A MEAN VALUE PROPERTY FOR POLYCALORIC FUNCTIONS

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Abstract. In this paper we prove a mean value property for polycaloric functions in one space dimensional case. The proof given here is a slight modification of that of the recent paper by F.Da Lio and L.Rodino [3] and seems more straightforward.

1 Introduction There are many papers that deal with a mean value property for polyharmonic functions (see [1, 2, 4, 6, 7] etc.). Especially, in 2011, G. Lysik ([7]) gave a simple and elegant proof of the following mean value property for polyharmonic functions and its inverse. Let $m \in \mathbf{N}$ and let U be a domain in $\mathbf{R}^N$. If $u \in C^{2m}(U)$ and $\Delta^m u = 0$, then for any ball $B_R(x) \subset U$ it holds

$$\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)dy = \sum_{k=0}^{m} \frac{\Delta^k u(x)}{4^k(\frac{N}{2}+1)_k k!}R^{2k} \tag{1.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \in \mathbf{N}$.

The main subject of this paper concerns the heat version of the result (1.1). First, we fix some terminologies. Let $U \subset \mathbf{R}^N$ be an open set and $U_T = U \times (0,T]$ denote a parabolic cylinder. We say that a function u defined on U_T is *caloric* if u is a solution of the linear heat equation $(\partial_t - \Delta_x)u(x,t) = 0$, $(x,t) \in U_T$, where $\Delta_x = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. Also, in this paper, u is called *polycaloric* if u is a solution of the equation $(\partial_t - \Delta_x)^m u(x,t) = 0, (x,t) \in U_T$ for some $m \in \mathbf{N}$. For fixed $x \in \mathbf{R}^N, t \in \mathbf{R}$, and $r > 0$, let

$$E(x,t;r) = \left\{ (y,s) \in \mathbf{R}^N \times \mathbf{R} \;\middle|\; s \le t, \Phi(x-y,t-s) \ge \frac{1}{r^N} \right\}$$

denote a heat ball with a top point (x,t), where

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{N/2}} \exp\left(-\dfrac{|x|^2}{4t}\right) & (x \in \mathbf{R}^N, t > 0) \\ 0 & (x \in \mathbf{R}^N, t < 0) \end{cases}$$

is the fundamental solution of the heat equation. Note that a heat ball is symmetric with respect to y_i-axis $(i = 1, \cdots, N)$ and

$$E(0,0;1) = \left\{ (y,s) \in \mathbf{R}^N \times \mathbf{R} \mid -\frac{1}{4\pi} \le s < 0, |y| \le \sqrt{2Ns\log(-4\pi s)} \right\}.$$

It is well known that caloric functions possess the mean value property. Namely, if u is caloric on U_T, then for each heat ball $E(x,t;r) \subset U_T$ it holds

$$u(x,t) = \frac{1}{4r^N}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \tag{1.2}$$

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(see [5]: p.p 53-54 Theorem 3, or [10]). There is also an inverse mean value property of caloric functions under certain conditions ([9]).

Heat version of the result (1.1) is also known. Namely, in 2006, F. Da Lio and L. Rodino [3] proved the following asymptotic expansion formula for the heat integral mean (1.2) as a power series with respect to the radius of the heat ball:

Let $u \in C^\infty(\mathbf{R}^{N+1})$ and $(x,t) \in \mathbf{R}^{N+1}$, then it holds

$$\begin{aligned}(1.3)\qquad &\frac{1}{4r^N}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \\ &= u(x,t) + \sum_{k=1}^{M} r^{2k} H_k u(x,t) + O\left(r^{2M+2}\right) \text{ as } r \to 0,\end{aligned}$$

where H_k is given by

$$(1.4)\qquad H_k u = \beta_{k,N}\left(\partial_t - \frac{N}{2k+N}\Delta_x\right)^{k-1}(\partial_t - \Delta_x)\,u$$

and

$$\beta_{k,N} = (-1)^k \frac{N}{k!}\frac{1}{(2k+N)}\left(\frac{N}{2k+N}\right)^{\frac{N}{2}+1}\left(\frac{1}{4\pi}\right)^k.$$

One of the key ideas in [3] is to introduce the differential operator H_k which is the k-th power of different heat operators whose diffusion coefficients depend on the iteration number k, though the exact meaning of H_k is less clear.

In this paper, we prove the formula (1.3) in [3] by another method, when the space dimension $N = 1$. We do not need to introduce the weighted power H_k and, in the author's opinion, the method seems more straightforward.

Theorem 1. *Let $N = 1$, $u \in C^\infty(U_T)$, $r > 0$ and $M \in \mathbf{N}$. Then we have*

$$\begin{aligned}&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \\ &= u(x,t) + \sum_{k=1}^{M}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t) \times C_{l,k} + O(r^{2M+2}) \text{ as } r \to 0, \\ &\text{where } C_{l,k} = \frac{(-1)^k}{(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\binom{k-1}{l}(2k)^l.\end{aligned}$$

Theorem 1 is the formula (1.3) in one space dimensional case.

Remark 2. *Theorem 1 was proved in [3]. Indeed, by using the binomial theorem, we get*

$$H_k u = \beta_{k,N}\sum_{l=0}^{k-1}\binom{k-1}{l}\left(\frac{2k}{2k+N}\right)^l\left(\frac{N}{2k+N}\right)^{k-1-l}(\partial_t - \Delta_x)^{k-l}(\partial_t)^l u.$$

Therefore we obtain Theorem 1 for general dimensional case. However we do not need to introduce the differential operator H_k (1.4) in one space dimensional case. An assumption of one space dimension is a technical problem due to obtain representations (2.7, 2.8) in Lemma 4 by factorizing $v^{(2k)}(0)$ $(k = 1, 2, \cdots)$ concretely (see §2). It seems to be difficult in higher dimension case.

We finally give mean value properties for the polycaloric equation (see Corollary 10 in §3) and the higher order heat equation (see Proposition 12 in §3). The author hopes that mean value properties are useful for getting qualitative properties of solutions for the polycaloric equation and the higher order heat equation.

2 Proof of theorem 1 In this section, we prove Theorem 1. We set $(x,t)=(0,0)$ to simplify the description. Let $u:\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ be a smooth function. Set $E(r)=E(0,0,r)$ and put

$$\phi(r)=\frac{1}{r^N}\iint_{E(r)}u(x,t)\frac{|x|^2}{t^2}dxdt=\iint_{E(1)}u(ry,r^2s)\frac{|y|^2}{s^2}dyds. \tag{2.1}$$

In the following, we will carry out the Maclaurin expansion of $\phi(r)$ with respect to $r\in\mathbf{R}$. Set $v(r)=u(x,t)=u(ry,r^2s)$ for $(y,s)\in\mathbf{R}^N\times\mathbf{R}$. By differentiating $\phi(r)$ directly, we have

$$\phi^{(n)}(0)=\iint_{E(1)}v^{(n)}(0)\frac{|y|^2}{s^2}dyds. \tag{2.2}$$

We use standard notations of multi-indices; for $y=(y_1,\cdots,y_N)\in\mathbf{R}^N$ and a multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)\in\mathbf{N}_0^N$, we write $y^\alpha=y_1^{\alpha_1}\cdots y_N^{\alpha_N}$ and $|\alpha|=\alpha_1+\cdots+\alpha_N$. Next lemma concerns the evaluation of $v^{(n)}(0)$ and is valid for general dimension $N\in\mathbf{N}$.

Lemma 3 ($v^{(n)}(0)$)**.** *For $k\in\mathbf{N}_0$, we obtain*

$$\phi^{(2k-1)}(0)=0, \tag{2.3}$$

$$v^{(2k)}(0)=\sum_{j=0}^{k}\sum_{|\beta|=k-j}(\partial_x^2)^\beta(\partial_t)^j u(0,0)\times A_{\beta,k}(y,s) \tag{2.4}$$

where

$$A_{\beta,k}(y,s)=\frac{(2k)!}{(2\beta)!j!}y^{2\beta}s^j.$$

Proof. Since $v(r)$ is a C^∞ function of r, for all $M\geq 1$ we have

$$v(r)=\sum_{n=0}^{2M+1}\frac{v^{(n)}(0)}{n!}r^n+O(r^{2M+2})\ \text{ as }\ r\to 0. \tag{2.5}$$

On the other hand, since $v(r)$ is a composed function of $u(x,t)$ and $x=ry, t=r^2s$, we have

$$\begin{aligned} v(r)&=\sum_{m=0}^{2M+1}\frac{1}{m!}\left((ry_1)\frac{\partial}{\partial x_1}+\cdots+(ry_N)\frac{\partial}{\partial x_N}+(r^2s)\frac{\partial}{\partial t}\right)^m u(0,0)+O(r^{2M+2})\\ &=\sum_{m=0}^{2M+1}\frac{1}{m!}\sum_{|\alpha|+j=m}\frac{m!}{\alpha_1!\cdots\alpha_N!j!}(ry)^\alpha(r^2s)^j(\partial_x^\alpha\partial_t^j)u(0,0)+O(r^{2M+2})\\ &=\sum_{m=0}^{2M+1}\sum_{|\alpha|+j=m}\frac{y^\alpha s^j}{\alpha!j!}\ (\partial_x^\alpha\partial_t^j)u(0,0)\times r^{|\alpha|+2j}+O(r^{2M+2}). \end{aligned} \tag{2.6}$$

By comparing the coefficients of r^n in the both expressions of (2.5) and (2.6), we obtain

$$\frac{v^{(n)}(0)}{n!}=\sum_{|\alpha|+2j=n}\frac{y^\alpha s^j}{\alpha!j!}\ (\partial_x^\alpha\partial_t^j)u(0,0).$$

Thus,

$$\phi^{(n)}(0) = \iint_{E(1)} v^{(n)}(0) \frac{|y|^2}{s^2} dyds$$
$$= \sum_{|\alpha|+2j=n} \frac{n!}{\alpha! j!} (\partial_x^\alpha \partial_t^j) u(0,0) \times \iint_{E(1)} y^\alpha s^j \frac{|y|^2}{s^2} dyds\,.$$

Since $E(1)$ is symmetric about y_i-axis($i = 1, \cdots, N$), $\iint_{E(1)} y^\alpha s^j \frac{|y|^2}{s^2} dyds$ vanishes when at least one α_i of $\alpha = (\alpha_1, \cdots, \alpha_N)$ is odd (i.e. when n is odd because $|\alpha| + 2j = n$). This proves (2.3). Next, we consider the case $\alpha = 2\beta$ for some $\beta \in \mathbf{N}_0^N$ and let $n = 2k\,(k \in \mathbf{N})$. Then we obtain

$$v^{(2k)}(0) = \sum_{2|\beta|+2j=2k} (\partial_x^2)^\beta (\partial_t)^j u(0,0) \times \frac{(2k)!}{(2\beta)! j!} y^{2\beta} s^j$$
$$= \sum_{j=0}^{k} \sum_{|\beta|=k-j} (\partial_x^2)^\beta (\partial_t)^j u(0,0) \times \frac{(2k)!}{(2\beta)! j!} y^{2\beta} s^j,$$

which implies (2.4).

□

Lemma 4 (Factorization)**.** *Let $N = 1$. Then*

$$v^{(2k)}(0) = \sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times B_{l,k}(y,s) \tag{2.7}$$

where

$$B_{l,k}(y,s) = (-1)^{k+l} \sum_{m=0}^{l} \begin{pmatrix} k-l+m \\ m \end{pmatrix} \times A_{k-l+m,k}(y,s) \tag{2.8}$$

for $0 \le l \le k$.

Proof. By the assumption $N = 1$ and (2.4), it is enough to prove that

$$\sum_{j=0}^{k} (\partial_x^2)^{k-j} (\partial_t)^j u(0,0) \times A_{k-j,k} = \sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times B_{l,k}. \tag{2.9}$$

We prove (2.9) by comparing the coefficients of $(\partial_x^2)^{k-j} (\partial_t)^j u(0,0)$ in both sides.

Since

$$\sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) B_{l,k}$$
$$= (\partial_t - \partial_x^2)^k u(0,0) B_{0,k} + (\partial_t - \partial_x^2)^{k-1} (\partial_t) u(0,0) B_{1,k} + \cdots + (\partial_t)^k u(0,0) B_{k,k},$$

the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.9) is given by

$$\begin{aligned}&(-1)^{k-j}\Bigg[\binom{k}{k-j}B_{0,k}+\binom{k-1}{k-j}B_{1,k}+\binom{k-2}{k-j}B_{2,k}+\cdots\\&\quad+\binom{k-j+1}{k-j}B_{j-1,k}+\binom{k-j}{k-j}B_{j,k}\Bigg]\\&=(-1)^{k-j}\sum_{l=0}^{j}\binom{k-l}{k-j}B_{l,k}.\end{aligned}$$

Inserting the definition of $B_{l,k}$ in (2.8) into this expression, we assure that the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.9) is given by

$$(-1)^{k-j}\sum_{l=0}^{j}\binom{k-l}{k-j}(-1)^{k+l}\sum_{m=0}^{l}\binom{k-l+m}{m}\times A_{k-l+m,k}. \tag{2.10}$$

Since

$$\begin{aligned}&\sum_{l=0}^{j}\binom{k-l}{k-j}(-1)^{k+l}\sum_{m=0}^{l}\binom{k-l+m}{m}A_{k-l+m,k}\\&=\binom{k}{k-j}(-1)^{k}\binom{k}{0}A_{k,k}\\&+\binom{k-1}{k-j}(-1)^{k+1}\Bigg[\binom{k-1}{0}A_{k-1,k}+\binom{k}{1}A_{k,k}\Bigg]\\&+\binom{k-2}{k-j}(-1)^{k+2}\Bigg[\binom{k-2}{0}A_{k-2,k}+\binom{k-1}{1}A_{k-1,k}+\binom{k}{2}A_{k,k}\Bigg]\\&+\cdots\\&+\binom{k-j}{k-j}(-1)^{k+j}\Bigg[\binom{k-j}{0}A_{k-j,k}+\cdots+\binom{k-1}{j-1}A_{k-1,k}+\binom{k}{j}A_{k,k}\Bigg],\end{aligned}$$

coefficients of $A_{k-i,k}$ for all $0\le i\le j-1$ in (2.10) is given by

$$\begin{aligned}&(-1)^{k-j}(-1)^{k+i}\sum_{n=0}^{j-i}(-1)^n\binom{k-i-n}{k-j}\binom{k-i}{n}\\&=(-1)^{i-j}\sum_{n=0}^{j-i}(-1)^n\binom{k-i}{k-j}\binom{j-i}{n}=0,\end{aligned}$$

where the last equality comes from $\sum_{n=0}^{p}(-1)^n\binom{p}{n}=(-1+1)^p=0$.

Then we prove that

$$\sum_{l=0}^{j}\binom{k-l}{k-j}(-1)^{k+l}\sum_{m=0}^{l}\binom{k-l+m}{m}A_{k-l+m,k}=\binom{k-j}{k-j}(-1)^{k+j}A_{k-j,k}. \tag{2.11}$$

Therefore, by (2.10) and (2.11), the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.9) is $A_{k-j,k}$. We have thus proved Lemma 4. □

From (2.2) and (2.7), we deduce

$$\phi^{(2k)}(0) = \sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times \iint_{E(1)} B_{l,k}(y,s) dyds \tag{2.12}$$

Note that, on the right hand side of (2.12), the heat operator $(\partial_t - \partial_x^2)$ acts on u except for $l = k$.

Lemma 5. *We put*

$$\tilde{C}_{l,k} = \iint_{E(1)} B_{l,k}(y,s) \frac{y^2}{s^2} dyds.$$

Then we get

$$\tilde{C}_{l,k} = \frac{(2k)!(-1)^k 4}{k!(4\pi)^k (2k+1)^{k+\frac{3}{2}}} \begin{pmatrix} k-1 \\ l \end{pmatrix} (2k)^l \tag{2.13}$$

for $0 \le l \le k-1$ *and* $\tilde{C}_{k,k} = 0$.

Proof. We prove Lemma 5 by simple calculations. First, by the definition of $B_{l,k}$ in (2.8)

$$B_{l,k} = (-1)^{k+l} \sum_{m=0}^{l} \begin{pmatrix} k-l+m \\ m \end{pmatrix} \frac{(2k)!}{(2k-2l+2m)!(l-m)!} y^{2k-2l+2m} s^{l-m}$$

for $0 \le l \le k$, we have

$$\tilde{C}_{l,k} = (-1)^{k+l} \sum_{m=0}^{l} \begin{pmatrix} k-l+m \\ m \end{pmatrix} \frac{(2k)!}{(2k-2l+2m)!(l-m)!} \iint_{E(1)} y^{2k-2l+2m+2} s^{l-m-2} dyds.$$

Direct calculation shows that

$$\begin{aligned}
\iint_{E(1)} y^{2k-2l+2m+2} s^{l-m-2} dyds &= \int_{s=-1/4\pi}^{s=0} s^{l-m-2} \int_{|y| \le \sqrt{2s \log(-4\pi s)}} y^{2k-2l+2m+2} dyds \\
&= \frac{2}{(2k-2l+2m+3)} \int_{-1/4\pi}^{0} s^{l-m-2} \left\{ 2s \log(-4\pi s) \right\}^{k-l+m+\frac{3}{2}} ds \\
&= \frac{(-1)^{l-m} 2^{k-l+m+\frac{3}{2}}}{(k-l+m+\frac{3}{2})(4\pi)^{k+\frac{1}{2}}} \int_0^\infty t^{k-l+m+\frac{3}{2}} \exp\left(-\left(k + \frac{1}{2} \right) t \right) dt \\
&= \frac{(-1)^{l-m} 4^{k-l+m} 2^3 \Gamma(k-l+m+\frac{3}{2})}{(4\pi)^k \sqrt{\pi} (2k+1)^{k-l+m+\frac{5}{2}}}
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. Thus, we get

$$\begin{aligned}
\tilde{C}_{l,k} &= \frac{(-1)^k (2k)! 4^{k-l} 8}{(4\pi)^k \sqrt{\pi} (2k+1)^{k-l+\frac{5}{2}} (k-l)!} \sum_{m=0}^{l} \frac{(-1)^m (k-l+m)! 4^m \Gamma(k-l+m+\frac{3}{2})}{m!(2k-2l+2m)!(l-m)!(2k+1)^m}. \\
&= \frac{(-1)^k (2k)! 4}{k!(4\pi)^k (2k+1)^{k-l+\frac{5}{2}}} \begin{pmatrix} k \\ l \end{pmatrix} \sum_{m=0}^{l} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \frac{2k-2l+2m+1}{(2k+1)^m},
\end{aligned}$$

where the last equality comes from the fact $\Gamma(s+1) = s\Gamma(s)$.

Since we have the following equation

$$\begin{aligned}
&(2k+1)^l \sum_{m=0}^{l} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \frac{2k-2l+2m+1}{(2k+1)^m} \\
&= (2k+1) \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-1)^m (2k+1)^{l-m} - 2 \sum_{m=0}^{l-1} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} (l-m)(2k+1)^{l-m} \\
&= (2k+1)(2k)^l - 2l(2k+1) \sum_{m=0}^{l-1} (-1)^m \begin{pmatrix} l-1 \\ m \end{pmatrix} (2k+1)^{l-m-1} \\
&= 2(k-l)(2k)^{l-1}(2k+1)
\end{aligned}$$

Therefore we obtain $\tilde{C}_{k,k} = 0$ and (2.13). □

From all Lemmas, we obtain

$$\phi^{(2k)}(0) = \sum_{l=0}^{k-1} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times \tilde{C}_{l,k} \quad (k = 1, 2, \ldots),$$

which proves Theorem 1.

3 A mean value property for polycaloric functions In this section, first we recall the well-known regularity property of (poly-) caloric functions.

Proposition 6 (caloric function is smooth)**.** *If $u : U_T \to \mathbf{R}$ is caloric, then $u \in C^\infty(U_T)$.*

Proof. See [5]: p.p 59-61 Theorem 8. □

Proposition 7 (polycaloric function is smooth)**.** *If $u : U_T \to \mathbf{R}$ is polycaloric, then $u \in C^\infty(U_T)$.*

Proof. Assume that there exists $m \in \mathbf{N}$ such that $(\partial_t - \Delta_x)^m u = 0$ in U_T. Then we find caloric functions $u_0, u_1, \cdots, u_{m-1} : U_T \to \mathbf{R}$ such that

$$u(x,t) = u_0(x,t) + tu_1(x,t) + \cdots + t^{m-1} u_{m-1}(x,t) \tag{3.1}$$

holds true, by proposition 1 in [8]. Indeed, for $j = 1, 2, \cdots, m$, we may choose

$$u_{m-j}(x,t) = \frac{1}{(m-j)!} \sum_{k=0}^{j-1} \frac{(-t)^k}{k!} (\partial_t - \Delta_x)^{m-j+k} u(x,t).$$

Therefore $u_0, u_1, \cdots, u_{m-1}$ are caloric and satisfy the equation (3.1). By proposition 6 and (3.1), we obtain $u \in C^\infty(U_T)$. □

By proposition 6 and proposition 7, we obtain several corollaries which are proved by F.Da Lio and L.Rodino [3] as follows. We do not need the additional assumption that u is smooth, after assuming that u is caloric or polycaloric.

corollary 8 (A mean value property for analytic functions. [3] Proposition 2.2)**.** *Let $N = 1$ and $u \in C^\infty(U_T)$. Assume that $(\partial_t - \partial_x^2)u(x,t)$ is an analytic function in U_T. Then $\phi(r)$ given in (2.1) is an analytic function of $r \in \mathbf{R}$ in a neighborhood of $r = 0$, and it holds*

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{\infty}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k} \\
&\textit{where } C_{l,k} = \frac{(-1)^k}{(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\left(\begin{array}{c} k-1 \\ l \end{array}\right)(2k)^l.
\end{aligned}$$

Remark 9. *If u is caloric on U_T, then $u \in C^\infty(U_T)$ and $(\partial_t - \partial_x^2)u(x,t)$ is obviously analytic in U_T and for each heat ball $E(x,t;r) \subset U_T$ the following equation holds:*

$$\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds = u(x,t).$$

Corollary 10 can be considered as the generalization of (1.1) to the polycaloric case.

corollary 10 (A mean value property for polycaloric functions)**.** *Let $N = 1$ and $(\partial_t - \partial_x^2)u(x,t)$ be an analytic function in U_T. If u is polycaloric on U_T (i.e.$(\partial_s - \partial_y^2)^m u(y,s) = 0$, $(y,s) \in U_T$, $m \in \mathbf{N}$), then for each heat ball $E(x,t;r) \subset U_T$ the following equality holds:*

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{m-1}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k} \\
&+ \sum_{k=m}^{\infty}\frac{r^{2k}}{k!}\sum_{l=k-m+1}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k}, \\
&\textit{where } C_{l,k} = \frac{(-1)^k}{(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\left(\begin{array}{c} k-1 \\ l \end{array}\right)(2k)^l.
\end{aligned}$$

Proof. This is a direct consequence of Theorem 1 and Proposition 7. □

corollary 11 ([3] Corollary 2.1)**.** *Let $N = 1$. Suppose that there exist $n_1 \geq 0$ and $n_2 \geq 1$ such that*

$$(\partial_t - \partial_x^2)(\partial_t)^{n_1}u = 0 \;\; \textit{and} \;\; (\partial_t - \partial_x^2)^{n_2}u = 0 \;\; \textit{in } U_T.$$

Then for all $r > 0$ we have

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{M}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k},
\end{aligned} \tag{3.2}$$

with $M = n_1 + n_2 - 1$ (when $n_1 = 0$ or $n_2 = 1$ the sum in the right-hand side of (3.2) does not appear).

Proof. Note that we get $u \in C^\infty(U_T)$, since u is polycaloric in U_T. See the proof of corollary 2.1 in [3]. $\square$

We finally give a mean value property for the higher order heat equation $\partial_t u + (-1)^m \Delta^m u = 0$ $(m \in \mathbf{N})$ for general dimension. In the proof, we use proposition 2.2 and a result in the proof of proposition 2.1 in [3].

Proposition 12 (A mean value property for the higher order heat equation)**.** *Let $u \in C^\infty(U_T)$ and $(\partial_t - \Delta_x)u(x,t)$ be an analytic function in U_T. Assume that u is a solution of the higher order heat equation $\partial_t u + (-1)^m \Delta^m u = 0$. Then for each heat ball $E(x,t;r) \subset U_T$ the following equality holds:*

$$\frac{1}{4r^N} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds = u(x,t) + \sum_{k=1}^{\infty} r^{2k} H_k u(x,t), \tag{3.3}$$

where H_k is given by

$$H_k u = \begin{cases} \dfrac{\rho_{k,N}}{k!} \displaystyle\sum_{h=0}^{k} (-1)^{k-h} \binom{k}{h} (N+2h) \left(\frac{N}{2k+N}\right)^h \Delta^{mk+(1-m)h} u, & (m : odd) \\ \dfrac{\rho_{k,N}}{k!} \displaystyle\sum_{h=0}^{k} \binom{k}{h} (N+2h) \left(\frac{N}{2k+N}\right)^h \Delta^{mk+(1-m)h} u, & (m : even) \end{cases}$$

$$\textit{where} \quad \rho_{k,N} = \frac{1}{2k+N} \left(\frac{N}{2k+N}\right)^{\frac{N}{2}+1} \left(\frac{1}{4\pi}\right)^k.$$

Proof. Let $p \in \mathbf{N}$. Note that u satisfies

$$\partial_t^p u = \begin{cases} \Delta^{pm} u, & (m : \text{odd}) \\ (-1)^p \Delta^{pm} u, & (m : \text{even}) \end{cases} \tag{3.4}$$

since u is a smooth solution of the higher order heat equation $\partial_t u + (-1)^m \Delta^m u = 0$. On the other hand, (3.3) holds by proposition 2.2 in [3], and according to a result in [3] (p,268, line 2 and 9), H_k is given by

$$H_k u = \frac{\rho_{k,N}}{k!} \sum_{h=0}^{k} (-1)^{k-h} \binom{k}{h} (N+2h) \left(\frac{N}{2k+N}\right)^h \Delta^h (\partial_t)^{k-h} u. \tag{3.5}$$

Finally, combining (3.4) and (3.5), we get the proposition 12. $\square$

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