A SEARCH GAME WITH A NON-ADDITIVE COST

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Abstract. We treat a zero-sum two-person game, what is called, a search game between the hider and the seeker, in which there is a cost for searching a region. If the seeker searches two regions, it is usual that the total cost for two regions is the sum of each cost for a region. However, there may be a saving of the setup cost for the second region when the seeker decides in advance two regions efficiently, and plans to change from one region to another region efficiently. If we take into mind this kind of saving, the cost may not be non-additive. In this paper, we analyze a search game when the cost is not necessarily additive.

1 Introduction In this paper we treat a zero-sum two-person game, what is called, a search game between the hider and the seeker. In a search game, there is a cost for searching a region. If the seeker searches two regions, it is usual that the total cost for two regions is the sum of each cost for a region. In this sense the cost is additive. It is possible to consider that each search cost for a region includes a setup cost. It is likely, however, that there is a saving of the setup cost for the second region when the seeker decides in advance two regions efficiently, and plans to change from one region to another region efficiently. If we take into mind this kind of saving, the cost may not be non-additive. In this paper, we analyze a search game when the cost is not necessarily additive. [4] considers an additive search cost but multiple objects. [2] constructs and analyzes another search game with non-additive costs. There exists an extensive literature on search games. For example, see [1] and [3].

2 Model and properties. Let $N = \{1, \dots, n\}$ be the set of boxes. Define a search game on N. The hider chooses a box $i \in N$ and hides an (immobile) object in that box. Without knowing the hider's choice, the seeker chooses an ordered partition $\mathcal{S} = \{S_1, \dots, S_k\}$ of N, first inspects the set of boxes S_1 , and he finds an object if i is in S_1. If i is not in S_1, then he does not find and he inspects the set of boxes S_2, and so on. We assume he finds an object certainly (with probability 1) if he examines the right set of boxes. Associated with an inspection of $S \subseteq N$ is the inspection cost $c(S)$. An interpretation of an inspection of a set of boxes is as follows. The cost $c(\{i\})$ for $i \in N$ may include some setup cost for beginning the search of the box i. If the searcher can save this setup cost by considering a set of boxes and by devising the method of search, then the cost for a set of boxes could be defined. Under this kind of consideration, it is reasonable to assume $c(\emptyset) = 0$ and to assume

$$
\begin{aligned}
c(S) + c(T) &\geq c(S \cup T), \quad \forall S, T \subseteq N, S, T \neq \emptyset, S \cap T = \emptyset, \\
c(S) &\geq c(T) \geq 0, \quad \forall T \subseteq S \subseteq N, T \neq \emptyset.
\end{aligned} \tag{1}
$$

The first inequality in (1) says that there may be some saving in cost by considering a search for the sets S and T simultaneously. The second is very usual.

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The set of all ordered partitions of N is denoted by Σ which is the set of all strategies for the seeker. The set of all strategies for the hider is N. When the hider and the seeker use strategies $i \in N$ and $\mathcal{S} = \{S_1, \ldots, S_k\} \in \Sigma$, and if $i \in S_j, 1 \le j \le k$, the cost for the seeker is $f(i, \mathcal{S}) = \sum_{\ell=1}^{j} c(S_\ell)$. The hider wishes to maximize it and the seeker wishes to minimize it by choosing $i \in N$ and $\mathcal{S} \in \Sigma$ respectively. We have a two-person zero-sum game $\Gamma(N, c)$ which can be expressed by a finite matrix. A mixed strategy for the hider is $p = (p_1, \ldots, p_n)$ which is a probability distribution over N where $\sum_{i \in N} p_i = 1, p_i \ge 0$ for all $i \in N$. We use the notation $p(S) \equiv \sum_{i \in S} p_i$ and $p|_S \equiv \{p_i/p(S)\}$ for all $S \subseteq N$. We let $p(\emptyset) = 0$. A mixed strategy for the seeker is a probability distribution over Σ, that is, $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma}$ where $\sum_{\mathcal{S} \in \Sigma} q(\mathcal{S}) = 1$ and $q(\mathcal{S}) \ge 0$ for all $\mathcal{S} \in \Sigma$. When the hider and the seeker use strategies p, q, the expected cost is expressed as $f(p, q)$.

Example 1.
In this example, for simplicity we restrict strategies for the seeker to ordered partitions $\mathcal{S} = \{S_1, \ldots, S_k\}$ such that

$$i \in S_\alpha, j \in S_\beta, \alpha < \beta \Longrightarrow i < j. \tag{2}$$

We assume that the hider knows this. Let $n = 2$. From (2) the strategies for the seeker are[1] $\mathcal{S}_1 = \{\overline{1}, \overline{2}\}, \mathcal{S}_2 = \{\overline{12}\}$. The payoff matrix for the hider is

Table 1

	$\mathcal{S}_1$	$\mathcal{S}_2$
1	$c(\overline{1})$	$c(\overline{12})$
2	$c(\overline{1}) + c(\overline{2})$	$c(\overline{12})$

By (1), a pair of optimal strategies is $(2, \mathcal{S}_2)$ and the value is $c(\overline{12})$.

Let $n = 3$. The strategies for the seeker are $\mathcal{S}_1 = \{\overline{1}, \overline{2}, \overline{3}\}, \mathcal{S}_2 = \{\overline{12}, \overline{3}\}, \mathcal{S}_3 = \{\overline{1}, \overline{23}\}, \mathcal{S}_4 = \{\overline{123}\}$. The payoff matrix for the hider is

Table 2

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
1	$c(\overline{1})$	$c(\overline{12})$	$c(\overline{1})$	$c(\overline{123})$
2	$c(\overline{1}) + c(\overline{2})$	$c(\overline{12})$	$c(\overline{1}) + c(\overline{23})$	$c(\overline{123})$
3	$c(\overline{1}) + c(\overline{2}) + c(\overline{3})$	$c(\overline{12}) + c(\overline{3})$	$c(\overline{1}) + c(\overline{23})$	$c(\overline{123})$

By (1), a pair of optimal strategies is $(3, \mathcal{S}_4)$ and the value is $c(\overline{123})$.
From the observation on $n = 2, 3$ in Example 1 we see easily a solution for $n \ge 2$ as follows.

Proposition 2.1. Let $n \ge 2$. Restrict the strategies for the seeker to ordered partitions which satisfy (2). A pair of optimal strategies is $(n, \{N\})$ and the value is $c(N)$. If $f(n, \mathcal{S}) > c(N)$ for all $\mathcal{S} \in \Sigma \setminus \{\{N\}\}$ then it is unique.

Proof: For any $\mathcal{S} = \{S_1, \ldots, S_k\}$ which satisfies (2),

$$f(n, \mathcal{S}) = \sum_{j=1}^{k} c(S_j) \ge c(N), \tag{3}$$

by (1). On the other hand, for any $i \in N$, it holds $f(i, \{N\}) = c(N)$. So a strategy pair $(n, \{N\})$ is a saddle point in the payoff matrix for the hider. Next assume that $f(n, \mathcal{S}) >$

[1] For simplicity, we write $\{1\}, \{1, 2\}$, etc. as $\overline{1}, \overline{12}$, etc.

$c(N)$ for all $\mathcal{S} \in \Sigma \setminus \{\{N\}\}$. Suppose $(i, \mathcal{T})$ is another saddle point. If $i = n$ then $\mathcal{T} \neq \{N\}$ and $f(n, \mathcal{T}) > c(N) = f(n, \{N\})$ which contradicts the fact that $(i, \mathcal{T})$ is a saddle point. If $i \neq n$ then $f(i, \mathcal{T}) \geq f(n, \mathcal{T}) > f(n, \{N\}) = f(i, \{N\})$, which implies $f(i, \mathcal{T}) > f(i, \{N\})$, contradicting the fact that $(i, \mathcal{T})$ is a saddle point. □

We can see the solution when the inspection cost is additive. This is an extreme case of the cost function.

Proposition 2.2. Assume the inspection cost is additive, that is, it satisfies

$$c(S) = \sum_{i \in S} c(i), \text{ for all } S \subseteq N. \tag{4}$$

An optimal strategy for the hider is

$$p_i = \frac{c(i)}{\sum_{j \in N} c(j)}, \quad \forall i \in N. \tag{5}$$

An optimal strategy for the seeker is to choose at random an ordered partition from the set $\Sigma^1 \equiv \{\{\{\pi(1)\}, \ldots, \{\pi(n)\}\} : \pi \text{ is a permutation on } N\}$. The value of the game is

$$\frac{1}{\sum_{j \in N} c(j)} \sum_{i=1}^{n} \sum_{j=1}^{i} c(i)c(j). \tag{6}$$

Proof: For any $\mathcal{S} = \{S_1, \ldots, S_k\}$, let $S_j = \{i_1^j, \ldots, i_{s_j}^j\}, i_1^j < \ldots < i_{s_j}^j$ for all $j = 1, \ldots, k$. Define $\mathcal{S}'$ by $\mathcal{S}' = \{\{i_1^1\}, \ldots, \{i_{s_1}^1\}, \ldots, \{i_1^j\}, \ldots, \{i_{s_j}^j\}, \ldots, \{i_1^k\}, \ldots, \{i_{s_k}^k\}\}$. For any $i \in N$, if $i = i_t^j \in S_j$, then

$$\begin{aligned} f(i, \mathcal{S}) &= c(S_1) + \ldots + c(S_{j-1}) + c(S_j) \\ &= c(S_1) + \ldots + c(S_{j-1}) + \sum_{\ell=1}^{s_j} c(i_\ell^j) \\ &\geq c(S_1) + \ldots + c(S_{j-1}) + \sum_{\ell=1}^{t} c(i_\ell^j) = f(i, \mathcal{S}'). \end{aligned} \tag{7}$$

This implies that $\mathcal{S}$ is dominated by $\mathcal{S}'$. So the seeker chooses from the set Σ^1. The hider knows this, and if he takes p defined by (5), then $f(p, \mathcal{S})$ equals to the quantity given in (6). If the seeker takes q which means that he chooses from the set Σ^1 at random, the expected inspection cost $f(i, q)$ is equals to the quantity given in (6) for all $i \in N$. □

In general, the inspection cost is not always additive. Suppose that p is a strategy for the hider. Suppose the seeker can guess this strategy. Let

$$F_p(N) \equiv \min\{f(p, \mathcal{S}') : \mathcal{S}' \in \Sigma\}. \tag{8}$$

with $F_p(\emptyset) = 0$. By the theory of dynamic programming, we have

$$F_p(N) = \min\{c(S) + p(N \setminus S) F_{p|_{N \setminus S}}(N \setminus S) : S \subseteq N, S \neq \emptyset\}, \tag{9}$$

where $p|_{N \setminus S}$ is a posteriori probability distribution on $N \setminus S$ after the seeker searches S. As an initial condition we have

$$F_p(\{i\}) = c(\{i\}), \forall i \in N. \tag{10}$$

If we can guess an optimal strategy for the hider, then by (9) and (10), we could calculate a best reply of the seeker, as in the next example.

Example 2. Assume that the inspection cost depends on the number of boxes in the set,that is,

$$c(S) = C(|S|), \quad \forall S \subseteq N, \tag{11}$$

where $C(\bullet)$ is a function on $\{0, 1, \ldots, n\}$. From (1), $C(\bullet)$ satisfies

$$\begin{aligned} &C(s) + C(t) \geq C(s+t), \quad \forall s,t : s+t \leq n, \\ &C(s) \geq C(t), \quad \forall s,t : n \geq s \geq t \geq 0. \end{aligned} \tag{12}$$

Then an optimal strategy for the hider is $p^e \equiv (\frac{1}{n}, \ldots, \frac{1}{n})$. We write as $F(s) \equiv F_{p^e}(S)$ for all $S \subseteq N$ such that $|S| = s$, since F depends only on the number of elements in S for every $S \subseteq N$. The equations (9) and (10) become

$$\begin{aligned} &F(n) = \min\{C(s) + \frac{n-s}{n} F(n-s) : 1 \leq s \leq n\}, \\ &F(1) = C(1), F(0) = 0. \end{aligned} \tag{13}$$

Let $G(s) \equiv sF(s)$ for $1 \leq s \leq n$. Then (13) becomes

$$G(n) = \min\{nC(s) + G(n-s) : 1 \leq s \leq n\}, \quad G(1) = C(1), G(0) = 0. \tag{14}$$

Case 1. $C(s) = \sqrt{s}$. By (14), we see

$$G(n) = \begin{cases} n\sqrt{n}, & \text{if } 1 \leq n \leq 3; \ (s = n) \\ n\sqrt{n-1} + 1, & \text{if } 4 \leq n \leq 11. \ (s = n-1) \end{cases} \tag{15}$$

For $n \geq 12$ we could calculate sequentially by (14).

Case 2. $C(s) = \log(s+1)$. By (14), $G(n) = n \log n + \log 2$ for $1 \leq n \leq 5$, by $s = n - 1$.

3 A search game with strictly monotonic cost function.

In this section we analyze an optimal strategy for the hider when the costs are strictly monotonic with respect to inclusion relation.

Proposition 3.1. For $i \in N$, assume the inspection cost satisfies

$$c(S) > c(S \setminus \{i\}), \text{ for all } S \text{ such that } i \in S. \tag{16}$$

Let p be an optimal strategy for the hider. Then $p_i > 0$.

Proof: Assume that the inspection cost satisfies (16) but $p_i = 0$. Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a best reply to p. Suppose $i \in S_j$. Let $\mathcal{S}' = \{S_1, \ldots, S_{j-1}, S_j \setminus \{i\}, S_{j+1}, \ldots, S_k, \{i\}\}$. Since $\mathcal{S}$ is a best reply, we have

$$0 \geq f(p, \mathcal{S}) - f(p, \mathcal{S}') = [c(S_j) - c(S_j \setminus \{i\})][p(S_j \setminus \{i\}) + \sum_{\ell=j+1}^{k} p(S_\ell)], \tag{17}$$

while, by (16), $c(S_j) - c(S_j \setminus \{i\}) > 0$. This and (17) imply $p(S_j \setminus \{i\}) + \sum_{\ell=j+1}^{k} p(S_\ell) = 0$. This implies $p(S_j) = p(S_{j+1}) = \ldots = p(S_k) = 0$. Define p' by

$$p'_x = \begin{cases} p_x - \varepsilon, & \text{if } p_x > 0; \\ p_x, & \text{if } x \neq i, p_x = 0; \\ \kappa\varepsilon, & \text{if } x = i, \end{cases} \tag{18}$$

where $\kappa \equiv |X|$ and $X \equiv \{x : p_x > 0\}$. We note $X \subset S_1 \cup \cdots \cup S_{j-1}$, $p'(S_j) = \kappa\varepsilon$ and $p'(S_{j+1}) = \ldots = p'(S_k) = 0$.

$$\begin{aligned} f(p', \mathcal{S}) &= \sum_{\ell=1}^{k} p'(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &= \sum_{\ell=1}^{j-1} [p(S_\ell) - |S_\ell \cap X|\varepsilon] \sum_{m=1}^{\ell} c(S_m) + \kappa\varepsilon[\sum_{m=1}^{j-1} c(S_m) + c(S_j)] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\kappa \sum_{m=1}^{j-1} c(S_m) + \kappa c(S_j) - \sum_{\ell=1}^{j-1} |S_\ell \cap X| \sum_{m=1}^{\ell} c(S_m)] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\kappa \sum_{m=1}^{j-1} c(S_m) + \kappa c(S_j) - \sum_{m=1}^{j-1} c(S_m) \sum_{\ell=m}^{j-1} |S_\ell \cap X|] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\sum_{m=1}^{j-1} c(S_m)[\kappa - \sum_{\ell=m}^{j-1} |S_\ell \cap X|] + \kappa c(S_j)] \\ &> \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) = f(p, \mathcal{S}). \end{aligned} \tag{19}$$

Let $\mathcal{T} = \{T_1, \ldots, T_b\}$ be any pure strategy for the seeker. If $f(p, \mathcal{T}) > f(p, \mathcal{S})$, then $f(p', \mathcal{T}) > f(p, \mathcal{S})$ by making $\varepsilon > 0$ sufficiently small. If $f(p, \mathcal{T}) = f(p, \mathcal{S})$ then $\mathcal{T}$ is a best reply to p, and there is a such that $i \in T_a$. For the same reason as in $\mathcal{S}$, we have $p(T_a) = \ldots = p(T_b) = 0$. Changing j, k to a, b and $\mathcal{S}$ to $\mathcal{T}$ in (19), we can see $f(p', \mathcal{T}) > f(p, \mathcal{T}) = f(p, \mathcal{S})$. Since the number of pure strategies for the seeker is finite, we obtain a better strategy p' for the hider. This contradicts the optimality of p. Hence, $p_i > 0$. □

It is easy to see that (16) holds for every $i \in N$ if and only if the inspection cost is strictly monotonic, that is, $c(S) > c(T)$ for all S, T such that $T \subset S$ and $S \neq T$. In practice, it is very likely that the inspection cost is strictly monotonic since it costs by all means if the seeker behaves. Let $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma}$ be a mixed strategy for the seeker, where $q(\mathcal{S})$ is the probability that he chooses $\mathcal{S}$. By the complementary slackness theorem in linear programming and Proposition 3.1, we obtain

Corollary 3.2. Assume the inspection cost is strictly monotonic. Let p be an optimal strategy for the hider. Then $p_i > 0$ for all $i \in N$. Let q be an optimal strategy for the seeker. Then $f(i,q) = v(N)$ for all $i \in N$, where $v(N)$ is the value of the game.

4 A search game with mass-effective cost function For $S \subseteq N$, let Σ^S be the set of all ordered partitions of S. A restricted game $\Gamma(S,c)^-$ on S is defined as follows. The set of strategies for the hider is S, the set of strategies for the seeker is $\Sigma^S \setminus \{\{S\}\}$, and the cost for a strategy pair $(i,\mathcal{S}), i \in S, \mathcal{S} \in \Sigma^S \setminus \{\{S\}\}$, is $f(i,\mathcal{S})$. The value of the game $\Gamma(S,c)^-$ is denoted by $v(S)^-$. In the restricted game on S, the strategy $\{S\}$ for the seeker is excluded from the strategy set in the original game $\Gamma(S,c)$ on S.

Proposition 4.1. Assume $c(N) < v(N)^-$. An optimal strategy for the seeker is $\{N\}$. An optimal strategy for the hider is p which is an optimal strategy for the hider in the restricted game $\Gamma(N,c)^-$. The value of the game is $c(N)$.

Proof: For every $i \in N$, we have $f(i,\{N\}) = c(N)$. Let p be an optimal strategy for the hider in the restricted game $\Gamma(N,c)^-$. Then $f(p,\mathcal{S}) \geq v(N)^- > c(N)$ for all $\mathcal{S} \in \Sigma^N \setminus \{\{N\}\}$. Furthermore, $f(p,\{N\}) = c(N)$. This completes the proof. □

The discussion in Proposition 4.1 could be extended to the restricted game on every $S \subseteq N$ if an optimal strategy for the hider has some property. For a mixed strategy p for the hider, we define a mixed strategy p^S for the hider on the game on $S \subseteq N$ by $p^S = p|_S$, which is a projection of p on the strategy space of the game on S.

Proposition 4.2. Suppose p is an optimal strategy for the hider in the game on N. Assume that p^S is an optimal strategy for the hider in the game on $S \subset N$. Assume $c(S) < v(S)^-$. Then the seeker can exclude from the consideration a strategy such as $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\}$ where $\mathcal{S}_2$ is an ordered partition of $N \setminus S$ and $\mathcal{S}_1$ is an ordered partition of S and $\mathcal{S}_1 \neq \{S\}$.

Proof: Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\}$ be a strategy for the seeker in the statement of Proposition 4.2.

$$\begin{aligned} f(p,\mathcal{S}) &= p(S)\sum_{i\in S} p_i^S f(i,\mathcal{S}) + \sum_{i\in N\setminus S} p_i f(i,\mathcal{S}) \\ &\geq p(S)v(S)^- + \sum_{i\in N\setminus S} p_i f(i,\mathcal{S}) \\ &> p(S)c(S) + \sum_{i\in N\setminus S} p_i f(i,\mathcal{S}) \\ &= f(p,\{\{S\},\mathcal{S}_2\}). \end{aligned} \tag{20}$$

This implies $\mathcal{S}$ is not a best reply to p. □

5 A game on a star network. In practice, it may cost the absurdity for the change of the box when we consider whether the seeker searches for another box j after having searched for a box i. In this case, the seeker will search a box $k \neq j$ after he has searched for a box i. This kind of things could be expressed by a network where nodes are boxes. Edges express changeability between boxes. This situation is expressed as a game on a network. In this section we treat this model.

Let $G = (N,E)$ be an undirected graph with the node set N and the edge set E. A subgraph $(S,E(S))$ is an undirected graph where $S \subseteq N$ and $E(S) = \{(i,j) \in E, i,j \in$

$S\} \subseteq E$. A subset $S \subseteq N$ is called connected if the subgraph $(S, E(S))$ is connected. A subset $\Sigma^* \subset \Sigma$ is defined by the set of all ordered partitions of N such that every element of each ordered partition is connected. Hereafter, we assume that the strategy space for the seeker is Σ^* when the game is on a network. If G is a complete graph, then $\Sigma^* = \Sigma$. Let G is a linear graph, that is, $E = \{(i, i+1) : 1 \leq i \leq n-1\}$. This model is the same as Example 1 in Section 1.

In this section we treat the case where the graph G is a tree in which $E = \{(1, i) : 2 \leq i \leq n\}$. It is easy to see that a subset S is connected if and only if $1 \in S$. It is possible to analyze if we assume a symmetry in cost as follows.

Assumption 1.

$$c(S) = c(S') \text{ and } c(S \cup \{1\}) = c(S' \cup \{1\}) \text{ for all } S, S' \subseteq N \setminus \{1\} : |S| = |S'|. \tag{21}$$

Since nodes in $N \setminus \{1\}$ are symmetric both in inspection cost and in position in the tree, it is easy to see that there is an optimal strategy $p = (p_1, \ldots, p_n)$ for the hider such that

$$x \equiv p_2 = \ldots = p_n \text{ and } y \equiv p_1 = 1 - (n-1)x, 0 \leq x \leq \frac{1}{n-1}. \tag{22}$$

This strategy is expressed as $p = p(x)$. A pure strategy in Σ^* for the seeker is expressed as

$$\{\{i_1\}, \ldots, \{i_k\}, S, \{i_{k+1}\}, \ldots, \{i_{n-|S|}\}\} \tag{23}$$

where $1 \in S$ and $i_1, \ldots i_k, i_{k+1}, \ldots, i_{n-|S|}$ is a permutation on $N \setminus S$. By the same symmetry as for the hider, it is easy to see that there is an optimal strategy $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma^*}$ such that $q(\mathcal{S}) = q(\mathcal{S}')$ if $\mathcal{S}$ is obtained from $\mathcal{S}'$ by a permutation on $N \setminus S$. From this observation, it suffices to restrict our attention to pure strategies $\mathcal{S}$ where $S = \{1, \ldots, s\}, s \equiv |S|$ and

$$\mathcal{S} = \mathcal{S}(s, k) \equiv \{\overline{s+1}, \ldots, \overline{s+k}, S, \overline{s+k+1}, \ldots, \overline{n}\}, \tag{24}$$

for $k = 0, \ldots, n-s$ and for $s = 1, \ldots, n$.

Lemma 5.1. For each $p = p(x)$,

$$f(p, \mathcal{S}(s, k)) = k[c - xc(S) - (n-s)cx] + c(S) + \frac{(n-s)(n-s+1)}{2}cx, \tag{25}$$

where $c(i) = c, \forall i \notin S$.

Proof: For $p = (p_1, \ldots, p_n)$,

$$\begin{aligned} f(p, \mathcal{S}(s,k)) = {} & p_{s+1}c(s+1) + \cdots + p_{s+k}[c(s+1) + \cdots + c(s+k)] \\ & + p(S)[c(s+1) + \cdots + c(s+k) + c(S)] \\ & + p_{s+k+1}[c(s+1) + \cdots + c(s+k) + c(S) + c(s+k+1)] \\ & + \cdots \\ & + p_n[c(s+1) + \cdots + c(s+k) + c(S) + c(s+k+1) + \cdots + c(n)]. \end{aligned} \tag{26}$$

From (21) and (22), we have

$$\begin{aligned} f(p, \mathcal{S}(s,k)) = {} & xc + \cdots + kcx + p(S)[kc + c(S)] \\ & + x[kc + c(S) + c] \\ & + \cdots \\ & + x[kc + c(S) + (n-s-k)c]. \end{aligned} \tag{27}$$

Since $p(S) = y + (s-1)x = 1-(n-s)x$, from (27), we have (25). □

The hider will consider that the seeker may choose (s,k) so that it minimizes $f(p, \mathcal{S}(s,k))$, given by (25).

Lemma 5.2. For each $p = p(x)$ and $1 \le s \le n$,

$$\min_{0 \le k \le n-s} \{f(p, \mathcal{S}(s,k))\} = \min\{f(p, \mathcal{S}(s,0)), f(p, \mathcal{S}(s, n-s))\}. \tag{28}$$

Proof: From (25), if $c - xc(S) - (n-s)cx > 0$, then $k = 0$ minimizes (25). If $c - xc(S) - (n-s)cx < 0$, then $k = n-s$ minimizes (25). □

For $p = p(x)$, let

$$a(x) \equiv \min_{1 \le s \le n} \{\min\{f(p, \mathcal{S}(s,0)), f(p, \mathcal{S}(s,n-s))\}\}. \tag{29}$$

The hider will choose x so that it maximizes $a(x)$. Here we note that $f(p, \mathcal{S}(s,0))$ is increasing in x and $f(p, \mathcal{S}(s,n-s))$ is decreasing in x as follows:

$$\begin{aligned} f(p, \mathcal{S}(s,0)) &= c(S) + \frac{(n-s)(n-s+1)}{2}cx, \\ f(p, \mathcal{S}(s,n-s)) &= (n-s)c + c(S) - x[(n-s)c(S) + \frac{(n-s)(n-s-1)}{2}c]. \end{aligned} \tag{30}$$

Suppose x^* maximizes $a(x)$. There are $s_1 \le \ldots \le s_\alpha$ and $t_1 \le \ldots \le t_\beta$ such that

$$\begin{aligned} a(x^*) &= f(p(x^*), \mathcal{S}(s_1,0)) = \cdots = f(p(x^*), \mathcal{S}(s_\alpha,0)) \\ &= f(p(x^*), \mathcal{S}(t_1, n-t_1)) = \cdots = f(p(x^*), \mathcal{S}(t_\beta, n-t_\beta)). \end{aligned} \tag{31}$$

By the complementary slackness theorem, there is q such that, for all $j \in N$,

$$\begin{aligned} &\sum_{i=1}^{\alpha} q(\mathcal{S}(s_i,0)) f(j, \mathcal{S}(s_i,0)) + \sum_{i=1}^{\beta} q(\mathcal{S}(t_i, n-t_i)) f(j, \mathcal{S}(t_i, n-t_i)) = a(x^*), \\ &q(\mathcal{S}) = 0, \text{for other } \mathcal{S}. \end{aligned} \tag{32}$$

In summary we have

Proposition 5.3. Under Assumption 1, an optimal strategy for the hider is $p(x^*)$ which is defined by (22) and (31). An optimal strategy for the seeker is q defined by (32). The value of the game is $a(x^*)$.

Let's illustrate the above argument by an example.

Example 3. Let $n = 3$ and $c(\overline{2}) = c(\overline{3}) = 2, c(\overline{12}) = c(\overline{13}) = 3, c(\overline{123}) = 4$. By (1), we have $1 \le c(\overline{1}) \le 3$. By (30),

$$\begin{aligned} &f(p, \mathcal{S}(1,0)) = c(\overline{1}) + 6x, f(p, \mathcal{S}(1,2)) = 4 + c(\overline{1}) - 2[c(\overline{1}) + 1]x, \\ &f(p, \mathcal{S}(2,0)) = 3 + 2x, f(p, \mathcal{S}(2,1)) = 5 - 3x, f(p, \mathcal{S}(3,0)) = 4. \end{aligned} \tag{33}$$

Suppose $c(\overline{1}) = 2$. By drawing a diagram, we see that $x^* = \frac{3}{8}$ maximizes $a(x)$ where $f(p, \mathcal{S}(2,0))$ and $f(p, \mathcal{S}(1,2))$ intersect. $a(\frac{3}{8}) = \frac{15}{4}$. The first in (32) becomes

$$\begin{aligned} f(1,q) &= q(\overline{2},\overline{3},\overline{1}) \times 6 + q(\overline{3},\overline{2},\overline{1}) \times 6 + q(\overline{12},\overline{3}) \times 3 + q(\overline{13},\overline{2}) \times 3 \\ f(2,q) &= q(\overline{2},\overline{3},\overline{1}) \times 2 + q(\overline{3},\overline{2},\overline{1}) \times 4 + q(\overline{12},\overline{3}) \times 3 + q(\overline{13},\overline{2}) \times 5 \\ f(3,q) &= q(\overline{2},\overline{3},\overline{1}) \times 4 + q(\overline{3},\overline{2},\overline{1}) \times 2 + q(\overline{12},\overline{3}) \times 5 + q(\overline{13},\overline{2}) \times 3. \end{aligned} \tag{34}$$

We find q so that these three are equal to $a(\frac{3}{8}) = \frac{15}{4}$:

$$q(\overline{2},\overline{3},\overline{1}) = q(\overline{3},\overline{2},\overline{1}) = \frac{1}{8}, \quad q(\overline{12},\overline{3}) = q(\overline{13},\overline{2}) = \frac{3}{8}. \tag{35}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{15}{4} = a(x^*)$.

Suppose $c(\overline{1}) = 1$. In the same way as above, we have $x^* = \frac{2}{5}$ and $a(\frac{2}{5}) = \frac{17}{5}$. The intersection of lines $f(p,\mathcal{S}(1,0))$ and $f(p,\mathcal{S}(1,2))$ is critical. So the seeker will choose

$$q(\overline{2},\overline{3},\overline{1}) = q(\overline{3},\overline{2},\overline{1}) = \frac{3}{10}, \quad q(\overline{1},\overline{2},\overline{3}) = q(\overline{1},\overline{3},\overline{2}) = \frac{1}{5}. \tag{36}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{17}{5} = a(x^*)$.

Suppose $c(\overline{1}) = 3$. We have $x^* = \frac{2}{5}$ and $a(\frac{2}{5}) = \frac{19}{5}$. The intersection of lines $f(p,\mathcal{S}(2,0)), f(p,\mathcal{S}(2,1))$ and $f(p,\mathcal{S}(1,2))$ is critical. So the seeker will choose

$$\begin{aligned} &q(\overline{2},\overline{3},\overline{1}) = q(\overline{3},\overline{2},\overline{1}), \quad q(\overline{3},\overline{12}) = q(\overline{2},\overline{13}), q(\overline{12},\overline{3}) = q(\overline{13},\overline{2}),\\ &q(\overline{12},\overline{3}) = \frac{3}{2}q(\overline{3},\overline{12}) + 4q(\overline{2},\overline{3},\overline{1}),\\ &10q(\overline{2},\overline{3},\overline{1}) + 5q(\overline{3},\overline{12}) = 1. \end{aligned} \tag{37}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{19}{5} = a(x^*)$.

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