A Note on Intuitionistic Fuzzy n-Racks

Guy Roger Biyogmam
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Abstract. In this paper we apply the concept of intuitionistic fuzzy sets to n-racks, $n \geq 2$. Several related results are established. In particular, we discuss some properties of normality and maximality of intuitionistic fuzzy n-racks using their (α, β)-cut sets.

1 Introduction In [3], the author introduced the category of n-racks as a generalization of racks [6], and studied n-subracks in [4]. Intuitionistic fuzzy sets were introduced by Krassimiri T. Atanassov [1] as a generalization of the concept of fuzzy sets introduced by Zadeh [9] in the 60s. They have been applied to several algebraic concepts such as equivalence relations [2], congruences [7] and groups [8]. In this work, we develop this concept on n-racks. In particular we extend some results established in [5] on fuzzy n-racks to intuitionistic fuzzy n-racks.

Let us recall a few definitions. A n-rack[1][3] $(R, [-, \ldots, -]_R)$ is a set R endowed with an n-ary operation $[-, \ldots, -]_R : R \times R \times \ldots \times R \longrightarrow R$ such that

- $\big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}]_R\big]_R = \big[[x_1, \ldots, x_{n-1}, y_1]_R, \ldots, [x_1, \ldots, x_{n-1}, y_n]_R\big]_R$ (This is the left distributive property of n-racks)
- For $a_1, \ldots, a_{n-1}, b \in R$, there is a unique $x \in R$ with $[a_1, \ldots, a_{n-1}, x]_R = b$.

If in addition there is a distinguish element $1 \in R$, such that $[1, \ldots, 1, y]_R = y$ and $[x_1 \ldots, x_{n-1}, 1]_R = 1$ for all $x_1, \ldots, x_{n-1} \in R$, then $(R, [-, \ldots, -]_R, 1)$ is said to be a pointed n-rack.

- A n-rack R is involutive if it further satisfies
$$\big[x_1, \ldots, x_{n-1}, [x_1, \ldots, x_{n-1}, y]\big] = y \text{ for all } x_1, \ldots, x_{n-1}, y \in R.$$
- A n-rack R is trivial if it further satisfies $[x_1, x_2, \ldots, x_{n-1}, y]_R = y$ for all $x_i, y \in R$.
- A n-rack is a n-quandle if it further satisfies $[x_1, x_2, \ldots, x_{n-1}, y]_R = y$ if $x_i = y$ for some $i \in \{1, 2, \ldots, n-1\}$.
- A non empty subset S of a n-rack (resp. pointed n-rack) R is called n-semisubrack of R if S is closed under the n-rack operation. S is called n-subrack of R if it has a n-rack structure (resp. pointed n-rack structure).

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[1] In this paper, we mean by a n-rack, a left n-rack.

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2 intuitionistic fuzzy n-subracks Recall from [1] that for a set R, an intuitionistic fuzzy set S in R is an object $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$, where $\mu_S : R \to [0,1]$ and $\nu_S : R \to [0,1]$ are two functions satisfying $0 \leq \mu_S(x) + \nu_S(x) \leq 1$ for all $x \in R$. Also $\mu_S(x)$ and $\nu_S(x)$ define respectively the degree of membership and the degree of non-membership of $x \in R$. We say that S is constant if μ_S or ν_S is constant. Note that when $\mu_S(x) + \nu_S(x) = 1$ for all $x \in R$, S is a fuzzy set. Also for two intuitionistic fuzzy sets $S_1 = \{\langle x, \mu_{S_1}(x), \nu_{S_1}(x)\rangle : x \in R\}$ and $S_2 = \{\langle x, \mu_{S_2}(x), \nu_{S_2}(x)\rangle : x \in R\}$, one says that $S_1 \subseteq S_2$ if and only if $\mu_{S_1}(x) \leq \mu_{S_2}(x)$ and $\nu_{S_1}(x) \geq \nu_{S_2}(x)$ for all $x \in R$. Throughout the paper, we consider only intuitionistic fuzzy sets that are not fuzzy sets.

Definition 2.1. *Let R be a n-rack. An intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ in R is said to be an intuitionistic fuzzy n-semisubrack of R if for any $x_1, \dots, x_n \in R$,*

i) $\mu_S([x_1, \dots, x_n]) \geq min\{\mu_S(x_1), \dots, \mu_S(x_n)\}$

ii) $\nu_S([x_1, \dots, x_n]) \leq max\{\nu_S(x_1), \dots, \nu_S(x_n)\}$

iii) $\mu_S(1) \geq \mu_S(x)$ *and* $\nu_S(1) \leq \nu_S(x)$ *for all* $x \in R$ *if the rack is pointed by 1.*

Definition 2.2. *[8] Let S be an intuitionistic fuzzy set of a set R. The $(\alpha, \beta) - cut$ of S is a crisp subset $C_{\alpha,\beta}(S)$ of S given by*

$$C_{\alpha,\beta}(S) = \{x \in R \ / \ \mu_S(x) \geq \alpha, \ \nu_S(x) \leq \beta\}$$

where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.

The following is a characterization of intuitionistic fuzzy n-semisubracks by means of $(\alpha, \beta) - cut$ sets.

Proposition 2.3. *Let R be a n-rack. The intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ is an intuitionistic fuzzy n-semisubrack of R if and only if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the $(\alpha, \beta) - cut$ of S is a n-semisubrack of R when it is non empty.*

Proof. $\Rightarrow$) Let $\alpha, \beta \in [0,1]$. Assume that $C_{\alpha,\beta}(S) \neq 0$ and let $\{a_i\}_{i=1,\dots,n-1} \subseteq C_{\alpha,\beta}(S)$. Then as S is an intuitionistic fuzzy n-semisubrack, we have

$$\mu_S([a_1, \dots, a_n]) \geq min\{\mu_S(a_1), \dots, \mu_S(a_{n-1}), \mu_S(a_n)\} \geq \alpha$$

and

$$\nu_S([a_1, \dots, a_n]) \leq max\{\nu_S(a_1), \dots, \nu_S(a_{n-1}), \nu_S(a_n)\} \leq \beta,$$

i.e. $[a_1, \dots, a_{n-1}, a_n] \in C_{\alpha,\beta}(S)$. So $C_{\alpha,\beta}(S)$ is closed under the n-rack operation and thus it is a n-semisubrack of R.

$\Leftarrow$) We proceed by contradiction. Assume S is not an intuitionistic fuzzy n-semisubrack of R. So there are $x_1, \dots, x_n \in R$ with either $\mu_S([x_1, \dots, x_n]) < min\{\mu_S(x_1), \dots, \mu_S(x_n)\}$ or $\nu_S([x_1, \dots, x_n]) > max\{\nu_S(x_1), \dots, \nu_S(x_n)\}$. Without loss of generality, consider the first case. Then setting

$$\alpha_0 = \frac{min\{\mu_S(x_1), \dots, \mu_S(x_n)\} + \mu_S([x_1, \dots, x_n])}{2}$$

yields to the compound inequality

$$0 \leq \mu_S([x_1, \dots, x_n]) < \alpha_0 \leq min\{\mu_S(x_1), \dots, \mu_S(x_n)\} \leq \mu_S(x_i)$$

for all $i = 1, \ldots, n$. Choose $\beta_0 \in [0,1]$ such that $\alpha_0 + \beta_0 \leq 1$ and $\nu_S(x_i) \geq \beta_0$ for all $i = 1, \ldots, n$. Hence $x_i \in C_{\alpha_0,\beta_0}(S)$ for all $i = 1, \ldots, n$ and $[x_1, \ldots, x_n] \notin C_{\alpha_0,\beta_0}(S)$. This contradicts the fact that $C_{\alpha_0,\beta_0}(S)$ is a n-semisubrack of R. The proof for the second case is similar.

□

Definition 2.4. *Let R be a n-rack. An intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ in R is said to be an intuitionistic fuzzy n-subrack of R if for any $x_1, \ldots, x_{n-1}, y \in R$,*

i) $\mu_S(y) \geq min\{\mu_S([x_1, \ldots, x_{n-1}, y]), \mu_S(x_1), \ldots, \mu_S(x_{n-1})\}$

ii) $\nu_S(y) \leq max\{\nu_S([x_1, \ldots, x_{n-1}, y]), \nu_S(x_1), \ldots, \nu_S(x_{n-1})\}$

iii) $\mu_S(1) \geq \mu_S(x)$ *and* $\nu_S(1) \leq \nu_S(x)$ *for all* $x \in R$ *if the rack is pointed by 1.*

Example 2.5. *Consider the $(t,s)-n$-rack M of example 2.3 in [3] with $n = 4$, $s = 1$, $t = 0$ and $M = \mathbb{N}$. Then M is a 4-rack with rack operation $[x_1, x_2, x_3, x_4] = x_1 + x_2 + x_3$. Define on M the intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ by*

$$\mu_S(x) = \begin{cases} \frac{1}{4}, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases} \quad \text{and} \quad \nu_S(x) = \begin{cases} 0, & \text{if } x \text{ is odd} \\ \frac{1}{4}, & \text{if } x \text{ is even} \end{cases}.$$

A case by case checking shows that S is an intuitionistic fuzzy 4-semisubrack. However, S is not an intuitionistic fuzzy 4-subrack because for $x_1 = 1$, $x_2 = 3$, $x_3 = 5$ and $x_4 = 2$, we have $\mu_S([x_1, x_2, x_3, x_4]) = \mu_S(9) = \frac{1}{4}$ and so
$\mu_S(x_4) = 0 < \frac{1}{4} = min\{\mu_S([x_1, x_2, x_3, x_4]), \mu_S(x_1), \mu_S(x_2), \mu_S(x_3)\}$.

Example 2.6. *Consider the quandle (containing the dihedral rack $D = \{a, b, c\}$ as a subquandle) $(R = \{1, a, b, c\}, \circ)$ whose Cayley table is given by:*

∘	1	a	b	c
1	1	a	b	c
a	1	a	c	b
b	1	c	b	a
c	1	b	a	c

It is easy to show that the intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ on R defined by

$$\mu_S(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1, a \\ \frac{1}{8}, & \text{if } x = b, c \end{cases} \quad \text{and} \quad \nu_S(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1, a \\ \frac{3}{4}, & \text{if } x = b, c \end{cases}$$

is an intuitionistic fuzzy subrack of R.

Theorem 2.7. *[4] A n-semisubrack S of a pointed n-rack $(R, [-, \ldots, -], 1)$ is a n-subrack if and only if for all $b \in R$, $[a_1, a_2, \ldots, a_{n-1}, b] \in S$ and $\{a_i\}_{i=1,\ldots,n-1} \subseteq S$ implies $b \in S$.*

The following is a characterization of intuitionistic fuzzy n-subracks by means of $(\alpha, \beta)-$*cut* sets.

Proposition 2.8. *Let R be a n-rack. The intuitionistic fuzzy set $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ is an intuitionistic fuzzy n-subrack of R if and only if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the $(\alpha, \beta) - cut$ of S is a n-subrack of R when it is non empty.*

Proof. $\Rightarrow$) Let $\alpha, \beta \in [0,1]$. Assume that $C_{\alpha,\beta}(S) \neq 0$ and let $\{a_i\}_{i=1,\dots,n-1} \subseteq C_{\alpha,\beta}(S)$ with $[a_1, \dots, a_{n-1}, b] \in C_{\alpha,\beta}(S)$. Then $\mu_S([a_1, \dots, a_{n-1}, b]) \geq \alpha$, $\mu_S(a_i) \geq \alpha$ and $\nu_S([a_1, \dots, a_{n-1}, b]) \leq \beta$, $\nu_S(a_i) \leq \beta$ for $i = 1, \dots, n-1$. Now as S is an intuitionistic fuzzy n-subrack of R, we have

$$\mu_S(b) \geq min\{\mu_S([a_1, \dots, a_{n-1}, b]), \mu_S(a_1), \dots, \mu_S(a_{n-1})\} \geq \alpha$$

and

$$\nu_S(b) \leq max\{\nu_S([a_1, \dots, a_{n-1}, b]), \nu_S(a_1), \dots, \nu_S(a_{n-1})\} \leq \beta,$$

i.e. $b \in C_{\alpha,\beta}(S)$. So $C_{\alpha,\beta}(S)$ is a n-subrack of R.
$\Leftarrow$) We proceed by contradiction. Assume S is not an intuitionistic fuzzy n-subrack of R. So there are $x_1^0, \dots, x_{n-1}^0, y_0 \in R$ with either

$$\mu_S(y_0) < min\{\mu([x_1^0, \dots, x_{n-1}^0, y_0]), \mu(x_1^0), \dots, \mu(x_{n-1}^0)\}$$

or

$$\nu_S(y_0) > max\{\nu([x_1^0, \dots, x_{n-1}^0, y_0]), \nu(x_1^0), \dots, \nu(x_{n-1}^0)\}.$$

Without loss of generality, consider the first case. Setting

$$\alpha_0 = \frac{min\{\mu_S([x_1^0, \dots, x_{n-1}^0, y_0]), \mu_S(x_1^0), \dots, \mu_S(x_{n-1}^0)\} + \mu_S(y_0)}{2}$$

yields to the compound inequality

$$0 \leq \mu_S(y_0) < \alpha_0 \leq min\{\mu_S([x_1^0, \dots, x_{n-1}^0, y_0]), \mu_S(x_1^0), \dots, \mu_S(x_{n-1}^0)\} \leq \mu_S(x_i^0).$$

Choose $\beta_0 \in [0,1]$ such that $\alpha_0 + \beta_0 \leq 1$ and $\nu_S(x_i^0) \geq \beta_0$ for all $i = 1, \dots, n-1$. So $[x_1^0, \dots, x_{n-1}^0, y_0] \in C_{\alpha_0,\beta_0}(S)$, $x_i^0 \in C_{\alpha_0,\beta_0}(S)$ for all $i = 1, \dots, n-1$ and $y_0 \notin C_{\alpha_0,\beta_0}(S)$. This contradicts by theorem 2.7 the fact that $C_{\alpha_0,\beta_0}(S)$ is a n-subrack of R. The proof for the second case is similar.

□

Remark 2.9. *If R is an involutive n-subrack, one shows by theorem 2.7 that n-semisubracks and n-subracks coincide. It follows by proposition 2.8 and proposition 2.3 that intuitionistic fuzzy n-subracks and intuitionistic fuzzy n-semisubracks coincide in involutive n-racks (thus in trivial n-racks).*

Proposition 2.10. *Let S be a n-subrack of R. Then S can be realized as a $(\alpha, \beta) - cut$ of some intuitionistic fuzzy n-subrack of R.*

Proof. Choose $r, s \in [0,1]$ with $s < r$. Consider the fuzzy set on R defined by

$$\mu_S(x) = \begin{cases} r, & \text{if } x \in S \\ s, & \text{else.} \end{cases} \quad and \quad \nu_S(x) = \begin{cases} s, & \text{if } x \in S \\ r, & \text{else.} \end{cases}$$

We claim that the set $\tilde{S} = \big\{\langle x, \mu_S, \nu_S \rangle : x \in R\big\}$ is an intuitionistic fuzzy n-subrack of R. In fact, a case by case checking shows that the inequalities
$\mu_S(x_n) \geq min\{\mu_S([x_1, \dots, x_{n-1}, x_n]), \mu_S(x_1), \dots, \mu_S(x_{n-1})\}$ and
$\nu_S(x_n) \leq max\{\nu_S([x_1, \dots, x_{n-1}, x_n]), \nu_S(x_1), \dots, \nu_S(x_{n-1})\}$ fail only if $x_n \notin S$, $[x_1, \dots, x_n] \in S$ and $x_i \in S$ for all $i = 1, \dots, n-1$. But this can't occur by theorem 2.7 as S is a n-subrack of R. Moreover, it is clear that for any choice of $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1, \alpha \leq r$ and $\beta \geq s$, we have $C_{\alpha,\beta}(\tilde{S}) = S$. □

Corollary 2.11. *Let S be a n-subrack of R. For each $\alpha, \beta \in (0,1]$ with $\alpha + \beta \leq 1$, there is an intuitionistic fuzzy n-subrack $\tilde{S} = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ of R with $C_{\alpha,\beta}(\tilde{S}) = S$.*

Proof. The result follows by the proof of Proposition 2.10. □

3 Normal and Maximal Intuitionistic Fuzzy n-Subracks Throughout this section, R denotes a pointed n-rack.

Definition 3.1. *A normal intuitionistic fuzzy n-subrack of R is an intuitionistic fuzzy n-subrack $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ of R such that $1 \in (\mu_S^{-1} + \nu_S^{-1})(1)$.*

Proposition 3.2. *Every intuitionistic fuzzy n-subrack of R can be embedded into a normal intuitionistic fuzzy n-subrack of R.*

Proof. Let S be an intuitionistic fuzzy n-subrack of R. If S is normal, there is nothing to prove. Otherwise, let $p, q \in [0,1]$ such that $\mu_S(1) \leq p$, $\nu_S(1) \geq q$ and $p+q = 1$. Consider on R the functions ζ_S and ζ'_S defined by $\zeta_S(x) = \mu_S(x) - \mu_S(1) + p$ and $\zeta'_S(x) = \nu_S(x) - \nu_S(1) + q$. Clearly, ζ_S and ζ'_S are well-defined, $(\zeta_S + \zeta'_S)(1) = 1$, $\zeta_S(x) \geq \mu_S(x)$ and $\zeta'_S(x) \leq \nu_S(x)$ for all $x \in R$. Also, for $x_1, x_2, \ldots, x_n \in R$ we have

$$\begin{aligned}
\zeta_S(x_n) &= \mu_S(x_n) - \mu_S(1) + p \\
&\geq min\{\mu_S([x_1, \ldots, x_n]), \mu_S(x_1), \ldots, \mu_S(x_{n-1})\} - \mu_S(1) + p \\
&\geq min\{\mu_S([x_1, \ldots, x_n]) - \mu_S(1) + p, \mu_S(x_1) - \mu_S(1) + p, \ldots, \mu_S(x_{n-1}) - \mu_S(1) + p\} \\
&\geq min\{\zeta_S([x_1, \ldots, x_n]), \zeta_S(x_1), \ldots, \zeta_S(x_{n-1})\},
\end{aligned}$$

$$\begin{aligned}
\zeta'_S(x_n) &= \nu_S(x_n) - \nu_S(1) + q \\
&\leq max\{\nu_S([x_1, \ldots, x_n]), \nu_S(x_1), \ldots, \nu_S(x_{n-1})\} - \nu_S(1) + q \\
&\leq max\{\nu_S([x_1, \ldots, x_n]) - \nu_S(1) + q, \nu_S(x_1) - \nu_S(1) + q, \ldots, \nu_S(x_{n-1}) - \nu_S(1) + q\} \\
&\leq max\{\zeta'_S([x_1, \ldots, x_n]), \zeta'_S(x_1), \ldots, \zeta'_S(x_{n-1})\},
\end{aligned}$$

and $\zeta_S(1) \geq \zeta_S(x)$ and $\zeta'_S(1) \leq \zeta'_S(x)$ for all $x \in R$ since $\mu_S(1) \geq \mu_S(x)$ and $\nu_S(1) \leq \nu_S(x)$ for all $x \in R$.

Hence the set $\{\langle x, \zeta_S(x), \zeta'_S(x)\rangle : x \in R\}$ is a normal intuitionistic fuzzy n-subrack containing S. □

Definition 3.3. *Let S_1 and S_2 be two intuitionistic fuzzy n-subracks of R. We say*[2] *that $S_1 \subseteq_{ae} S_2$ if the set $\{x \in R \;/\; \mu_{S_1}(x) \geq \mu_{S_2}(x),\ \nu_{S_1}(x) \leq \nu_{S_2}(x)\} = \{1\}$.*

Remark 3.4. *It is not hard to check that this relation is an order. Under this relation, the intuitionistic fuzzy set $\{\langle x, \zeta_S(x), \zeta'_S(x)\rangle : x \in R\}$ above in the proof of proposition 3.2 is the smallest normal intuitionistic fuzzy n-subrack of R containing S. Denote it $\bar{S} = \{\langle x, \bar{\mu}_S(x), \bar{\nu}_S(x)\rangle : x \in R\}$.*

Definition 3.5. *When $p = \frac{1}{2}$ and $q = \frac{1}{2}$, $\bar{S}$ is called the normal closure of S.*

Definition 3.6. *A non constant intuitionistic fuzzy n-subrack of R is said to be maximal if its normal closure is maximal among normal intuitionistic fuzzy n-subracks of R.*

Theorem 3.7. *Every maximal intuitionistic fuzzy n-subrack of R is normal.*

[2] Read $S_1 \subseteq_{ae} S_2$ as "$S_1 \subseteq S_2$" almost everywhere

Proof. Let S be a maximal intuitionistic fuzzy n-subrack of R. If $\mu_S(1)+\nu_S(1)=1$, then S is normal and $S=\bar{S}$. Assume $\mu_S(1)+\nu_S(1)\neq 1$ and define an intuitionistic fuzzy set S_0 on R by $S_0=\{\langle x,\zeta_{S_0}(x),\zeta'_{S_0}(x)\rangle : x\in R\}$ with $\zeta_{S_0}(x)=\frac{\mu_S(x)+\mu_S(1)}{2}$ and $\zeta'_{S_0}(x)=\frac{\nu_S(x)+\nu_S(1)}{2}$. Clearly, S_0 is an intuitionistic fuzzy n-subrack of R since for $x_1,x_2,\ldots,x_n\in R$ we have

$$\begin{aligned}\zeta_{S_0}(x_n)&=\frac{\mu_S(x_n)+\mu_S(1)}{2}\\&\geq\frac{min\{\mu_S([x_1,\ldots,x_n]),\mu_S(x_1),\ldots,\mu_S(x_{n-1})\}+\mu_S(1)}{2}\\&\geq min\{\frac{\mu_S([x_1,\ldots,x_n])+\mu_S(1)}{2},\frac{\mu_S(x_1)+\mu_S(1)}{2},\ldots,\frac{\mu_S(x_{n-1})+\mu_S(1)}{2}\}\\&\geq min\{\zeta_{S_0}([x_1,\ldots,x_n]),\zeta_{S_0}(x_1),\ldots,\zeta_{S_0}(x_{n-1})\},\end{aligned}$$

$$\begin{aligned}\zeta'_{S_0}(x_n)&=\frac{\nu_S(x_n)+\nu_S(1)}{2}\\&\leq\frac{max\{\nu_S([x_1,\ldots,x_n]),\nu_S(x_1),\ldots,\nu_S(x_{n-1})\}+\nu_S(1)}{2}\\&\leq max\{\frac{\nu_S([x_1,\ldots,x_n])+\nu_S(1)}{2},\frac{\nu_S(x_1)+\nu_S(1)}{2},\ldots,\frac{\nu_S(x_{n-1})+\nu_S(1)}{2}\}\\&\leq max\{\zeta'_{S_0}([x_1,\ldots,x_n]),\zeta'_{S_0}(x_1),\ldots,\zeta'_{S_0}(x_{n-1})\}.\end{aligned}$$

and $\zeta_{S_0}(1)\geq\zeta_{S_0}(x)$ and $\zeta'_{S_0}(1)\leq\zeta'_{S_0}(x)$ for all $x\in R$ since $\mu_S(1)\geq\mu_S(x)$ and $\nu_S(1)\leq\nu_S(x)$ for all $x\in R$. Moreover, $\zeta_{S_0}(1)=\mu_S(1)$, $\zeta'_{S_0}(1)=\nu_S(1)$ and $\mu_S(x_0)<\mu_S(1)$ and $\nu_S(x_0)>\nu_S(1)$ for some $x_0\in R$ as S is non constant. Let $\bar{S}_0=\{\langle x,\bar{\zeta}_{S_0}(x),\bar{\zeta}'_{S_0}(x)\rangle : x\in R\}$ be the normal closure of S_0. Then

$$\bar{\zeta}_{S_0}(x_0)=\zeta_{S_0}(x_0)-\zeta_{S_0}(1)+\frac{1}{2}=\zeta_{S_0}(x_0)-\mu_S(1)+\frac{1}{2}>\mu_S(x_0)-\mu_S(1)+\frac{1}{2}=\bar{\mu}_S(x_0)$$

and

$$\bar{\zeta}'_{S_0}(x_0)=\zeta'_{S_0}(x_0)-\zeta'_{S_0}(1)+\frac{1}{2}=\zeta'_{S_0}(x_0)-\nu_S(1)+\frac{1}{2}<\nu_S(x_0)-\nu_S(1)+\frac{1}{2}=\bar{\nu}_S(x_0).$$

This contradicts the maximality of $\bar{S}$ among the normal intuitionistic fuzzy n-subracks of R. Hence $\mu_S(1)+\nu_S(1)=1$ and S is normal. ☐

Theorem 3.8. *If S is a maximal intuitionistic fuzzy n-subrack of R, then*

$$Im(\mu_s+\nu_S)=\{0,1\}.$$

Proof. Assume S is a maximal intuitionistic fuzzy n-subrack of R. Then $\mu_S(1)+\nu_S(1)=1$ and $S=\bar{S}$ by theorem 3.7. Now let $x\in R$ with $0<\mu_S(x)+\nu_S(x)<1$. Define an intuitionistic fuzzy set S_0 on R by $S_0=\{\langle x,\zeta_{S_0}(x),\zeta'_{S_0}(x)\rangle : x\in R\}$ with $\zeta_{S_0}(x)=\frac{\mu_S(x)+\frac{1}{2}}{2}$ and $\zeta'_{S_0}(x)=\frac{\nu_S(x)+\frac{1}{2}}{2}$. Clearly, S_0 is an intuitionistic fuzzy n-subrack of R by the proof of theorem 3.7. Moreover S_0 is normal as S is normal. In addition, $\bar{\zeta}_{S_0}(x)=\zeta_{S_0}(x)>\mu_S(x)=\bar{\mu}_S(x)$ since $0<\mu_S(x)<\frac{1}{2}$ for all $x\in R$, and $\bar{\zeta}'_{S_0}(x)=\zeta'_{S_0}(x)<\nu_S(x)=\bar{\nu}_S(x)$ since $\nu_S(x)>\frac{1}{2}$ for all $x\in R$. Thus $\bar{S}\subseteq_{ae}\bar{S}_0$ because the set $\{x\in R\ /\ \bar{\zeta}_{S_0}(x)\geq\bar{\mu}_{S_2}(x),\ \bar{\zeta}'_{S_0}(x)\leq\bar{\nu}_{S_2}(x)\}\neq\{1\}$. This contradicts the maximality of $\bar{S}$ among the normal intuitionistic fuzzy n-subracks of R. Hence $\mu_S(1)+\nu_S(1)=0$ or $\mu_S(1)+\nu_S(1)=1$. ☐

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Communicated by *Junzo Watada*

Department of Mathematics,

Southwestern Oklahoma State University,

100 CampusDrive, Weatherford,OK73096,USA

guy.biyogmam@swosu.edu