LOCAL COMPLETENESS AND PARETO EFFICIENCY IN PRODUCT SPACES

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ABSTRACT. In this paper we study the Pareto efficiency with respect to a locally nuclear cone in the product of two locally convex spaces with the restricted assumption that only some related sets are locally complete.

1. Introduction

Kuhn and Tucker in their famous paper [14] considered "proper solutions" for vector maximum problems and introduced the concept of proper efficient points. Hurwicz [7] introduced the notion of a proper maximal point with respect to ordering cones to characterize maximal points as solutions for optimization problems. Isac [8, 9, 11] used a method based on a general existence theorem for critical points of dynamical systems to obtain several general results on the existence of solutions of the general optimization problem in sequentially complete locally convex spaces. He introduced the concept of nuclear cone [9] in a locally convex space, intimately related to Pareto efficiency [8, 9, 10, 11, 12]. Also he defined a nuclear cone in a product of two locally convex spaces [12] to obtain maximal point theorems and a vectorial Ekeland type theorem.

After it was discovered, the Ekeland's principle [5] has had many different applications and extensions [6, 10, 11, 12, 20]. Qiu [21, 22, 23] and Bosch, García et al.[2, 3, 4] found some extensions of Ekeland's variational principle and Pareto efficiency assuming only local completeness conditions. In this paper by adapting ideas of Isac [12] we extend Pareto efficiency respect to locally nuclear cones in the product of two locally convex spaces only assuming that some related sets are locally complete. Also we stablish a vectorial Ekeland type theorem for locally complete spaces.

2. Preliminaries

Througout this paper (E, τ) will denote a locally convex space E, with topology τ generated by a family of seminorms $\{\rho_\alpha : \alpha \in \Lambda\}$ with Λ a set of indexes. A disk B in E is a closed, bounded and absolutely convex set. We denote by (E_B, ρ_B) the linear span of B endowed with the topology defined by the Minkowski functional associated with B. If (E_B, ρ_B) is complete then B is called a Banach disk. E' will denote the topological dual of (E, τ) and $(E_B, \rho_B)'$ will denote the topological dual of E_B with respect to the norm ρ_B.

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A sequence $(x_n)_n$ in E is said to be locally convergent or Mackey convergent to an element x in E if there exists a disk B in E such that the sequence converges to x in E_B with respect to ρ_B. A sequence is called locally Cauchy or Mackey Cauchy if it is a ρ_B-Cauchy sequence in E_B for a certain disk B in E.

Let C be a non-void subset of E. A point x is a local limit point of C if there is a sequence in C that is locally convergent to x. A set C is locally closed if every local limit point of C belongs to C.

A subset A of a space E is said to be locally complete if every local Cauchy sequence in A converges locally to a point of A. It is clear that every locally complete subset of a space is locally closed. For the whole space (E, τ), it is locally complete if and only if every disk B in E is in a Banach disk. And a locally closed subset A of a locally complete space E is locally complete. For more details on local completeness see [13, 17].

A closed pointed convex cone in a locally convex space is a nonempty subset $K \subset E$ such that:

(1) K is a closed convex subset,
(2) $K + K \subset K$,
(3) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}^+$,
(4) $K \cap (-K) = \{0\}$.

If a closed, pointed convex cone $K \subset E$ is given we can define an ordering in E by $x \preceq_K y$ if and only if $y - x \in K$. For more details on order and cones see [16].

If $A \subset E$ is a nonempty subset we say that $a \in A$ is an efficient (maximal) point of A if $A \cap (a + K) = \{a\}$. We denote by $E(A; K)$ the set of efficient points of A with respect to K.

We say that $\Gamma : A \to 2^A$ is a dynamical system (in the generalized sense) if for every $x \in A$, $\Gamma(x)$ is a nonempty subset of A, and $x^* \in A$ is a critical point for Γ if $\Gamma(x^*) = \{x^*\}$. We can see easily that $\Gamma_A(x) = A \cap (x + K)$ for every $x \in A$ is a dynamical system. Note that for $y \in \Gamma_A(x) = A \cap (x + K)$ and $z \in \Gamma_A(y) = A \cap (y + K)$, we have $z \in A \cap ((x + K) + K) = A \cap (x + K)$. So $\Gamma_A(y) \subset \Gamma_A(x)$. The reader can verify that an element $x^* \in A$ is an efficient point of A if and only if x^* is a critical point of Γ_A. Following Aubin-Siegel, Muntean, Petrusel, Rus and Yao a critical point for a dynamical system is also known in the literature as an end or stationary point (see [1]) or a strict fixed point for a set valued operator (see [15, 18, 19]).

In [9] G. Isac introduced the concept of nuclear cone. The cone $K \subset (E, \tau)$ is said to be nuclear if for every ρ_α in the family of seminorms wich defines the topology τ there exists $f_\alpha \in E'$ such that $\rho_\alpha(x) \leq f_\alpha(x)$, for every $x \in K$. In [2], is proved the following

Corollary 1. *Let (E, τ) be a locally convex space and $K \subset E$ a closed, pointed convex cone. Suppose that there exists a non-zero Banach disk D in E and $f \in (E_D, \rho_D)'$, such that $K \cap E_D \neq \{0\}$ and $\rho_D(x) \leq f(x)$, for every $x \in K \cap E_D$. Suppose that for a nonempty locally closed subset $B \subset E$ we have $B \cap E_D \neq \emptyset$ and that f is bounded above in $B \cap E_D$; then for every $x_0 \in B \cap E_D$, there exists an element $x^* \in E(B; K)$ such that $x^* \in x_0 + K$.*

Note that in this corollary the property of nuclearity is applied *locally* to the cone in the space (E_D, ρ_D). Motivated by this condition, we say that a closed, pointed convex cone $K \subset E$ is locally nuclear with respect to the disk $D \subset E$ if there exists $f \in (E_D, \rho_D)'$ such that $\rho_D(x) \leq f(x)$, for every $x \in K \cap E_D$.

3. Main Results

In [2], the author and C. Bosch proved the following theorem

Theorem 1. *Let (E,τ) be a locally convex space, $A \subset E$ a nonempty subset and $K \subset E$ a closed, pointed convex cone. Suppose there exists A_0, a nonempty subset of A, such that:*
a) A_0 is locally complete,
b) $\Gamma_A(A_0) \subset A_0$,
c) There exists a Banach disk $D \subset E$ and $f \in (E_D, \rho_D)'$ such that $A_0 \subset E_D$ and
i) $\rho_D(v) \leq f(v)$, for $v \in K(A_0) = \{v \in K : v = v_1 - v_2;\ v_1, v_2 \in A_0\} \subset E_D$
ii) $\sup\{f(x) : x \in A_0\} < \infty$
Then $E(A,K)$ is nonempty.

We note, from the local completeness of A in the proof of this theorem, that is sufficient to ask B to be a disk, that is, the completeness of (E_B, ρ_B) is unnecessary. Now, from this theorem and the locally nuclear property for a cone, we obtain

Corollary 2. *Let (E,τ) be a locally convex space, $A \subset E$ a nonempty subset and $B \subset E$ a disk such that $A \subset E_B$. Let $K \subset E$ be a closed, pointed convex cone locally nuclear respect to B. Suppose there exists $x_0 \in A$ such that $A \cap (x_0 + K)$ is locally complete and bounded. Then $E(A,K)$ is nonempty.*

Theorem 2. *Let (E,τ) be a locally convex space, $K \subset E$ a closed, pointed convex cone and $A \subset E$ a locally complete subset. Suppose that given $x_0 \in A$ there exists a sequence $(x_n)_n \in A$ such that $x_{n+1} \in \Gamma_A(x_n) \setminus \{x_n\}$, for every $n \in \mathbb{N}$. Suppose there exists a disk $D \subset E$ such that $\lim_n R_D(\Gamma_A(x_n)) = 0$, where $R_D(\Gamma_A(x_n)) = \sup\{\rho_D(x-y) : x, y \in \Gamma(x_n)\}$. Then there exists $x^* \in E(A;K)$ such that $x_0 \preceq_K x^*$.*

Proof. According to the hypothesis, $x_{n+k} \in \Gamma(x_n)$ for every $n \in \mathbb{N}$, $k \in \mathbb{N}$. Since $\lim_n R_D(\Gamma_A(x_n)) = 0$, then there exists $n_0 \in \mathbb{N}$ such that $\Gamma_A(x_{n_0+k}) \subset E_D$, for every $k \in \mathbb{N}$. Then $\rho_D(x_{n+k} - x_n) \leq R_D(\Gamma_A(x_n))$ for $n \geq n_0$ and $k \in \mathbb{N}$. So, the sequence $(x_n)_n \in A$ is locally Cauchy. Since A is locally complete, there exists $x^* \in A$ such that $\rho_D(x_n - x^*)$ converges to zero. Clearly, $x^* \in \bigcap_{n \in \mathbb{N}\cup\{0\}} \Gamma_A(x_n)$. Since $\lim_n R_D(\Gamma_A(x_n)) = 0$ then $R_D(\Gamma(x^*)) = 0$ and $\{x^*\} = \Gamma(x^*) = \bigcap_{n \in \mathbb{N}\cup\{0\}} \Gamma(x_n)$. And $x^* \in \Gamma_A(x_0)$ implies $x^* - x_0 \in K$, so $x_0 \preceq_K x^*$. □

Let (E,τ), (F,τ') be locally convex spaces and suppose that F is ordered by a closed, pointed convex cone $K \subset F$. Let $B \subset E$ and $D \subset F$ be disks. So, the space $E_B \times F_D$ is a normed space endowed with the topology generated by $\rho_B + q_D$. Let $K_D = K \cap F_D$ and suppose $K_D \neq \{0\}$. Find $k_0 \in K_D$ such that $q_D(k_0) = 1$. Consider the set $K_D^* = \{f \in (F_D, q_D)' : f(y) \geq 0 \text{ for every } y \in K_D\}$ and $\psi \in K_D^*$ such that $\psi(k_0) = 1$. Let $1 > \varepsilon > 0$. In $E_B \times F_D$ consider the set

$$K(\varepsilon, B, D) = \left\{(x,y) \in E_B \times F_D\colon\ y + \sqrt{\varepsilon}\,(\rho_B(x) + q_D(y))\,k_0 \in -K_D\right\}.$$

Proposition 1. *The set $K(\varepsilon,B,D)$ is a non-trivial, closed, pointed and nuclear cone in $(E_B \times F_D, \rho_B + q_D)$.*

Proof. Let (x,y); $(u,v) \in K(\varepsilon,B,D)$.

Since $\rho_B(x+u)+q_D(y+v) \le \rho_B(x)+\rho_B(u)+q_D(y)+q_D(v)$,
then $(\, y+v)+\sqrt{\varepsilon}\,(\rho_B(x+u)+q_D(y+v))\,k_0$
$=(y+v)+\sqrt{\varepsilon}\,(\rho_B(x)+\rho_B(u)+q_D(y)+q_D(v)-\gamma)\,k_0$, for some $\gamma \ge 0$,
$=(y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0)+\,(v+\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0)-\gamma k_0$
$\in -K_D-K_D-K_D=-K_D$.

Then $(x,y)+(u,v) \in K(\varepsilon,B,D)$.

Let $\lambda \in \mathbb{R}^+$ and $(x,y) \in K(\varepsilon,B,D)$, so $\lambda(x,y) \in K(\varepsilon,B,D)$, since
$\lambda y+\sqrt{\varepsilon}\,(\rho_B(\lambda x)+q_D(\lambda y))\,k_0=\lambda\,(\,y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0)$
$\in \lambda(-K_D)=-K_D$.

Note that, $K_D \cap (-K_D)=(K\cap(-K))\cap F_D=\{0\}$.

Let $(x_n,y_n) \in K(\varepsilon,B,D)$ such that $(x_n,y_n)\to(x_0,y_0)$ with respect to the norm ρ_B+q_D. Then $x_n \to x_0$ respect to ρ_B and respect to τ and $y_n \to y_0$ respect to q_D and respect to τ'. Then $y_n+\sqrt{\varepsilon}\,(\rho_B(x_n)+q_D(y_n))\,k_0 \in -K_D$ converges to $y_0+\sqrt{\varepsilon}\,(\rho_B(x_0)+q_D(y_0))\,k_0$ respect to τ'. And $y_0+\sqrt{\varepsilon}\,(\rho_B(x_0)+q_D(y_0))\,k_0$ belongs to $-K_D=-K\cap F_D$ since K is τ'-closed in F and then $-K\cap F_D$ is q_D-closed in F_D.

Let $x \in E_B\setminus\{0\}$ and $y \in K_D\setminus\{0\}$. Since $\rho_B(x)>0$ and $q_D(y)>0$ then $y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0 \in K_D\setminus\{0\}$, that is $(x,y)\in(E_B,F_D)\setminus K(\varepsilon,B,D)$. Recall that $q_D(k_0)=1$, then $-k_0+\sqrt{\varepsilon}q_D(-k_0)k_0=-(1-\sqrt{\varepsilon})k_0 \in -K_D$ and $k_0+\sqrt{\varepsilon}q_D(k_0)k_0=(1+\sqrt{\varepsilon})k_0 \in K_D$. That means, $(0,-k_0)\in K(\varepsilon,B,D)$ and $(0,k_0)\notin K(\varepsilon,B,D)$. So, $K(\varepsilon,B,D)$ is a non-trivial, pointed and closed cone in $(E_B\times F_D,\rho_B+q_D)$.

Let us see it is nuclear in this space. For $\pi_2: E_B\times F_D \to F_D$, where $\pi_2(x,y)=y$ and $\psi \in K_D^*$ such that $\psi(k_0)=1$. Let $\Psi: E_B\times F_D\to\mathbb{R}$, given by $\Psi(x,y)=\psi\circ\pi_2(x,y)=\psi(y)$, So, $\Psi\in(E_B\times F_D,\rho_B+q_D)'$. Then for every $(u,v)\in K(\varepsilon,B,D)$ there exists $k\in K_D$ such that $v+\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0=-k\in -K_D$. Then $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0=-v-k$ and applying ψ we obtain $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))=\psi\,(\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0)=-\psi(v)-\psi(k)\le-\psi(v)=-\Psi(u,v)$.

Let $T: E_B\times F_D\to\mathbb{R}$, such that $T=-\Psi$. Then $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\le T(u,v)$, for every $(u,v)\in K(\varepsilon,B,D)$. So, $T\in(K(\varepsilon,B,D))^*$ and $K(\varepsilon,B,D)$ is nuclear in $(E_B\times F_D,\rho_B+q_D)$. □

Theorem 3. *Let (E,τ), (F,τ') be locally convex spaces and suppose that F is ordered by a closed, pointed convex cone $K\subset F$. Let $A\subset E\times F$ be a nonempty locally complete subset and $B\subset E$, $D\subset F$ disks such that $A\subset E_B\times F_D$. Let $k_0\in K_D=K\cap F_D$ be an element such that $q_D(k_0)=1$. For $1>\varepsilon>0$ consider $K(\varepsilon,B,D)$. Suppose there exists $z_0\in F_D$ such that $\{y\in F_D\colon (x,y)\in A$ for some $x\in E_B\}\subset z_0+K_D$. Then for every $(x_0,y_0)\in A$ there exists $(x^*,y^*)\in A$ satisfying*

i) $(x^*,y^*)\in A\cap[(x_0,y_0)+K(\varepsilon,B,D)]$

ii) $A\cap[(x^*,y^*)+K(\varepsilon,B,D)]=\{(x^*,y^*)\}$

Proof. Let $T\in[K(\varepsilon,B,D)]^*$ be as the constructed in Proposition 5. Then
$K(\varepsilon,B,D)\subset\{(x,y)\in E_B\times F_D\colon \sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\le T(x,y)\}$
$=\{(x,y)\in E_B\times F_D\colon \Psi(x,y)+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\le 0\}$;
for $\Psi(x,y)=-T(x,y)=\psi(y)$.

Let $(u,v)\in A$, then $v\in z_0+K_D$, and $v-z_0\in K_D$. Since $\psi\in K_D^*$, then $\psi(v-z_0)\ge 0$ and $\Psi(u,v)=\psi(v)\ge\psi(z_0)$. Hence Ψ is bounded from below on A. Consider the generalized dynamical system $\Gamma: A\to 2^A$, such that $\Gamma(x,y)=$

$A \cap [(x,y)+K(\varepsilon,B,D)]$. We will define an inductive sequence in A. Starting from (x_0,y_0), suppose $(x_n,y_n)\in A$ is defined and $\Gamma(x_{k+1},y_{k+1})\subset\Gamma(x_k,y_k)$, for $k=0,1,...,n-1$. If we have $\Gamma(x_n,y_n)=\{(x_n,y_n)\}$, then we have finished. So, if we have $\Gamma(x_n,y_n)\neq\{(x_n,y_n)\}$ for every $n\in\mathbb{N}$, we have to find (x^*,y^*).

For every $(x,y)\in\Gamma(x_n,y_n)\setminus\{(x_n,y_n)\}$, $\rho_B(x_n-x)+q_D(y_n-y)>0$.
Since $K(\varepsilon,B,D)\subset\{(x,y)\in E_B\times F_D\colon\ \Psi(x,y)+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\leq 0\}$
then $\Psi(x,y)-\Psi(x_n,y_n)+\sqrt{\varepsilon}\,(\rho_B(x-x_n)+q_D(y-y_n))\leq 0$,
for $(x,y)\in\Gamma(x_n,y_n)\setminus\{(x_n,y_n)\}$. So,

$$\Psi(x,y)\leq\Psi(x_n,y_n)-\sqrt{\varepsilon}\,(\rho_B(x-x_n)+q_D(y-y_n))<\Psi(x_n,y_n), \tag{3.1}$$

for $(x,y)\in\Gamma(x_n,y_n)\setminus\{(x_n,y_n)\}$. Then

$$0<\Psi(x_n,y_n)-\Psi(x,y)\leq\Psi(x_n,y_n)-\inf_{(x,y)\in\Gamma(x_n,y_n)}\Psi(x,y).$$

So, for every $n\in\mathbb{N}\cup\{0\}$ there exists $(x_{n+1},y_{n+1})\in\Gamma(x_n,y_n)$ such that

$$\Psi(x_{n+1},y_{n+1})<\inf_{(x,y)\in\Gamma(x_n,y_n)}\Psi(x,y)+\frac{1}{2}\left[\Psi(x_n,y_n)-\inf_{(x,y)\in\Gamma(x_n,y_n)}\Psi(x,y)\right],$$

And $\Gamma(x_{k+1},y_{k+1})\subset\Gamma(x_k,y_k)$, for every $k\in\mathbb{N}\cup\{0\}$. And from the previous inequality, for $(s,t)\in\Gamma(x_{k+1},y_{k+1})$ we have

$$\begin{aligned}\Psi(x_{k+1},y_{k+1})-\Psi(s,t)&\leq\Psi(x_{k+1},y_{k+1})-\inf_{(v,w)\in\Gamma(x_{k+1},y_{k+1})}\Psi(v,w)\\ \leq\Psi(x_{k+1},y_{k+1})-\inf_{(v,w)\in\Gamma(x_k,y_k)}\Psi(v,w)&\leq\frac{1}{2}\left[\Psi(x_k,y_k)-\inf_{(v,w)\in\Gamma(x_k,y_k)}\Psi(v,w)\right]\\ &\leq\frac{1}{2}\left[\Psi(x_0,y_0)-\inf_{(v,w)\in\Gamma(x_k,y_k)}\Psi(v,w)\right]\end{aligned}$$

by (3.1), since $(x_k,y_k)\in\Gamma(x_0,y_0)$. So, for $n\in\mathbb{N}$ and $(s,t)\in\Gamma(x_n,y_n)$ we have

$$\begin{aligned}\Psi(x_n,y_n)-\Psi(s,t)&\leq\frac{1}{2}\left[\Psi(x_0,y_0)-\inf_{(v,w)\in\Gamma(x_{n-1},y_{n-1})}\Psi(v,w)\right]\\ &\leq\frac{1}{2^2}\left[\Psi(x_0,y_0)-\inf_{(v,w)\in\Gamma(x_{n-2},y_{n-2})}\Psi(v,w)\right]\leq\cdots\\ \cdots&\leq\frac{1}{2^n}\left[\Psi(x_0,y_0)-\inf_{(v,w)\in\Gamma(x_0,y_0)}\Psi(v,w)\right]\end{aligned}$$

Recall $K(\varepsilon,B,D)\subset\{(x,y)\in E_B\times F_D\colon\ \Psi(x,y)+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\leq 0\}$.
If $(s,t)\in\Gamma(x_n,y_n)=A\cap[(x_n,y_n)+K(\varepsilon,B,D)]$ then
$(s-x_n,t-y_n)\in K(\varepsilon,B,D)$, which implies

$$\Psi(s-x_n,t-y_n)+\sqrt{\varepsilon}\,(\rho_B(s-x_n)+q_D(t-y_n))\leq 0.$$

Then for $(s,t)\in\Gamma(x_n,y_n)$ and for every $n\in\mathbb{N}$ we have

$$\begin{aligned}\rho_B(s-x_n)+q_D(t-y_n)&\leq\frac{1}{\sqrt{\varepsilon}}\left[\Psi(x_n,y_n)-\Psi(s,t)\right]\\ &\leq\frac{1}{\sqrt{\varepsilon}}\frac{1}{2^n}\left[\Psi(x_0,y_0)-\inf_{(v,w)\in\Gamma(x_0,y_0)}\Psi(v,w)\right].\end{aligned}$$

Since $(x_{k+1},y_{k+1})\in\Gamma(x_n,y_n)=A\cap[(x_n,y_n)+K(\varepsilon,B,D)]$, for every $n\in\mathbb{N}\cup\{0\}$, then $\rho_B(x_{n+1}-x_n)+q_D(y_{n+1}-y_n)\leq R_{B\times D}\left(\Gamma(x_n,y_n)\right)$ which is small if n is large enough. So, the sequence $(x_n,y_n)\in A$ is a locally Cauchy sequence with respect to $\rho_B(\cdot)+q_D(\cdot)$ and convergent to some $(x^*,y^*)\in A$, since A is locally complete. Then by Theorem 4, $(x^*,y^*)\in E(A,K(\varepsilon,B,D))$. □

Let (E,τ) and (F,τ') be locally convex spaces. F ordered by a closed pointed convex cone K. Recall $f:E\to F$ is bounded from below if there exists $z_*\in K$ such that $f(x)\succeq_K z_*$, for every $x\in E$, that is, $f(E)\subset z_*+K$. Also, an element $f(x_\varepsilon)$ is an approximately efficient point of $f(E)$ with respect to K, $k_0\in K$ and $\varepsilon\in[0,1)$ if $f(E)\cap[f(x_\varepsilon)-\varepsilon k_0-(K\setminus\{0\})]=\emptyset$. The set of approximately efficient points of $f(E)$ with respect to K, $k_0\in K$ and $\varepsilon\in[0,1)$ is denoted by $Eff\,(f(E),K_{\varepsilon k_0})$ where $K_{\varepsilon k_0}=\varepsilon k_0+K$. Note that $Eff\,(f(E),K_{\varepsilon k_0})=E\,(f(E),-K)$ (minimal), for $\varepsilon=0$.

As an application of the previous Theorem, under these conditions, we prove the following vectorial Ekeland type Theorem.

Theorem 4. *Let (E,τ), (F,τ') be locally complete locally convex spaces and F ordered by a closed, pointed convex cone K. Let $k_0,z_*\in K$ and $f:E\to F$ be such that $f(x)\succeq_K z_*$, for every $x\in E$, and assume $Graph(f)=\{(x,f(x)):x\in E\}$ is locally closed in $E\times F$. Let $\varepsilon\in(0,1)$ and $f(x_0)\in Eff\,(f(E),K_{\varepsilon k_0})$. Let $D\subset F$ be a non-zero disk such that $k_0,z_*,f(x_0)\in F_D$. Then for every disk non-zero $B\subset E$ such that $x_0\in E_B$ and $f(E_B)\subset z_*+K_D=z_*+(K\cap F_D)$, there exists $x_\varepsilon\in E_B$ satisfying:*

1. $f(x_\varepsilon)\in f(x_0)-\sqrt{\varepsilon}\rho_B(x_\varepsilon-x_0)k_0-K_D$

2. $f(x_\varepsilon)\in E\left(f^{B,D}_{\varepsilon k_0}(E_B),-K_D\right)$; where

$$f^{B,D}_{\varepsilon k_0}(x)=f(x)+\sqrt{\varepsilon}\left[\rho_B(x-x_\varepsilon)+q_D(f(x)-f(x_\varepsilon))\right]k_0$$

for every $x\in E_B$.

Proof. We may assume D is a disk such that $q_D(k_0)=1$. In order to apply the previous theorem, we verify those hypotheses. For $\varepsilon\in(0,1)$ consider the corresponding $K(\varepsilon,B,D)$. As the locally complete set A, now consider $A=\{(x,f(x)):x\in E_B\}\subset E_B\times F_D$ which we will denote by $Graph(f^{B,D})$, and $(x_0,f(x_0))\in Graph(f^{B,D})$. Since $Graph(f)$ is locally closed in the locally complete space $(E\times F)$ then $Graph(f^{B,D})$ is locally complete. Recall B and D are Banach disks, since E and F are locally complete. Then according to the previous theorem, there exists $(x_\varepsilon,f(x_\varepsilon))\in Graph(f^{B,D})$ such that

i) $(x_\varepsilon,f(x_\varepsilon))\in Graph(f^{B,D})\cap[(x_0,f(x_0))+K(\varepsilon,B,D)]$

ii) $Graph(f^{B,D})\cap[(x_\varepsilon,f(x_\varepsilon))+K(\varepsilon,B,D)]=\{(x_\varepsilon,f(x_\varepsilon))\}$.

From (ii), for $x\in E_B\setminus\{x_\varepsilon\}$ we have $(x,f(x))-(x_\varepsilon,f(x_\varepsilon))\notin K(\varepsilon,B,D)$, that is $f(x)-f(x_\varepsilon)+\sqrt{\varepsilon}\left[\rho_B(x-x_\varepsilon)+q_D(f(x)-f(x_\varepsilon))\right]k_0\notin -K_D$.

Then $f^{B,D}_{\varepsilon k_0}(x)\notin f^{B,D}_{\varepsilon k_0}(x_\varepsilon)-K_D$, for every $x\in E_B\setminus\{x_\varepsilon\}$.

Hence $f^{B,D}_{\varepsilon k_0}(E_B)\cap\left[f^{B,D}_{\varepsilon k_0}(x_\varepsilon)-(K_D\setminus\{0\})\right]=\emptyset$, and $f^{B,D}_{\varepsilon k_0}(x_\varepsilon)=f(x_\varepsilon)$ is a minimal efficient point, according to (2).

To see (1), from (i) we have $(x_\varepsilon,f(x_\varepsilon))\in(x_0,f(x_0))+K(\varepsilon,B,D)$. Then $f(x_\varepsilon)-f(x_0)+\sqrt{\varepsilon}\left[\rho_B(x_0-x_\varepsilon)+q_D(f(x_0)-f(x_\varepsilon))\right]k_0\in -K_D$. Then $f(x_\varepsilon)+\sqrt{\varepsilon}\left[\rho_B(x_0-x_\varepsilon)\right]k_0\in f(x_0)-\sqrt{\varepsilon}\left[q_D(f(x_0)-f(x_\varepsilon))\right]k_0-K_D\subset f(x_0)-K_D-K_D$. Hence $f(x_\varepsilon)\in f(x_0)-\sqrt{\varepsilon}\rho_B(x_0-x_\varepsilon)k_0-K_D$. □

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