On Relative Extreme Amenability

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Abstract. The purpose of this paper is to study the notion of *relative extreme amenability* for pairs of topological groups. We give a characterization by a fixed point property on universal spaces. In addition we introduce the concepts of an *extremely amenable interpolant* as well as *maximally relatively extremely amenable* pairs and give examples. It is shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant. The theory is applied to generalize results of [KPT05] relating to the application of Fraïssé theory to theory of Dynamical Systems. In particular, new conditions enabling to characterize universal minimal spaces of automorphism groups of Fraïssé structures are given.

1 Introduction The goal of this paper is to study the notion of *relative extreme amenability*: a pair of topological groups $H \subset G$ is called relatively extremely amenable if whenever G acts continuously on a compact space, there is an H-fixed point. This notion was isolated by the second author while investigating transfer properties between Fraïssé theory and dynamical systems along the lines of [KPT05], and the corresponding results appears in [NVT13]. We now provide a short description of the contents of the present article and some of the results. Section 2 contains notation. Subsection 3.1 recalls the notion of *universal spaces.* In subsection *3.2* it is shown that (G, H) is relatively extremely amenable if and only if there exists a universal G-space with a H-fixed point. In subsection 3.3 the notion of *extremely amenable interpolant* is introduced and an example of a non trivial interpolant is given. Subsection 3.4 contains technical lemmas. In subsection *3.5* the notions of *maximal relative extreme amenability* and *maximal extreme amenability* are introduced and illustrated. It is also shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant and that $Aut(\mathbb{Q}, <)$ is maximally extremely amenable in S_∞. Subsections 3.6 and 3.7 deal with applications to a beautiful theory developed in [KPT05] - the application of Fraïssé theory to the theory of Dynamical Systems. In subsection 3.6 the following theorem is shown (see subsection for the definitions of the various terms appearing in the statement):

Theorem 1. *Let* $\{<\} \subset L$,$L_0 = L \setminus \{<\}$ *be signatures,* K_0 *a Fraïssé class in* L_0, K *an order Fraïssé expansion of* K *in* L, $F_0 = Flim(K_0)$, $F = Flim(K)$. *Let* $G_0 = Aut(F_0)$ *and* $G = Aut(F)$. *Denote* $<^F = <_0$ *and* $X_K = \overline{G_0 <_0}$. (G_0, G) *is relatively extremely amenable and* $Fix_{X_K}(G)$ *is transitive w.r.t* X_K if and only if X_K is the universal minimal space of G_0.

In subsection 3.7 the *weak ordering property* is introduced and it is proven that if (G_0, G) is relatively extremely amenable then the weak ordering property implies the ordering property. Finally in subsection 3.8 a question is formulated.

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2 Preliminaries We denote by (G, X) a topological dynamical system (t.d.s), where G is a (Hausdorff) topological group and X is a compact (Hausdorff) topological space. We may also refer to X as a G-space. If it is desired to distinguish a specific point $x_0 \in X$, we write (G, X, x_0). Given a continuous action (G, X) and $x \in X$, denote by $Stab_G(x) = Stab(x) = \{g \in G \,|\, gx = x\} \subset G$, the subgroup of elements of G fixing x, and for $H \subset G$ denote by $Fix_X(H) = Fix(H) = \{x \in X \,|\, \forall h \in H\, hx = x\} \subset X$, the set of elements of X, fixed by H. Note that $Fix_X(H)$ is a closed set. Given a linear order $<$ on a set D, we denote by $<^*$ the linear ordering defined on D by $a <^* b \Leftrightarrow b < a$ for all $a, b \in D$.

3 Results

3.1 Universal spaces. Let G be a topological group. The topological dynamical system (t.d.s.) (G, X) is said to be **minimal** if X and $\emptyset$ are the only G-invariant closed subsets of X. By Zorn's lemma each G-space contains a minimal G-subspace. (G, X) is said to be **universal** if any minimal G-space Y is a G-factor of X. One can show there exists a minimal and universal G-space U_G unique up to isomorphism. (G, U_G) is called the **universal minimal space of** G (for existence and uniqueness see for example [Usp02] , or the more recent [GL13]). (G, X, x_0) is said to be **transitive** if $\overline{Gx_0} = X$. One can show there exists a transitive t.d.s (G, A_G, a_0), unique up to isomorphism, such that for any transitive t.d.s (G, Y, y_0), there exists a G-equivariant mapping $\phi_Y : (G, A_G, a_0) \to (G, Y, y_0)$ such that $\phi(a_0) = y_0$. (G, A_G, a_0) is called the **greatest ambit**. Because any minimal subspace of A_G is isomorphic to the universal minimal space, A_G is universal. Note that if A_G is not minimal (e.g., this is the case if A_G is not distal see [dV93] IV(4.35)), then it is an example of a *non-minimal* universal space.

3.2 A Characterization of Relative Extreme Amenability Recall the following classical definition (originating in [Mit66]):

Definition 3.2.1. *Let G be a topological group. G is called* ***extremely amenable*** *if any t.d.s (G, X) has a G-fixed point, i.e. there exists $x_0 \in X$, such that for every $g \in G$, $gx_0 = x_0$.*

It is easy to see that for G to be extremely amenable is equivalent to $U_G = \{*\}$. Here is a generalization of the previous definition which appears in [NVT13]:

Definition 3.2.2. *Let G be a topological group and $H \subset G$, a subgroup. The pair (G, H) is called* ***relatively extremely amenable*** *if any t.d.s (G, X) has a H-fixed point, i.e. there exists $x_0 \in X$, such that for every $h \in H$, $hx_0 = x_0$.*

Proposition 3.2.3. *Let G be a topological group and $H \subset G$, a subgroup. The following conditions are equivalent:*

1. The pair (G, H) is relatively extremely amenable.
2. U_G has a H-fixed point.
3. There exists a universal G-space T_G and $t_0 \in T_G$ which is H-fixed.

Proof. (1)⇒(2). If (G, H) is relatively extremely amenable, then by definition (G, U_G) has a H-fixed point.

(2)⇒(3). Trivial.

(3)⇒(1). Let X be a minimal G-space. By universality of T_G, there exists a surjective G-equivariant mapping $\phi : (G, T_G) \to (G, X)$. Denote $x = \phi(t_0)$. Clearly for every $h \in H$, $hx = h\phi(t_0) = \phi(ht_0) = \phi(t_0) = x$ □

It is well-known that a non-compact locally compact group cannot be extremely amenable. Here is a strengthening of this fact:

Proposition 3.2.4. *Let G be a non-compact locally compact group and $\{e\} \subsetneq H \subset G$, a subgroup. The pair (G, H) is not relatively extremely amenable.*

Proof. By Veech's Theorem ([Vee77]) G acts freely on U_G. Now use Proposition 3.2.3(2). □

3.3 Extremely Amenable Interpolants

Definition 3.3.1. *Let G be a topological group and $H \subset G$, a subgroup. An extremely amenable group E is called an* ***extremely amenable interpolant*** *for the pair (G, H) if $H \subset E \subset G$.*

The following lemma is trivial:

Lemma 3.3.2. *Let G be a topological group and $H \subset G$, a subgroup. If there exists an extremely amenable interpolant for the pair (G, H), then (G, H) is relatively extremely amenable.*

Here is an example of a non trivial extremely amenable interpolant E for a pair (G, H), in the sense that neither $E = G$, nor $E = H$:

Example 3.3.3. *Let Q be the Hilbert cube. Recall that by a result of Uspenskij (Theorem 9.18 of [Kec95]), $Homeo(Q)$, equipped with the compact-open topology, is a universal Polish group, in the sense that any Polish group embeds inside it through a homomorphism. Let $Homeo_+(I)$ be the group of increasing homeomorphisms of the interval I, equipped with the compact-open topology. By a result of Pestov (see [Pes98]) $Homeo_+(I)$ is extremely amenable. Let $\phi : Homeo_+(I) \hookrightarrow Homeo(Q)$ be an embedding through a homomorphism. Let $f : I \to I$ given by $f(x) = x^2$. Notice $f \in Homeo_+(I)$. Denote $G = Homeo(Q)$, $E = \phi(Homeo_+(I))$ and $H = \phi(\{f^n \,|\, n \in \mathbb{Z}\})$. Notice $H \subsetneq E \subsetneq G$. E is clearly an extremely amenable interpolant for (G, H), but G (which acts homogeneously on Q) and H (which is isomorphic to $\mathbb{Z}$) are not extremely amenable.*

A natural question is if any relatively extremely amenable pair has an extremely amenable interpolant. Theorem 3.5.8 in the next subsection answers the question in the negative.

3.4 Order fixing groups Let S_∞ be the permutation group of the integers $\mathbb{Z}$, equipped with the pointwise convergence topology. Let F be an infinite countable set and fix a bijection $F \simeq \mathbb{Z}$. Let $LO(F) \subset \{0,1\}^{F \times F}$, be the space of linear orderings on F, equipped with the pointwise convergence topology. Under the above mentioned bijection $LO(F)$ becomes an S_∞-space. By Theorem 8.1 of [KPT05] $U_{S_\infty} = LO(F)$. Notice that we consider F as a set and not a topological space. In this subsection we will use $F = \mathbb{Z}$ and $F = \mathbb{Q}$, considered as infinitely countable sets with convenient enumerations (bijections) and the corresponding dynamical systems $(S_\infty, LO(\mathbb{Z}))$ and $(S_\infty, LO(\mathbb{Q}))$.

Lemma 3.4.1. *Let $<\in LO(\mathbb{Z})$ be the usual linear order on $\mathbb{Z}$, i.e. the order for which $n < n + 1$ for every $n \in \mathbb{Z}$. Then*

1. $Stab_{\mathbb{Z}}(<) = \{T_a | \, a \in \mathbb{Z}\}$, *where* $T_a : \mathbb{Z} \to \mathbb{Z}$ *is given by* $T_a(x) = x + a$.

2. $Fix_{LO(\mathbb{Z})}(Stab_{\mathbb{Z}}(<)) = \{<, <^*\}$.

Proof. (1) Let $T \in Stab(<)$. Denote $a = T(0)$. Notice that for all $x > 1$, $T(x) > T(1) > a$ and for all $x < 0$, $T(x) < a$. As T is onto we must have $T(1) = a + 1$. Similarly for all $x \in \mathbb{Z}$, $T(x) = x + a$, which implies $T = T_a$.

(2) Let $\prec \in Fix_{LO(\mathbb{Z})}(Stab(<))$. We claim that $\prec = <$ or $\prec = <^*$. Indeed $0 \prec 1$ or $1 \prec 0$. In the first case applying $T_a \in Stab(<)$, we have for all $a \in \mathbb{Z}$, $a \prec a + 1$. This implies $\prec = <$. Similarly in the second case for all $a \in \mathbb{Z}$, $a + 1 \prec a$ which implies $\prec = <^*$. □

Let $< \in LO(\mathbb{Q})$ be the usual order on $\mathbb{Q}$. In the following lemma, we follow the standard convention and write $Aut(\mathbb{Q}, <)$ instead of $Stab_{S_\infty}(<) \subset S_\infty$.

Lemma 3.4.2. *Let* $< \in LO(\mathbb{Q})$ *be the usual linear order on* $\mathbb{Q}$*, then*

$$Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, < \ )) = \{<, <^*\}.$$

Proof. Let $\prec \in Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <))$. Note that $0 \prec 1$ or $1 \prec 0$. In the first case, let $q_1, q_2 \in \mathbb{Q}$ with $q_1 < q_2$ and define $T : \mathbb{Q} \to \mathbb{Q}$ with $Tx = (q_2 - q_1)x + q_1$. Note that $T \in Aut(\mathbb{Q}, <)$. Hence, $q_1 = T(0) \prec T(1) = q_2$. As the argument works for any $q_1' < q_2'$ we have $\prec = <$. The second case is similar and implies $\prec = <^*$. □

3.5 Maximally Relatively Extremely Amenable Pairs

Proposition 3.5.1. *Let* G *be a topological group, then there exists a subgroup* $H \subset G$*, such that* (G, H) *is relatively extremely amenable and there exists no subgroup* $H \subset H' \subset G$*, such that* (G, H') *is relatively extremely amenable.*

Proof. By Zorn's lemma it is enough to show that any chain w.r.t. inclusion $\{G_\alpha\}_{\alpha \in A}$ such that (G, G_α) is relatively extremely amenable, has a maximal element. Note that if $G_\alpha \subset G_{\alpha'}$, then $Fix_{U_G}(G_{\alpha'}) \subset Fix_{U_G}(G_\alpha)$. In particular for any finite collection $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, we have $\bigcap_{i=1}^n Fix_{U_G}(G_{\alpha_i}) \neq \emptyset$, which implies by a standard compactness argument $\bigcap_{\alpha \in A} Fix_{U_G}(G_\alpha) \neq \emptyset$. This in turn implies that $Fix_{U_G}(\bigcup_{\alpha \in A} G_\alpha) \neq \emptyset$, which finally implies $(G, \bigcup_{\alpha \in A} G_\alpha)$ is relatively extremely amenable by Proposition 3.2.3(2). □

Definition 3.5.2. *A pair* (G, H) *as in Proposition 3.5.1 is called* ***maximally relatively extremely amenable****.*

Similarly to the previous theorem and definition we have:

Proposition 3.5.3. *Let* G *be a topological group, then there exists a subgroup* $H \subset G$*, such that* H *is extremely amenable and there exists no subgroup* $H \subset H' \subset G$*, such that* H' *is extremely amenable.*

Proof. By Zorn's lemma it is enough to show that any chain w.r.t. inclusion $\{G_\alpha\}_{\alpha \in A}$ such that $G_\alpha \subset G$ and G_α is extremely amenable, has a maximal element. Let $(\bigcup_{\alpha \in A} G_\alpha, X)$ be a dynamical system. By assumption for any $\alpha \in A$, $Fix_X(G_\alpha) \neq \emptyset$. In addition if $G_\alpha \subset G_{\alpha'}$, then $Fix_X(G_{\alpha'}) \subset Fix_X(G_\alpha)$. We now continue as in the proof of Theorem 3.5.1 to conclude $\bigcup_{\alpha \in A} G_\alpha$ is extremely amenable. □

Definition 3.5.4. *A subgroup* $H \subset G$ *as in Proposition 3.5.3 is called* ***maximally extremely amenable in*** G*.*

Remark 3.5.5. *It was pointed out in [Pes02] that if H is second countable (Hausdorff) group then there always exists an extremely amenable group G such that $H \subset G$. Indeed by [Usp90] $H \subset Iso(\mathbb{U})$ the group of isometries of Urysohn's universal complete separable metric space $\mathbb{U}$, equipped with the compact-open topology, and by [Pes02], $Iso(\mathbb{U})$ is extremely amenable.*

Theorem 3.5.6. *Let $G = S_\infty$ be the permutation group of the integers, equipped with the pointwise convergence topology. Let $<$ be the usual order on $\mathbb{Z}$ and $H = Stab_{\mathbb{Z}}(<) \subset G$. The pair (G, H) is maximally relatively extremely amenable.*

Proof. By Theorem 8.1 of [KPT05] $U_G = LO(\mathbb{Z})$, the space of linear orderings on $\mathbb{Z}$. By Proposition 3.2.3(2) (G, H) is relatively extremely amenable. Assume that there exists a subgroup E, with $H \subset E \subset G$ such that (G, E) is a relatively extremely amenable. Evoking again Proposition 3.2.3(2), there exists $\prec \in U_G$, so that $E \subset Stab(\prec)$. As $H \subset E \subset Stab(\prec)$, conclude by Lemma 3.4.1(2) that $\prec \in \{<, <^*\}$. As $H = Stab(<) = Stab(<^*)$, we conclude in both cases $E = H$. □

Lemma 3.5.7. *If (G, H) is maximally relatively extremely amenable and neither G nor H are extremely amenable, then (G, H) does not admit an extremely amenable interpolant.*

Proof. Assume for a contradiction that there exists an extremely amenable subgroup E, with $H \subset E \subset G$. Notice that (G, E) is relatively extremely amenable which constitutes a contradiction with the fact that (G, H) is maximally relatively extremely amenable. □

Theorem 3.5.8. *There exists a relatively extremely amenable pair (G, H) which which does not admit an extremely amenable interpolant.*

Proof. Let $G = S_\infty$ be the permutation group of the integers, equipped with the pointwise convergence topology. Let $<$ be the usual order on $\mathbb{Z}$ and $H = Stab(<) \subset G$. By Theorem 3.5.6 (G, H) is maximally relatively extremely amenable. Clearly G is not extremely amenable as $U_G \neq \{*\}$. By Lemma 3.4.1(1) $H = \{T_a \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$, where the second equivalence is as topological groups. This implies H is not extremely amenable. Now invoke Lemma 3.5.7. □

Theorem 3.5.9. *$Aut(\mathbb{Q}, <)$ is maximally extremely amenable in S_∞.*

Proof. By [Pes98] $Aut(\mathbb{Q}, <)$ is extremely amenable. Now we can proceed as in the proof of Theorem 3.5.8 using Lemma 3.4.2. □

Remark 3.5.10. *Even though the previous result never appeared in print, Todor Tsankov pointed out that it can be derived from an earlier result by Cameron. Indeed, the article [Cam76] allows a complete description of the closed subgroups G of S_∞ containing $Aut(\mathbb{Q})$ (essentially, there are only five of them, see [BP11] for an explicit description) and it can be verified that among those, only $Aut(\mathbb{Q})$ is extremely amenable.*

3.6 Applications in Fraïssé Theory The following two sections deal with applications Fraïssé Theory. Two general references for this theory are [Fra00] and [Hod93]. We follow the exposition and notation of [KPT05].

Let $\{<\} \subset L, L_0 = L \setminus \{<\}$ be signatures, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L, $F = Flim(K)$ the Fraïssé limit of K. By Theorem 5.2$(ii) \Rightarrow (i)$ of [KPT05], if we denote $F_0 = Flim(K_0)$ then $F_0 = F|L_0$. Let $G_0 = Aut(F_0)$ and $G = Aut(F)$. Denote $<^F = <_0$, i.e. $<_0$ is the linear order corresponding to the symbol $<$ in F, and let $X_K = \overline{G_0 <_0}$ (X_K is called set of K-*admissible* linear orderings of F in [KPT05]). In [KPT05], two combinatorial properties for K have considerable importance in order to compute universal minimal spaces. Those are called *ordering property* and *Ramsey property*:

Definition 3.6.1. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L, $F = Flim(K)$ the Fraïssé limit of K. We say that K satisfies the **ordering property** (relative to K_0) if for every $A_0 \in K_0$, there is $B_0 \in K_0$, such that for every linear ordering $\prec$ on A_0 and linear ordering $\prec'$ on B_0, if $A = \langle A_0, \prec\rangle \in K$ and $B = \langle B_0, \prec'\rangle \in K$, then there is an embedding $A \hookrightarrow B$.*

Definition 3.6.2. *Let $\{<\} \subset L$ be a signature and K be an order Fraïssé class in L. We say that K satisfies the **Ramsey property** if, for every positive $k \in \mathbb{N}$, every $A \in K$ and every $B \in K$, there exists $C \in K$ such that for every k-coloring of the substructures of C which are isomorphic to A, there is a substructure $\tilde{B}$ of C which is isomorphic to B and such that all substructures of $\tilde{B}$ which are isomorphic to A receive the same color.*

Those two properties are relevant because they capture dynamical properties of X_K. For example, Theorem 7.4 of [KPT05] states that the minimality of X_K is equivalent to K having the ordering property, and Theorem 10.8 of [KPT05] states that X_K being universal and minimal is equivalent to K having the ordering and Ramsey properties. Those results naturally led the authors of [KPT05] to ask whether X_K being universal is equivalent to K having the Ramsey property. This question is precisely the reason for which the concept of relative extremely amenability was introduced. Recall that by Theorem 4.7 of [KPT05], the Ramsey property of K is equivalent to G being extremely amenable. In [NVT13], it is shown that the universality of X_K is equivalent to (G_0, G) being relatively extremely amenable. However, it is still unknown whether (G_0, G) being relatively extremely amenable is really weaker than G being extremely amenable (see Section 3.8 for more about this aspect).

Remark 3.6.3. *The reason for which only* order *expansions (i.e. $\{<\} \subset L$,$L_0 = L \setminus \{<\}$, and $<$ is interpreted as a linear order) were considered in [KPT05] is that, at the time where the article was written, expanding the signature by such a symbol was sufficient in order to obtain Ramsey property and ordering property in all known practical cases. However, we know now that there are some cases where expanding the language with more symbols is necessary (E.g.* circular tournaments *and* boron tree structures*, whose Ramsey-type properties have been respectiveley analyzed by Laflamme, Nguyen Van Thé and Sauer in [LNVTS10], and by Jasiński in [Jas13]). The description of the corresponding universal minimal spaces is very similar to what is obtained in [KPT05] and will appear in a forthcoming paper. For the sake of clarity, we will only treat here the case of order expansions, which extends to the general case without difficulty.*

3.7 The weak ordering property. Theorem 10.8 of [KPT05] states that K has the ordering and Ramsey properties if and only if X_K is the universal minimal space of G_0. The purpose of this section is to show that the combinatorial assumptions made on K can actually be slightly weakened. We start with a generalization of the notion of transitivity mentioned in subsection 3.1.

Definition 3.7.1. *Let G be a topological group and X a G-space. $Y \subset X$ is said to be **transitive w.r.t** X if and only if for any $y \in Y$, $\overline{Gy} = X$.*

Proposition 3.7.2. *Let G_0 be a topological group and let T_{G_0} be G_0-universal. Let $x \in T_{G_0}$ and let $G = Stab_{G_0}(x) \subset G_0$. T_{G_0} is minimal if and only if $Fix_{T_{G_0}}(G)$ is transitive w.r.t T_{G_0}.*

Proof. If T_{G_0} is minimal then T_{G_0} is transitive w.r.t itself and trivially $Fix_{T_{G_0}}(G) \subset T_{G_0}$ is transitive w.r.t T_{G_0}. To prove the inverse direction, let $M \subset T_{G_0}$ be a G_0-minimal space. By Proposition 3.2.3(3), (G_0, G) is relatively extremely amenable and therefore there exists $t_0 \in M \cap Fix_{T_{G_0}}(G)$. As $Fix_{T_{G_0}}(G)$ is transitive w.r.t T_{G_0}, conclude $T_{G_0} = \overline{G_0 t_0} \subset M$, so $T_{G_0} = M$ is minimal. □

The previous proposition enables us to prove the following equivalence:

Theorem 3.7.3. *(G_0, G) is relatively extremely amenable and $Fix_{X_K}(G)$ is transitive w.r.t X_K if and only if X_K is the universal minimal space of G_0.*

Proof. As indicated previously, the universality of X_K is equivalent to the fact that (G_0, G) is relatively extremely amenable. By Proposition 3.7.2, given that X_K is universal, the minimality of X_K is equivalent to the fact that $Fix_{X_K}(G)$ is transitive w.r.t X_K .

Remark 3.7.4. *By Theorem 3.2.3(3) $(S_\infty, Aut(\mathbb{Q}, <))$ is relatively extremely amenable. By Lemma 3.4.2 $Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <)) = \{<, <^*\}$. As $LO(\mathbb{Q}) = \overline{S_\infty <} = \overline{S_\infty <^*}$, we have that $Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <))$ is transitive w.r.t $LO(\mathbb{Q})$. By Theorem 3.7.3, it follows that $Aut(\mathbb{Q}, <)$ is extremely amenable. It should be noted that in [KPT05], one obtains the same results but in reverse order: one concludes $LO(\mathbb{Q})$ is the universal minimal space of G, using the fact that G_0 is extremely amenable.*

□

We are now going to show how to reformulate Theorem 3.7.3 in terms of combinatorics.

Definition 3.7.5. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. We say that (K_0, K) has the **relative Ramsey property** if for every positive $k \in \mathbb{N}$, every $A_0 \in K_0$ and every $B \in K$, there exists $C \in K_0$ such that for every k-coloring of the substructures of C_0 isomorphic to A_0, there is an embedding $\phi : B|L_0 \hookrightarrow C_0$ such that for any two substructures $\tilde{A}, \tilde{A}'$ of B_0 isomorphic to A_0, $\phi(\tilde{A})$ and $\phi(\tilde{A}')$ receive the same color whenever $\tilde{A}$ and $\tilde{A}'$ support isomorphic structures in B.*

In what follows, the relative Ramsey property will appear naturally because of the following fact (see [NVT13]):

Claim 3.7.6. *(G_0, G) is relatively extremely amenable iff (K_0, K) has the relative Ramsey property.*

We will also need the following variant of the notion of ordering property:

Definition 3.7.7. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. We say that K satisfies the **weak ordering property** relative to K_0 if for every $A_0 \in K_0$, there is $B_0 \in K_0$, such that for every linear ordering $\prec$ on A_0 with $A = \langle A, \prec\rangle \in K$ and linear ordering $\prec' \in Fix_{X_K}(G)$ we have $A \hookrightarrow \langle B_0, \prec' |B_0\rangle$.*

The following claim appears in the proof of Theorem 7.4 of [KPT05]:

Claim 3.7.8. *Let $<$ be a linear ordering on F_0. Then $<_0 \in \overline{G_0 <}$ if and only if for every $A \in K$ there is a finite substructure C_0 of F_0 such that $C = \langle C_0, < |C_0\rangle \cong A$.*

Proposition 3.7.9. *Assume K satisfies the weak ordering property relative to K_0, and that (K_0, K) has the relative Ramsey property. Then K satisfies the ordering property.*

Proof. Again, the universality of X_K is equivalent to the fact that (G_0, G) is relatively extremely amenable, which is in turn equivalent to (K_0, K) having the relative Ramsey property. By Theorem 7.4 of [KPT05] the minimality of X_K is equivalent to the ordering property of K (relative to K_0). By Proposition 3.7.2 in order to establish X_K is minimal, it is enough to show that $Fix_{X_K}(G)$ is transitive w.r.t X_K. Let $<\in Fix_{X_K}(G)$. It is enough to show $<_0 \in \overline{G<}$. Fix $A \in K$. As K satisfies the weak ordering property, there is B_0 as in Definition 3.7.7 such that $A \hookrightarrow \langle B_0, < |B_0\rangle$. Using the same argument as in the proof of Theorem 7.4 of [KPT05], we notice that there is a substructure C of B isomorphic to A. Denote $C_0 = C|L_0$ and notice $C = \langle C_0, < |C_0\rangle \cong A$. We now use Claim 3.7.8. ☐

Theorem 3.7.10. *K has the weak ordering property and (K_0, K) has the relative Ramsey property if and only if X_K is the universal minimal space of G_0.*

Proof. By Theorem 10.8 of [KPT05], if X_K is the universal minimal space of G_0 then K satisfies the ordering property, a fortiori, K satisfies the weak ordering property. In addition K satisfies the Ramsey property which implies (K_0, K) has the relative Ramsey property. The reverse direction follows from Proposition 3.7.9. ☐

3.8 A question. We mentioned previously that the concept of relative extreme amenability was introduced in order to know whether X_K being universal is equivalent to K having the Ramsey property. By Theorem 4.7 of [KPT05], the Ramsey property of K is equivalent to G being extremely amenable. We still do not know the answer to the following question from [KPT05]:

Question 3.8.1. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. Does universality for X_K imply that G is extremely amenable (equivalently, that K has the Ramsey property)?*

Moreover, in view of the notions we introduced previously, we ask:

Question 3.8.2. *Assume the previous question has a negative answer. Does there exist an extremely amenable interpolant for the pair (G_0, G)?*

As a final comment, and in view of Remark 3.6.3, it should be mentioned that Question 3.8.1 has a negative answer when K is not an order expansion of K_0, see [NVT13].

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