and its inverse function is $(1-t+t\sqrt{x})^2$, which is operator monotone.

Finally we give an example of operator monotone function F satisfying $F(1) = 0$ and $F'(1) = 1$ that does not induce an operator mean:

Example. Let $F(x) = 2(\sqrt{2(x+1)} - 2)$. Then we have $F(1) = 0$, $F'(1) = 0$ and

$$F^{-1}(z) = \frac{(z+4)^2}{8} - 1.$$

It follows that

$$g_t(y) = \frac{\left(2\sqrt{y} - \frac{1-t}{t}\left(\sqrt{2(1+y)} - 2\sqrt{y}\right)\right)^2}{2} - y$$

and hence

$$t^2 g_t(y) = (3-2t)y + (1-t)^2 - 2(1-t)\sqrt{2(1+y)y}.$$

Thus we have

$$g_{\frac{1}{2}}(y) = 8y + 1 - 4\sqrt{2(1+y)y}.$$

This function is convex and $g_{\frac{1}{2}}(y) = 0$ for $y = \frac{2\pm\sqrt{2}}{8}$. But, even in the region $y > \frac{2+\sqrt{2}}{8}$ where g_t is monotone and positive, the inverse function

$$f_{\frac{1}{2}}(x) = \frac{2(x+1) + \sqrt{2x^2+12x+2}}{8}$$

is not operator monotone. In fact, a concave function $h(x) = \sqrt{x^2+6x+1}$ is not operator monotone (and hence not operator concave) since $\operatorname{Arg} z < \operatorname{Arg} h(z)$ for some z in the upper half complex plane: Consider $z = re^{it}$ for $t = \frac{3\pi}{4}$. Then

$$h(z)^2 = 1 + 6r\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) - ir^2,$$

$\mathrm{Re} z^2 = 0$ and $\mathrm{Re}\, h(z)^2 = 1 - 3\sqrt{2}r$. It follows that $\operatorname{Arg} f(z) > \frac{3\pi}{4}$ for $r < \frac{1}{3\sqrt{2}}$, which shows h is not operator monotone.

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Department of Art and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

E-mail address, fujii@cc.osaka-kyoiku.ac.jp