Peripheral spectrum for $A \times B$

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Abstract.

We discuss the idea of peripheral spectrum and related concepts such as Maximum modulus set, peak sets etc. for a function algebra. We study the interrelation of them. We further study these concepts for the Cartesian product $A \times B$ of two function algebras.

1 Introduction The spectrum of an element of a Banach algebra unveils the algebraic structure of the Banach algebras. However, sometimes a subset, the peripheral spectrum of the spectrum suffices for the purpose. This concept was introduced in [1].

We shall assume throughout that A is a function algebra on a compact Hausdorff space X.

Definition 1.1 *Let A be a function algebra on X. For $f \in A$, the peripheral spectrum is the set, $\sigma_\pi(f) = \sigma(f) \cap \{z \in \mathbb{C} : |z| = \|f\|\}$, where $\sigma(f)$ is the spectrum of f, and the set $\{z \in \mathbb{C} : |z| = \|f\|\}$ is the circle centered at origin and having radius $\|f\|$, denoted by $\Gamma_{\|f\|}$.*

To emphasize on the algebra we denote the peripheral spectrum with respect to algebra A by $\sigma_{\pi_A}(f)$.

Remarks 1.2 *(1) $\sigma_\pi(f)$ is a nonempty compact subset of $\sigma(f)$.*
(2) The concept of peripheral spectrum can be defined for any Banach algebra. However, it is non-empty only if the spectral radius $r(f)$ equals the norm $\|f\|$.
e.g. Take $A = C^1[0,1]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and $f(t) = t, t \in [0,1]$.

2 Peripheral spectrum and peaking functions We have studied certain properties for the Cartesian product of two function algebras [2]. Let A and B be function algebras on X and Y respectively. Then $A \times B$ with coordinatewise operations and $\|(f,g)\| = \max\{\|f\|_\infty, \|g\|_\infty\}$ is a function algebra on $X + Y$. It was proved in general setting [3], that $\sigma((f,g)) = \sigma(f) \cup \sigma(g)$, $\forall f \in A, g \in B$. Here we discuss peripheral spectrum and related concepts for $A \times B$.

Theorem 2.1 *For $h = (f,g) \in A \times B$,*
(a) $\sigma_{\pi_{A\times B}}(h) \subset \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$
(b) $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$ iff $\|f\| = \|g\|$
(c) $\sigma_{\pi_{A\times B}}(h) = \begin{cases} \sigma_{\pi_A}(f), & \text{if } \|f\| > \|g\|; \\ \sigma_{\pi_B}(g), & \text{if } \|f\| < \|g\|. \end{cases}$

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Proof. (a) Let $\lambda \in \sigma_{\pi_{A\times B}}(h)$. Now

$$\begin{aligned}\sigma_{\pi_{A\times B}}(h) &= \sigma_{A\times B}((f,g)) \cap \{z \in \mathbb{C} : |z| = \|(f,g)\|\} \\ &= [\sigma_A(f) \cup \sigma_B(g)] \cap \{z \in \mathbb{C} : |z| = \|(f,g)\|\}\end{aligned}$$

Then $\lambda \in \sigma_A(f)$ or $\lambda \in \sigma_B(g)$. Also $|\lambda| = \|f\|$ or $|\lambda| = \|g\|$ or $|\lambda| = \|f\| = \|g\|$. Suppose $\lambda \in \sigma_A(f)$ and $\|(f,g)\| = \|f\|$. Then clearly $\lambda \in \sigma_{\pi_A}(f)$. If $\|(f,g)\| = \|g\|$, then $|\lambda| = \|g\| \geq \|f\|$ and as $\lambda \in \sigma_A(f)$, $|\lambda| \leq \|f\|$. So $|\lambda| = \|f\|$. So $\lambda \in \sigma_{\pi_A}(f)$. Thus whenever $\lambda \in \sigma_A(f)$, $\lambda \in \sigma_{\pi_A}(f)$.

Similarly, if $\lambda \in \sigma_B(g)$, then $\lambda \in \sigma_{\pi_B}(g)$.

Thus $\lambda \in \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$. Hence $\sigma_{\pi_{A\times B}}(h) \subset \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$.

(b) Now assume that $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$. If $\lambda \in \sigma_{\pi_A}(f)$, then $|\lambda| = \|f\|$ and also $|\lambda| = \|h\|$, as $\lambda \in \sigma_{\pi_{A\times B}}(h)$. So $\|f\| = \|h\|$. Similarly, if $\lambda \in \sigma_{\pi_B}(g)$, we get $\|g\| = \|h\|$. Thus $\|f\| = \|g\|$.

Conversely, suppose $\|f\| = \|g\|$. Then clearly $\|f\| = \|g\| = \|h\|$. Now if $\lambda \in \sigma_{\pi_A}(f)$, then $\lambda \in \sigma_A(f)$ and $|\lambda| = \|f\|$. But then $\lambda \in \sigma_{A\times B}(h)$ with $|\lambda| = \|h\|$. So $\lambda \in \sigma_{\pi_{A\times B}}(h)$. Hence $\sigma_{\pi_A}(f) \subset \sigma_{\pi_{A\times B}}(h)$.

Similarly, we get $\sigma_{\pi_B}(g) \subset \sigma_{\pi_{A\times B}}(h)$. Thus $\sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g) \subset \sigma_{\pi_{A\times B}}(h)$. Combining with (a), we get $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$.

(c) Suppose $\|f\| > \|g\|$. Then $\|h\| = \|f\| > \|g\|$. So $\Gamma_{\|f\|} = \Gamma_{\|h\|} = \Gamma$ (say). Now

$$\begin{aligned}\sigma_{\pi_{A\times B}}(h) &= \sigma_{A\times B}(h) \cap \Gamma \\ &= [\sigma_A(f) \cup \sigma_B(g)] \cap \Gamma \\ &= [\sigma_A(f) \cap \Gamma] \cup [\sigma_B(g) \cap \Gamma] \\ &= \sigma_{\pi_A}(f)\end{aligned}$$

as $\sigma_B(g) \cap \Gamma = \emptyset$, because $\|g\| < \|f\| = \|h\|$.

Similarly, $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_B}(g)$, if $\|f\| < \|g\|$.

Definition 2.2 *[1] Let A be a function algebra on a compact Hausdorff space X. Then for $f \in A$, the peripheral range, $Ran_{\pi_A}(f)$ is defined as,*

$$\begin{aligned}Ran_{\pi_A}(f) &= f(X) \cap \{z \in \mathbb{C} : |z| = \|f\|\} \\ &= f(X) \cap \Gamma_{\|f\|},\end{aligned}$$

where $f(X)$ is the range of f.

Remarks 2.3 *(1) Since $\sigma_{\pi_A}(f) \subset bd\sigma_A(f) \subset \widehat{f}(\partial_A) = f(\partial_A) \subset f(X)$ for a function algebra A on X, we have $\sigma_{\pi_A}(f) = Ran_{\pi_A}(f)$, $\forall f \in A$ [1], where ∂_A is the Šilov boundary for A.*
(2) Suppose A and B are function algebras on X with $A \subset B$. Then for $f \in A$, $\sigma_B(f) \subset \sigma_A(f)$ and the inclusion may be proper. However, by (1) above, $\sigma_{\pi_B}(f) = \sigma_{\pi_A}(f)$, $\forall f \in A$.

Definition 2.4 *[4] Let A be a function algebra on X and $f \in A$. The set of all x in X at which f attains its maximum modulus is called the maximum modulus set and is denoted by $E(f)$, i.e.,*

$$E(f) = \{x \in X : |f(x)| = \|f\|\}.$$

Remark 2.5 *It is clear from the Definitions 1.1 and 2.4 that $E(f) = f^{-1}(\sigma_{\pi_A}(f))$, for $f \in A$.*

Theorem 2.6 *For $h = (f, g) \in A \times B$,*
(a) $E(h) \subset E(f) \cup E(g)$
(b) $E(h) = E(f) \cup E(g)$ iff $\|f\| = \|g\|$
(c) $E(h) = \begin{cases} E(f), & \text{if } \|f\| > \|g\|; \\ E(g), & \text{if } \|f\| < \|g\|. \end{cases}$

Proof. (a) Let $z_0 \in E(h) = \{z \in X + Y : |h(z)| = \|h\|\}$. Then $|h(z_0)| = \|h\|$. If $z_0 \in X$, then $h(z_0) = f(z_0)$. Therefore $|f(z_0)| = |h(z_0)| = \|h\| \le \|f\| \le \|h\|$. Therefore we must have $|f(z_0)| = \|f\|$. So $z_0 \in E(f)$.

Similarly, if $z_0 \in Y$, then $z_0 \in E(g)$. Thus $E(h) \subset E(f) \cup E(g)$.

(b) Suppose that $E(h) = E(f) \cup E(g)$. Also assume that $\|f\| > \|g\|$. Then $\|h\| = \|f\|$. Let $y \in E(g)$. Then $|h(y)| = |g(y)| = \|g\| < \|h\|$, i.e., $y \notin E(h)$ which is not possible. Therefore we must have $\|f\| = \|g\| = \|h\|$.

Conversely, suppose that $\|f\| = \|g\| = \|h\|$ and let $z_0 \in E(f) \cup E(g)$. If $z_0 \in E(f)$, then $z_0 \in X \subset X + Y$ and $|h(z_0)| = |f(z_0)| = \|f\| = \|h\|$, i.e., $z_0 \in E(h)$.

Similarly, if $z_0 \in E(g)$, then $z_0 \in E(h)$. Thus $E(f) \cup E(g) \subset E(h)$. Combining with (a), we get $E(h) = E(f) \cup E(g)$.

(c) Suppose $\|f\| > \|g\|$. Then $\|h\| = \|f\|$. Let $z_0 \in E(h)$. Then if $z_0 \in Y$, we get $\|h\| = |h(z_0)| = |g(z_0)| \le \|g\| < \|f\|$ which is a contradiction. So we must have $z_0 \in X$. So $z_0 \in E(f)$. Thus $E(h) \subset E(f)$.

Conversely, let $z_0 \in E(f)$. Then as above, we get $E(f) \subset E(h)$. Hence $E(f) = E(h)$. Thus $E(f) = E(h)$, if $\|f\| > \|g\|$.

Similarly, $E(h) = E(g)$, if $\|f\| < \|g\|$.

Remark 2.7 *Since $E(f) = f^{-1}(\sigma_{\pi_A}(f))$, we can prove Theorem 2.6 using Theorem 2.1, directly also.*

Definition 2.8 *[4] Let A a function algebra on X. For $x \in X$ define,*

$$\mathcal{E}_x(A) = \{f \in A : |f(x)| = \|f\|\} = \{f \in A : x \in E(f)\}.$$

For a fixed $f \in A$ and $g \in B$ we define, $A_g = \{f \in A : \|f\| \le \|g\|\}$ and $B_f = \{g \in B : \|g\| \le \|f\|\}$.

Theorem 2.9 *For $z \in X+Y$, $\mathcal{E}_z(A\times B) = \begin{cases} \bigcup\{(f,g) : f \in \mathcal{E}_z(A),\ g \in B_f\}, & \text{if } z \in X; \\ \bigcup\{(f,g) : g \in \mathcal{E}_z(B),\ f \in A_g\}, & \text{if } z \in Y. \end{cases}$*

Proof. Let $h = (f, g) \in \mathcal{E}_z(A \times B)$. Then $|h(z)| = \|h\|$. If $z \in X$, then $h(z) = f(z)$. So $|f(z)| = |h(z)| = \|h\| = \|f\|$. Thus $|f(z)| = \|f\|$. So $f \in \mathcal{E}_z(A)$ and $\|h\| = \|f\| \ge \|g\|$, i.e., $g \in B_f$. Thus $\mathcal{E}_z(A \times B) \subset \bigcup\{(f, g) : f \in \mathcal{E}_z(A),\ g \in B_f\}$.

Conversely, suppose that $h = (f, g) \in \bigcup\{(f, g) : f \in \mathcal{E}_z(A),\ g \in B_f\}$. Then $|f(z)| = \|f\|$ and $\|f\| \ge \|g\|$. Now $|h(z)| = |f(z)| = \|f\| = \|h\|$. So $h \in \mathcal{E}_z(A \times B)$. Thus $\mathcal{E}_z(A \times B) = \bigcup\{(f, g) : f \in \mathcal{E}_z(A),\ g \in B_f\}$.

Similarly, if $z \in Y$, then $\mathcal{E}_z(A \times B) = \bigcup\{(f, g) : g \in \mathcal{E}_z(B),\ f \in A_g\}$.

Next we relate peaking function of A and B with that of $A \times B$.

Definition 2.10 *[1] Let A be a function algebra on X. An element $f \in A$ is called a peaking function for A if $\sigma_{\pi_A}(f) = \{1\}$, i.e., $\|f\| = 1$ and $|f(x)| < 1$ whenever $f(x) \neq 1$.*

In this case, $E(f) = \{x \in X : f(x) = 1\} = f^{-1}\{1\}$ is called the peak set of f.

The set of all peaking functions in A is denoted by $\mathscr{P}(A)$.

In general, $\mathscr{P}(A\times B) \neq \mathscr{P}(A)\times\mathscr{P}(B)$, as the following example shows. Let $A = B = (\mathbb{C}, |.|)$ and $h = (f, g) = (1, \frac{1}{2}) \in A \times B$. Then $\sigma_{\pi_{A\times B}}(h) = \{1\}$. So $h \in \mathscr{P}(A \times B)$ and $\sigma_{\pi_A}(f) = \{1\}, \sigma_{\pi_B}(g) = \{\frac{1}{2}\}$. Hence $f \in \mathscr{P}(A)$ but $g \notin \mathscr{P}(B)$. So $h \notin \mathscr{P}(A) \times \mathscr{P}(B)$.

Hence $\mathscr{P}(A \times B) \neq \mathscr{P}(A) \times \mathscr{P}(B)$.

Thus $\mathscr{P}(A \times B) \not\subset \mathscr{P}(A) \times \mathscr{P}(B)$. However, we get $\mathscr{P}(A \times B) \supset \mathscr{P}(A) \times \mathscr{P}(B)$ from the following result.

Now let us denote $U_A = \{f \in A : \|f\| \leq 1\}$, $S_A = \{f \in A : \|f\| = 1\}$, $U_B = \{g \in B : \|g\| \leq 1\}$, $S_B = \{g \in B : \|g\| = 1\}$.

Theorem 2.11 $\mathscr{P}(A \times B) = [\mathscr{P}(A) \times (U_B \setminus S_B)] \cup [(U_A \setminus S_A) \times \mathscr{P}(B)] \cup [\mathscr{P}(A) \times \mathscr{P}(B)]$.

Proof. Let $h = (f, g) \in \mathscr{P}(A \times B)$. Then $\sigma_{\pi_{A \times B}}(h) = \{1\}$, i.e., $\|h\| = 1$. If $\|f\| > \|g\|$, then $\|f\| = \|h\| = 1$ and $\|g\| < 1$. Therefore $g \in U_B \setminus S_B$ and $\sigma_{\pi_A}(f) = \{1\}$, by Theorem 2.1 (c), i.e., $f \in \mathscr{P}(A)$. Thus $h \in \mathscr{P}(A) \times (U_B \setminus S_B)$.

If $\|f\| < \|g\|$, then by similar argument we get $h \in (U_A \setminus S_A) \times \mathscr{P}(B)$.

If $\|f\| = \|g\|$, then also by similar argument we get $h \in \mathscr{P}(A) \times \mathscr{P}(B)$.

Thus $\mathscr{P}(A \times B) \subset [\mathscr{P}(A) \times (U_B \setminus S_B)] \cup [(U_A \setminus S_A) \times \mathscr{P}(B)] \cup [\mathscr{P}(A) \times \mathscr{P}(B)]$.

Conversely, let $h \in [\mathscr{P}(A) \times (U_B \setminus S_B)] \cup [(U_A \setminus S_A) \times \mathscr{P}(B)] \cup [\mathscr{P}(A) \times \mathscr{P}(B)]$. Suppose $h = (f, g) \in \mathscr{P}(A) \times (U_B \setminus S_B)$ with $f \in \mathscr{P}(A)$ and $g \in (U_B \setminus S_B)$. Then $\sigma_{\pi_A}(f) = \{1\}$, i.e., $\|f\| = 1$ and $\|g\| < 1$. So $\|h\| = 1$. Thus $\|f\| > \|g\|$. Then by Theorem 2.1 (c), $\sigma_{\pi_{A \times B}}(h) = \{1\}$. So $h \in \mathscr{P}(A \times B)$.

Similarly, if $h \in (U_A \setminus S_A) \times \mathscr{P}(B)$, then also $h \in \mathscr{P}(A \times B)$.

Suppose $h = (f, g) \in \mathscr{P}(A) \times \mathscr{P}(B)$. Then $\sigma_{\pi_A}(f) = \{1\} = \sigma_{\pi_B}(g)$, i.e., $\|f\| = \|g\| = 1$. Hence $\|h\| = 1$. Then by Theorem 2.1 (b), $\sigma_{\pi_{A \times B}}(h) = \{1\}$, i.e., $h \in \mathscr{P}(A \times B)$. Hence $\mathscr{P}(A \times B) \supset [\mathscr{P}(A) \times (U_B \setminus S_B)] \cup [(U_A \setminus S_A) \times \mathscr{P}(B)] \cup [\mathscr{P}(A) \times \mathscr{P}(B)]$. Hence the result.

Note that in above result the sets on right hand side are mutually disjoint.

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