On Relative Extreme Amenability

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Abstract. The purpose of this paper is to study the notion of relative extreme amenability for pairs of topological groups. We give a characterization by a fixed point property on universal spaces. In addition we introduce the concepts of an extremely amenable interpolant as well as maximally relatively extremely amenable pairs and give examples. It is shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant. The theory is applied to generalize results of [KPT05] relating to the application of Fraïssé theory to theory of Dynamical Systems. In particular, new conditions enabling to characterize universal minimal spaces of automorphism groups of Fraïssé structures are given.

1 Introduction The goal of this paper is to study the notion of relative extreme amenability: a pair of topological groups \( H \subseteq G \) is called relatively extremely amenable if whenever \( G \) acts continuously on a compact space, there is an \( H \)-fixed point. This notion was isolated by the second author while investigating transfer properties between Fraïssé theory and dynamical systems along the lines of [KPT05], and the corresponding results appears in [NVT13]. We now provide a short description of the contents of the present article and some of the results. Section 2 contains notation. Subsection 3.1 recalls the notion of universal spaces. In subsection 3.2 it is shown that \((G, H)\) is relatively extremely amenable if and only if there exists a universal \( G \)-space with a \( H \)-fixed point. In subsection 3.3 the notion of extremely amenable interpolant is introduced and an example of a non trivial interpolant is given. Subsection 3.4 contains technical lemmas. In subsection 3.5 the notions of maximal relative extreme amenability and maximal extreme amenability are introduced and illustrated. It is also shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant and that \( \text{Aut}(\mathbb{Q},<) \) is maximally extremely amenable in \( S_{\infty} \). Subsections 3.6 and 3.7 deal with applications to a beautiful theory developed in [KPT05] - the application of Fraïssé theory to the theory of Dynamical Systems. In subsection 3.6 the following theorem is shown (see subsection for the definitions of the various terms appearing in the statement):

Theorem 1. Let \( \{<\} \subset L, L_0 = L \setminus \{<\} \) be signatures, \( K_0 \) a Fraïssé class in \( L_0 \), \( K \) an order Fraïssé expansion of \( K \) in \( L \), \( F_0 = \text{Flim}(K_0) \), \( F = \text{Flim}(K) \). Let \( G_0 = \text{Aut}(F_0) \) and \( G = \text{Aut}(F) \). Denote \( \leq^F = \leq_0 \) and \( X_K = G_0 \leq_0 \). \((G_0, G)\) is relatively extremely amenable and \( \text{Fix}_{X_K}(G) \) is transitive w.r.t \( X_K \) if and only if \( X_K \) is the universal minimal space of \( G_0 \).

In subsection 3.7 the weak ordering property is introduced and it is proven that if \((G_0, G)\) is relatively extremely amenable then the weak ordering property implies the ordering property. Finally in subsection 3.8 a question is formulated.

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2 Preliminaries

We denote by \((G, X)\) a topological dynamical system (t.d.s), where \(G\) is a (Hausdorff) topological group and \(X\) is a compact (Hausdorff) topological space. We may also refer to \(X\) as a \(G\)-space. If it is desired to distinguish a specific point \(x_0 \in X\), we write \((G, X, x_0)\). Given a continuous action \((G, X)\) and \(x \in X\), denote by \(\text{Stab}(x) = \{g \in G \mid gx = x\} \subset G\), the subgroup of elements of \(G\) fixing \(x\), and for \(H \subset G\) denote by \(\text{Fix}_H(X) = \{x \in X \mid \forall h \in H \: hx = x\} \subset X\), the set of elements of \(X\), fixed by \(H\). Note that \(\text{Fix}_X(H)\) is a closed set. Given a linear order \(\prec\) on a set \(D\), we denote by \(\prec^*\) the linear ordering defined on \(D\) by \(a \prec^* b \iff b \prec a\) for all \(a, b \in D\).

3 Results

3.1 Universal spaces. Let \(G\) be a topological group. The topological dynamical system (t.d.s) \((G, X)\) is said to be minimal if \(X\) and \(\emptyset\) are the only \(G\)-invariant closed subsets of \(X\). By Zorn’s lemma each \(G\)-space contains a minimal \(G\)-subspace. \((G, X)\) is said to be universal if any minimal \(G\)-space \(Y\) is a \(G\)-factor of \(X\). One can show there exists a minimal and universal \(G\)-space \(U_G\) unique up to isomorphism. \((G, U_G)\) is called the universal minimal space of \(G\) (for existence and uniqueness see for example [Usp02], or the more recent [GL13]). \((G, X, x_0)\) is said to be transitive if \(G x_0 = X\). One can show there exists a transitive t.d.s \((G, A_G, a_0)\), unique up to isomorphism, such that for any transitive t.d.s \((G, Y, y_0)\), there exists a \(G\)-equivariant mapping \(\phi_Y : (G, A_G, a_0) \to (G, Y, y_0)\) such that \(\phi(a_0) = y_0\). \((G, A_G, a_0)\) is called the greatest ambit. Because any minimal subspace of \(A_G\) is isomorphic to the universal minimal space, \(A_G\) is universal. Note that if \(A_G\) is not minimal (e.g., this is the case if \(A_G\) is not distal see [dV93] IV(4.35)), then it is an example of a non-minimal universal space.

3.2 A Characterization of Relative Extreme Amenability

Recall the following classical definition (originating in [Mit66]):

**Definition 3.2.1.** Let \(G\) be a topological group. \(G\) is called extremely amenable if any t.d.s \((G, X)\) has a \(G\)-fixed point, i.e. there exists \(x_0 \in X\), such that for every \(g \in G\), \(gx_0 = x_0\).

It is easy to see that for \(G\) to be extremely amenable is equivalent to \(U_G = \{\ast\}\). Here is a generalization of the previous definition which appears in [NVT13]:

**Definition 3.2.2.** Let \(G\) be a topological group and \(H \subset G\), a subgroup. The pair \((G, H)\) is called relatively extremely amenable if any t.d.s \((G, X)\) has a \(H\)-fixed point, i.e. there exists \(x_0 \in X\), such that for every \(h \in H\), \(hx_0 = x_0\).

**Proposition 3.2.3.** Let \(G\) be a topological group and \(H \subset G\), a subgroup. The following conditions are equivalent:

1. The pair \((G, H)\) is relatively extremely amenable.
2. \(U_G\) has a \(H\)-fixed point.
3. There exists a universal \(G\)-space \(T_G\) and \(t_0 \in T_G\) which is \(H\)-fixed.
Proof. (1)⇒(2). If \((G, H)\) is relatively extremely amenable, then by definition \((G, U_G)\) has a \(H\)-fixed point.

(2)⇒(3). Trivial.

(3)⇒(1). Let \(X\) be a minimal \(G\)-space. By universality of \(T_G\), there exists a surjective 
\(G\)-equivariant mapping \(\phi : (G, T_G) \to (G, X)\). Denote \(x = \phi(t_0)\). Clearly for every \(h \in H\), 
\[hx = h\phi(t_0) = \phi(ht_0) = \phi(t_0) = x\]
It is well-known that a non-compact locally compact group cannot be extremely amenable. Here is a strengthening of this fact:

**Proposition 3.2.4.** Let \(G\) be a non-compact locally compact group and \(\{e\} \subseteq H \subset G\), a 
subgroup. The pair \((G, H)\) is not relatively extremely amenable.

**Proof.** By Veech’s Theorem ([Vee77]) \(G\) acts freely on \(U_G\). Now use Proposition 3.2.3(2).

### 3.3 Extremely Amenable Interpolants

**Definition 3.3.1.** Let \(G\) be a topological group and \(H \subset G\), a subgroup. An extremely 
amenable group \(E\) is called an **extremely amenable interpolant** for the pair \((G, H)\) if
\(H \subset E \subset G\).

The following lemma is trivial:

**Lemma 3.3.2.** Let \(G\) be a topological group and \(H \subset G\), a subgroup. If there exists 
an extremely amenable interpolant for the pair \((G, H)\), then \((G, H)\) is relatively extremely 
amenable.

Here is an example of a non trivial extremely amenable interpolant \(E\) for a pair \((G, H)\), in 
the sense that neither \(E = G\), nor \(E = H\):

**Example 3.3.3.** Let \(Q\) be the Hilbert cube. Recall that by a result of Uspenskij (Theorem 
9.18 of [Kec95]), \(\text{Homeo}(Q)\), equipped with the compact-open topology, is a universal Polish 
group, in the sense that any Polish group embeds inside it through a homomorphism. Let 
\(\text{Homeo}_+(I)\) be the group of increasing homeomorphisms of the interval \(I\), equipped 
with the compact-open topology. By a result of Pestov (see [Pes98]) \(\text{Homeo}_+(I)\) is extremely 
amenable. Let \(\phi : \text{Homeo}_+(I) \hookrightarrow \text{Homeo}(Q)\) be an embedding through a homomorphism.

Let \(f : I \to I\) given by \(f(x) = x^2\). Notice \(f \in \text{Homeo}_+(I)\). Denote \(G = \text{Homeo}(Q)\), 
\(E = \phi(\text{Homeo}_+(I))\) and \(H = \phi(\{f^n \mid n \in \mathbb{Z}\})\). Notice \(H \subset E \subset G\). \(E\) is clearly an 
extremely amenable interpolant for \((G, H)\), but \(G\) (which acts homogeneously on \(Q\)) and \(H\) 
(which is isomorphic to \(\mathbb{Z}\)) are not extremely amenable.

A natural question is if any relatively extremely amenable pair has an extremely amenable 
interpolant. Theorem 3.5.8 in the next subsection answers the question in the negative.

### 3.4 Order fixing groups

Let \(S_{\infty}\) be the permutation group of the integers \(\mathbb{Z}\), equipped 
with the pointwise convergence topology. Let \(F\) be an infinite countable set and fix a 
bijection \(F \approx \mathbb{Z}\). Let \(\text{LO}(F) \subset \{0, 1\}^{F \times F}\), be the space of linear orderings on \(F\), 
equipped with the pointwise convergence topology. Under the above mentioned bijection \(\text{LO}(F)\) 
becomes an \(S_{\infty}\)-space. By Theorem 8.1 of [KPT05] \(U_{S_{\infty}} = \text{LO}(F)\). Notice that we consider 
\(F\) as a set and not a topological space. In this subsection we will use \(F = \mathbb{Z}\) and \(F = \mathbb{Q}\), 
considered as infinitely countable sets with convenient enumerations (bijections) and the 
corresponding dynamical systems \((S_{\infty}, \text{LO}(\mathbb{Z}))\) and \((S_{\infty}, \text{LO}(\mathbb{Q}))\).

**Lemma 3.4.1.** Let \(<\subseteq \text{LO}(\mathbb{Z})\) be the usual linear order on \(\mathbb{Z}\), i.e. the order for which 
\(n < n + 1\) for every \(n \in \mathbb{Z}\). Then
1. \( \text{Stab}_2(\leq) = \{ T_a | a \in \mathbb{Z} \} \), where \( T_a : \mathbb{Z} \to \mathbb{Z} \) is given by \( T_a(x) = x + a \).

2. \( \text{Fix}_{LO(\mathbb{Z})}(\text{Stab}_2(\leq)) = \{ \leq, \leq^* \} \).

**Proof.** (1) Let \( T \in \text{Stab}(\leq) \). Denote \( a = T(0) \). Notice that for all \( x > 1 \), \( T(x) > T(1) = a \), and for all \( x < 0 \), \( T(x) < a \). As \( T \) is onto we must have \( T(1) = a + 1 \). Similarly for all \( x \in \mathbb{Z}, T(x) = x + a \), which implies \( T = T_a \).

(2) Let \( \prec \in \text{Fix}_{LO(\mathbb{Z})}(\text{Stab}(\leq)) \). We claim that \( \leq \preceq \) or \( \leq \leq^* \). Indeed \( 0 < 1 \) or \( 1 < 0 \).

In the first case applying \( T_a \in \text{Stab}(\leq) \), we have for all \( a \in \mathbb{Z}, a = a + 1 \). This implies \( \leq \preceq \). Similarly in the second case for all \( a \in \mathbb{Z}, a + 1 < a \) which implies \( \leq \leq^* \). □

Let \( \prec \in LO(\mathbb{Q}) \) be the usual order on \( \mathbb{Q} \). In the following lemma, we follow the standard convention and write \( Aut(\mathbb{Q}, \prec) \) instead of \( \text{Stab}_{S_\prec}(\prec) \subset S_\infty \).

**Lemma 3.4.2.** Let \( \prec \in LO(\mathbb{Q}) \) be the usual linear order on \( \mathbb{Q} \), then

\[
\text{Fix}_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, \prec)) = \{ \prec, \prec^* \}.
\]

**Proof.** Let \( \prec \in \text{Fix}_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, \prec)) \). Note that \( 0 < 1 \) or \( 1 < 0 \). In the first case, let \( q_1, q_2 \in \mathbb{Q} \) with \( q_1 < q_2 \) and define \( T : \mathbb{Q} \to \mathbb{Q} \) with \( T x = (q_2 - q_1)x + q_1 \). Note that \( T \in Aut(\mathbb{Q}, \prec) \). Hence, \( q_1 = T(0) < T(1) = q_2 \). As the argument works for any \( q_1' < q_2 \), we have \( \prec \leq \prec \). The second case is similar and implies \( \leq \leq^* \). □

### 3.5 Maximally Relatively Extremely Amenable Pairs

**Proposition 3.5.1.** Let \( G \) be a topological group, then there exists a subgroup \( H \subset G \), such that \( (G, H) \) is relatively extremely amenable and there exists no subgroup \( H \subset H' \subset G \), such that \( (G, H') \) is relatively extremely amenable.

**Proof.** By Zorn’s lemma it is enough to show that any chain w.r.t. inclusion \( \{ G_\alpha \}_{\alpha \in A} \) such that \( (G, G_\alpha) \) is relatively extremely amenable, has a maximal element. Note that if \( G_\alpha \subset G_{\alpha'} \), then \( \text{Fix}_{U_G}(G_{\alpha'}) \subset \text{Fix}_{U_G}(G_{\alpha}) \). In particular for any finite collection \( \alpha_1, \alpha_2, \ldots, \alpha_n \in A \), we have \( \bigcap_{\alpha=1}^{n} \text{Fix}_{U_G}(G_{\alpha}) \neq \emptyset \), which implies by a standard compactness argument \( \bigcap_{\alpha \in A} \text{Fix}_{U_G}(G_{\alpha}) \neq \emptyset \). This in turn implies that \( \text{Fix}_{U_G}(\bigcup_{\alpha \in A} G_{\alpha}) \neq \emptyset \), which finally implies \( (G, \bigcup_{\alpha \in A} G_{\alpha}) \) is relatively extremely amenable by Proposition 3.2.3(2). □

**Definition 3.5.2.** A pair \( (G, H) \) as in Proposition 3.5.1 is called maximally relatively extremely amenable.

Similarly to the previous theorem and definition we have:

**Proposition 3.5.3.** Let \( G \) be a topological group, then there exists a subgroup \( H \subset G \), such that \( H \) is extremely amenable and there exists no subgroup \( H \subset H' \subset G \), such that \( H' \) is extremely amenable.

**Proof.** By Zorn’s lemma it is enough to show that any chain w.r.t. inclusion \( \{ G_\alpha \}_{\alpha \in A} \) such that \( G_\alpha \subset G \) and \( G_\alpha \) is extremely amenable, has a maximal element. Let \( (\bigcup_{\alpha \in A} G_{\alpha}, X) \) be a dynamical system. By assumption for any \( \alpha \in A \), \( \text{Fix}_X(G_{\alpha}) \neq \emptyset \). In addition if \( G_\alpha \subset G_{\alpha'} \), then \( \text{Fix}_X(G_{\alpha'}) \subset \text{Fix}_X(G_{\alpha}) \). We now continue as in the proof of Theorem 3.5.1 to conclude \( \bigcup_{\alpha \in A} G_{\alpha} \) is extremely amenable. □

**Definition 3.5.4.** A subgroup \( H \subset G \) as in Proposition 3.5.3 is called maximally extremely amenable in \( G \).
Remark 3.5.5. It was pointed out in [Pes02] that if \( H \) is second countable (Hausdorff) group then there always exists an extremely amenable group \( G \) such that \( H \subset G \). Indeed by [Usp90] \( H \subset \text{Iso}(\mathbb{U}) \) the group of isometries of Urysohn’s universal complete separable metric space \( \mathbb{U} \), equipped with the compact-open topology, and by [Pes02], \( \text{Iso}(\mathbb{U}) \) is extremely amenable.

Theorem 3.5.6. Let \( G = S_\infty \) be the permutation group of the integers, equipped with the pointwise convergence topology. Let \( < \) be the usual order on \( \mathbb{Z} \) and \( H = \text{Stab}_\mathbb{Z}(<) \subset G \). The pair \( (G, H) \) is maximally relatively extremely amenable.

Proof. By Theorem 8.1 of [KPT05] \( U_G = \text{LO}(\mathbb{Z}) \), the space of linear orderings on \( \mathbb{Z} \). By Proposition 3.2.3(2) \( (G, H) \) is relatively extremely amenable. Assume that there exists a subgroup \( E \), with \( H \subset E \subset G \) such that \( (G, E) \) is a relatively extremely amenable. Evoking again Proposition 3.2.3(2), there exists \( < \in U_G \) so that \( E \subset \text{Stab}(<) \). As \( H \subset E \subset \text{Stab}(<) \), conclude by Lemma 3.4.1(2) that \( < \in \{ <, <^* \} \). As \( H = \text{Stab}(<) = \text{Stab}(<^*), we conclude in both cases \( E = H \).

Lemma 3.5.7. If \( (G, H) \) is maximally relatively extremely amenable and neither \( G \) nor \( H \) are extremely amenable, then \( (G, H) \) does not admit an extremely amenable interpolant.

Proof. Assume for a contradiction that there exists an extremely amenable subgroup \( E \), with \( H \subset E \subset G \). Notice that \( (G, E) \) is relatively extremely amenable which constitutes a contradiction with the fact that \( (G, H) \) is maximally relatively extremely amenable.

Theorem 3.5.8. There exists a relatively extremely amenable pair \( (G, H) \) which does not admit an extremely amenable interpolant.

Proof. Let \( G = S_\infty \) be the permutation group of the integers, equipped with the pointwise convergence topology. Let \( < \) be the usual order on \( \mathbb{Z} \) and \( H = \text{Stab}(<) \subset G \). By Theorem 3.5.6 \( (G, H) \) is maximally relatively extremely amenable. Clearly \( G \) is not extremely amenable as \( U_G \neq \{ * \} \). By Lemma 3.4.1(1) \( H = \{ T_a | a \in \mathbb{Z} \} \cong \mathbb{Z} \), where the second equivalence is as topological groups. This implies \( H \) is not extremely amenable. Now invoke Lemma 3.5.7.

Theorem 3.5.9. \( \text{Aut}(\mathbb{Q}, <) \) is maximally extremely amenable in \( S_\infty \).

Proof. By [Pes98] \( \text{Aut}(\mathbb{Q}, <) \) is extremely amenable. Now we can proceed as in the proof of Theorem 3.5.8 using Lemma 3.4.2.

Remark 3.5.10. Even though the previous result never appeared in print, Todor Tsankov pointed out that it can be derived from an earlier result by Cameron. Indeed, the article [Cam76] allows a complete description of the closed subgroups of \( S_\infty \) containing \( \text{Aut}(\mathbb{Q}) \) (essentially, there are only five of them, see [BP11] for an explicit description) and it can be verified that among those, only \( \text{Aut}(\mathbb{Q}) \) is extremely amenable.

3.6 Applications in Fraïssé Theory The following two sections deal with applications to Fraïssé Theory. Two general references for this theory are [Fra00] and [Hod93]. We follow the exposition and notation of [KPT05].

Let \( \{ < \} \subset L, L_0 = L \setminus \{ < \} \) be signatures, \( K_0 \) a Fraïssé class in \( L_0 \), \( K \) an order Fraïssé expansion of \( K_0 \) in \( L \), \( F = \text{Flim}(K) \) the Fraïssé limit of \( K \). By Theorem 5.2(ii) \( \Rightarrow \) (i) of [KPT05], if we forget \( F_0 = \text{Flim}(K_0) \) then \( F_0 = F \upharpoonright L_0 \). Let \( G_0 = \text{Aut}(F_0) \) and \( G = \text{Aut}(F) \). Denote \( <_F \equiv <_0 \), i.e. \( <_0 \) is the linear order corresponding to the symbol \( < \) in \( F \), and let \( X_K = \mathcal{G}_0 <_0 \) (\( X_K \) is called set of \( K \)-admissible linear orderings of \( F \) in [KPT05]). In [KPT05], two combinatorial properties for \( K \) have considerable importance in order to compute universal minimal spaces. Those are called ordering property and Ramsey property:
Definition 3.6.1. Let \( \{<\} \subset L \) be a signature, \( L_0 = L \setminus \{<\} \), \( K_0 \) a \( \mathcal{F} \)-Fraïssé class in \( L_0 \), \( K \) an order \( \mathcal{F} \)-Fraïssé expansion of \( K_0 \) in \( L \), \( F = \text{Flim}(K) \) the \( \mathcal{F} \)-Fraïssé limit of \( K \). We say that \( K \) satisfies the \textbf{ordering property} (relative to \( K_0 \)) if for every \( A_0 \in K_0 \), there is \( B_0 \in K_0 \), such that for every linear ordering \( < \) on \( A_0 \) and linear ordering \( <' \) on \( B_0 \), if \( A = \langle A_0, < \rangle \in K \) and \( B = \langle B_0, <' \rangle \in K \), then there is an embedding \( A \hookrightarrow B \).

Definition 3.6.2. Let \( \{<\} \subset L \) be a signature and \( K \) an order \( \mathcal{F} \)-Fraïssé class in \( L \). We say that \( K \) satisfies the \textbf{Ramsey property} if, for every positive \( k \in \mathbb{N} \), every \( A \in K \) and every \( B \in K \), there exists \( C \in K \) such that for every \( k \)-coloring of the substructures of \( C \) which are isomorphic to \( A \), there is a substructure \( \hat{B} \) of \( C \) which is isomorphic to \( B \) and such that all substructures of \( \hat{B} \) which are isomorphic to \( A \) receive the same color.

Those two properties are relevant because they capture dynamical properties of \( X_K \). For example, Theorem 7.4 of [KPT05] states that the minimality of \( X_K \) is equivalent to \( K \) having the ordering property, and Theorem 10.8 of [KPT05] states that \( X_K \) being universal and minimal is equivalent to \( K \) having the ordering and Ramsey properties. Those results naturally led the authors of [KPT05] to ask whether \( X_K \) being extremely amenable (see Section 3.8 for more about this aspect).

Remark 3.6.3. The reason for which only order expansions (i.e. \( \{<\} \subset L, L_0 = L \setminus \{<\} \), and \( < \) is interpreted as a linear order) were considered in [KPT05] is that, at the time where the article was written, expanding the signature by such a symbol was sufficient in order to obtain Ramsey property and ordering property in all known practical cases. However, we know now that there are some cases where expanding the signature by such a symbol was sufficient in order to obtain Ramsey property and ordering property in all known practical cases. However, we know now that there are some cases where expanding the signature by such a symbol was sufficient in order to obtain Ramsey property and ordering property in all known practical cases.

3.7 The weak ordering property. Theorem 10.8 of [KPT05] states that \( K \) has the ordering and Ramsey properties if and only if \( X_K \) is the universal minimal space of \( G_0 \). The purpose of this section is to show that the combinatorial assumptions made on \( K \) can actually be slightly weakened. We start with a generalization of the notion of transitivity mentioned in subsection 3.1.

Definition 3.7.1. Let \( G \) be a topological group and \( X \) a \( G \)-space. \( Y \subset X \) is said to be transitive \textit{w.r.t} \( X \) if and only if for any \( y \in Y \), \( \overline{Gy} = X \).

Proposition 3.7.2. Let \( G_0 \) be a topological group and let \( T_{G_0} \) be \( G_0 \)-universal. Let \( x \in T_{G_0} \) and let \( G = \text{Stab}_{G_0}(x) \subset G_0 \). \( T_{G_0} \) is minimal if and only if \( \text{Fix}_{T_{G_0}}(G) \) is transitive \textit{w.r.t} \( T_{G_0} \).

Proof. If \( T_{G_0} \) is minimal then \( T_{G_0} \) is transitive \textit{w.r.t} itself and trivially \( \text{Fix}_{T_{G_0}}(G) \subset T_{G_0} \) is transitive \textit{w.r.t} \( T_{G_0} \). To prove the inverse direction, let \( M \subset T_{G_0} \) be a \( G_0 \)-minimal space. By Proposition 3.2.3(3), \( (G_0, G) \) is relatively extremely amenable and therefore there exists \( t_0 \in M \cap \text{Fix}_{T_{G_0}}(G) \). As \( \text{Fix}_{T_{G_0}}(G) \) is transitive \textit{w.r.t} \( T_{G_0} \), conclude \( T_{G_0} = G_0t_0 \subset M \), so \( T_{G_0} = M \) is minimal. \( \square \)
The previous proposition enables us to prove the following equivalence:

**Theorem 3.7.3.** $(G_0, G)$ is relatively extremely amenable and $\text{Fix}_{X_K}(G)$ is transitive w.r.t $X_K$ if and only if $X_K$ is the universal minimal space of $G_0$.

**Proof.** As indicated previously, the universality of $X_K$ is equivalent to the fact that $(G_0, G)$ is relatively extremely amenable. By Proposition 3.7.2, given that $X_K$ is universal, the minimality of $X_K$ is equivalent to the fact that $\text{Fix}_{X_K}(G)$ is transitive w.r.t $X_K$.

**Remark 3.7.4.** By Theorem 3.2.3(3) $(S_\infty, \text{Aut}(\mathbb{Q}, <))$ is relatively extremely amenable. By Lemma 3.4.2 $\text{Fix}_{\text{LO}(\mathbb{Q})}(\text{Aut}(\mathbb{Q}, <)) = \{<, <^*\}$. As $\text{LO}(\mathbb{Q}) = S_\infty < = S_\infty <^*$, we have that $\text{Fix}_{\text{LO}(\mathbb{Q})}(\text{Aut}(\mathbb{Q}, <))$ is transitive w.r.t $\text{LO}(\mathbb{Q})$. By Theorem 3.7.3, it follows that $\text{Aut}(\mathbb{Q}, <)$ is extremely amenable. It should be noted that in [KPT05], one obtains the same results but in reverse order: one concludes $\text{LO}(\mathbb{Q})$ is the universal minimal space of $G$, using the fact that $G_0$ is extremely amenable.

We are now going to show how to reformulate Theorem 3.7.3 in terms of combinatorics.

**Definition 3.7.5.** Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, $K_0$ a Fraïssé class in $L_0$, $K$ an order Fraïssé expansion of $K_0$ in $L$. We say that $(K_0, K)$ has the **relative Ramsey property** if for every positive $k \in \mathbb{N}$, every $A_0 \in K_0$ and every $B \in K$, there exists $C \in K_0$ such that for every $k$-coloring of the substructures of $C_0$ isomorphic to $A_0$, there is an embedding $\phi : B[L_0 \leftarrow C_0]$ such that for any two substructures $A, A'$ of $B_0$ isomorphic to $A_0$, $\phi(A)$ and $\phi(A')$ receive the same color whenever $A$ and $A'$ support isomorphic structures in $B$.

In what follows, the relative Ramsey property will appear naturally because of the following fact (see [NVT13]):

**Claim 3.7.6.** $(G_0, G)$ is relatively extremely amenable iff $(K_0, K)$ has the relative Ramsey property.

We will also need the following variant of the notion of ordering property:

**Definition 3.7.7.** Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, $K_0$ a Fraïssé class in $L_0$, $K$ an order Fraïssé expansion of $K_0$ in $L$. We say that $K$ satisfies the **weak ordering property** relative to $K_0$ if for every $A_0 \in K_0$, there is $B_0 \in K_0$, such that for every linear ordering $<$ on $A_0$ with $A = \langle A, < \rangle \in K$ and linear ordering $<^* \in \text{Fix}_{X_K}(G)$ we have $A \leftrightarrow \langle B_0, <^* \mid B_0 \rangle$.

The following claim appears in the proof of Theorem 7.4 of [KPT05]:

**Claim 3.7.8.** Let $< \in L$ be a linear ordering on $F_0$. Then $< \in F_0$ if and only if for every $A \in K$ there is a finite substructure $C_0$ of $F_0$ such that $C = \langle C_0, < \mid C_0 \rangle \simeq A$.

**Proposition 3.7.9.** Assume $K$ satisfies the weak ordering property relative to $K_0$, and that $(K_0, K)$ has the relative Ramsey property. Then $K$ satisfies the ordering property.

**Proof.** Again, the universality of $X_K$ is equivalent to the fact that $(G_0, G)$ is relatively extremely amenable, which is in turn equivalent to $(K_0, K)$ having the relative Ramsey property. By Theorem 7.4 of [KPT05] the minimality of $X_K$ is equivalent to the ordering property of $K$ (relative to $K_0$). By Proposition 3.7.2 in order to establish $X_K$ is minimal, it is enough to show that $\text{Fix}_{X_K}(G)$ is transitive w.r.t $X_K$. Let $< \in \text{Fix}_{X_K}(G)$. It is enough
to show $<_0 \in \mathcal{G} \smallsetminus \mathcal{G}$. Fix $A \in K$. As $K$ satisfies the weak ordering property, there is $B_0$ as in Definition 3.7.7 such that $A \rightarrow \langle B_0, < \rangle B_0$. Using the same argument as in the proof of Theorem 7.4 of [KPT05], we notice that there is a substructure $C$ of $B$ isomorphic to $A$. Denote $C_0 = C|L_0$ and notice $C = (C_0, < | C_0) \cong A$. We now use Claim 3.7.8.

**Theorem 3.7.10.** $K$ has the weak ordering property and $(K_0, K)$ has the relative Ramsey property if and only if $X_K$ is the universal minimal space of $G_0$.

**Proof.** By Theorem 10.8 of [KPT05], if $X_K$ is the universal minimal space of $G_0$ then $K$ satisfies the ordering property, a fortiori, $K$ satisfies the weak ordering property. In addition $K$ satisfies the Ramsey property which implies $(K_0, K)$ has the relative Ramsey property. The reverse direction follows from Proposition 3.7.9.

### 3.8 A question.
We mentioned previously that the concept of relative extreme amenability was introduced in order to know whether $X_K$ being universal is equivalent to $K$ having the Ramsey property. By Theorem 4.7 of [KPT05], the Ramsey property of $K$ is equivalent to $G$ being extremely amenable. We still do not know the answer to the following question from [KPT05]:

**Question 3.8.1.** Let $\langle \rangle \subset L$ be a signature, $L_0 = L \setminus \langle \rangle$, $K_0$ a Fraïssé class in $L_0$, $K$ an order Fraïssé expansion of $K_0$ in $L$. Does universality for $X_K$ imply that $G$ is extremely amenable (equivalently, that $K$ has the Ramsey property)?

Moreover, in view of the notions we introduced previously, we ask:

**Question 3.8.2.** Assume the previous question has a negative answer. Does there exist an extremely amenable interpolant for the pair $(G_0, G)$?

As a final comment, and in view of Remark 3.6.3, it should be mentioned that Question 3.8.1 has a negative answer when $K$ is not an order expansion of $K_0$, see [NVT13].

**References**


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