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Deformations of finite hypergroups

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Abstract. The purpose of the present paper is to introduce q-deformations of finite groups of low order, for examples, cyclic groups, symmetric groups, dihedral groups and the quaternion group in the category of hypergroups. Moreover we discuss q-deformations of certain finite hypergroups.

1 Introduction

We investigate q-deformations of finite groups and finite hypergroups in the category of hypergroups. It is known that there is no q-deformations of finite groups in the category of quantum groups ([24]). However we introduce that there are many q-deformations of finite groups in the category of hypergroups.

Hypergroups $\mathbb{Z}_q(2)$ of order two with a parameter q $(0 < q \leq 1)$ are interpreted as q-deformations of the cyclic group $\mathbb{Z}_2$. This fact is our motivation that we started to investigate q-deformations of finite groups and finite hypergroups.

In section 3, we discuss q-deformations of the cyclic group $\mathbb{Z}_3$ of order three and the cyclic group $\mathbb{Z}_4$ of order four. In section 4, we discuss q-deformations of the symmetric group S_3, the dihedral group D_4 and quaternion group Q_4. These q-deformations are obtained by applying a notion of a semi-direct product hypergroup introduced by H. Heyer and S. Kawakami (see [5]).

Moreover we study q-deformations of certain finite hypergroups of low order, the orbital hypergroups $\mathcal{K}^\alpha(\mathbb{Z}_3)$ of $\mathbb{Z}_3$ and $\mathcal{K}^\alpha(\mathbb{Z}_4)$ of $\mathbb{Z}_4$, the character hypergroups $\mathcal{K}(\widehat{S_3})$ of S_3, $\mathcal{K}(\widehat{D_4})$ of D_4 and $\mathcal{K}(\widehat{Q_4})$ of Q_4, the conjugacy class hypergroups $\mathcal{K}(S_3)$ of S_3, $\mathcal{K}(D_4)$ of D_4 and $\mathcal{K}(Q_4)$ of Q_4 in section 5.

2 Preliminaries

For a finite set $K = \{c_0, c_1, \cdots, c_n\}$, we denote by $M^b(K)$ and $M^1(K)$, the set of all complex valued measures on K and the set of all non-negative probability measures on K respectively, namely

$$M^b(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} \ : \ a_j \in \mathbb{C} \ \ (j = 0, 1, 2, \cdots, n) \right\},$$

$$M^1(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} \ : \ a_j \geq 0 \ \ (j = 0, 1, 2, \cdots, n), \ \sum_{j=0}^{n} a_j = 1 \right\}$$

where the symbol δ_c stands for the Dirac measure in $c \in K$. For $\mu = a_0\delta_{c_0} + a_1\delta_{c_1} + \cdots + a_n\delta_{c_n} \in M^b(K)$, the *support* of μ is

$$\mathrm{supp}(\mu) := \{c_j \in K \ : \ a_j \neq 0 \ (j = 0, 1, 2, \cdots, n)\}.$$

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Axiom A *finite hypergroup* $K = (K, M^b(K), \circ, *)$ consists of a finite set $K = \{c_0, c_1, \cdots, c_n\}$ together with an associative product (called convolution) $\circ$ and an involution $*$ in $M^b(K)$ satisfying the following conditions.

(1) The space $(M^b(K), \circ, *)$ is an associative $*$-algebra with unit δ_{c_0}.
(2) For $c_i, c_j \in K$, the convolution $\delta_{c_i} \circ \delta_{c_j}$ belongs to $M^1(K)$.
(3) There exists an involutive bijection $c_i \mapsto c_i^*$ on K such that $\delta_{c_i^*} = \delta_{c_i}^*$. Moreover $c_j = c_i^*$ if and only if $c_0 \in \mathrm{supp}(\delta_{c_i} \circ \delta_{c_j})$ for all $c_i, c_j \in K$.

A finite hypergroup K is called *commutative* if the convolution $\circ$ on $M^b(K)$ is commutative.

Let K and L be finite hypergroups. A mapping $\varphi : K \to L$ is called a (*hypergroup*) *homomorphism* of K into L if there exists a $*$-homomorphism $\tilde{\varphi}$ of $M^b(K)$ into $M^b(L)$ as $*$-algebras such that $\delta_{\varphi(c)} = \tilde{\varphi}(\delta_c)$. If $\tilde{\varphi}$ is bijective, φ is called an *isomorphism* of K onto L. In the case that $L = K$, an isomorphism $\varphi : K \to K$ is called an *automorphism* of K. The set of all automorphisms of K becomes a group and it is denoted by $\mathrm{Aut}(K)$. Let G be a finite group. A homomorphism $\alpha : G \to \mathrm{Aut}(K)$ is called an action of G on K.

For a commutative hypergroup K, a complex-valued function χ on K is called a *character* if χ is linearly extendable on $M^b(K)$ to be $\tilde{\chi}(\delta_{c_i}) = \chi(c_i)$ and satisfying that $\tilde{\chi}(\delta_{c_0}) = 1$, $\tilde{\chi}(\delta_{c_i} \circ \delta_{c_j}) = \tilde{\chi}(\delta_{c_i})\tilde{\chi}(\delta_{c_j})$ and $\tilde{\chi}(\delta_{c_i}^*) = \overline{\tilde{\chi}(\delta_{c_i})}$ for all $c_i, c_j \in K$. We denote the trivial character by χ_0. Let $\hat{K}$ be the set of all characters of K. A convolution on $\hat{K}$ is defined by multiplication of functions on K. Then $\hat{K}$ becomes a signed hypergroup and the duality $\hat{\hat{K}} \cong K$ holds.

Conjugacy class hypergroup Let G be a finite group. For $g \in G$, put $\alpha_g(k) = Ad_g(k) = gkg^{-1}$ $(k \in G)$. Then α is an action of G on G. Hence we obtain the orbital hypergroup $\mathcal{K}^\alpha(G)$ which we denote by $\mathcal{K}(G)$ which is called a conjugacy class hypergroup of G.

Character hypergroup For a finite group G, $\hat{G} = \{\pi_0, \pi_1, \cdots, \pi_m\}$ is the set of the all equivalence classes of irreducible representations of G. For $\pi_j \in \hat{G}$, a character χ_j associated with π_j is defined by

$$\chi_j(g) = \frac{1}{\dim \pi_j} \mathrm{tr}(\pi_j(g)).$$

Then $\mathcal{K}(\hat{G}) = \{\chi_0, \chi_1, \cdots, \chi_m\}$ becomes a commutative hypergroup with unit χ_0 by the multiplication of functions on G.

Hypergroup join For two finite hypergroups $H = \{h_0, h_1, \cdots, h_m\}$ and $L = \{\ell_0, \ell_1, \cdots, \ell_k\}$, a hypergroup join

$$H \vee L = \{h_0, h_1, \cdots, h_m, \ell_1, \cdots, \ell_k\}$$

is defined by the convolution $\diamond$ whose structure equations are

$$\begin{aligned} &\delta_{h_i} \diamond \delta_{h_j} = \delta_{h_i} \circ \delta_{h_j}, \quad \delta_{h_i} \diamond \delta_{\ell_j} = \delta_{\ell_j}, \\ &\delta_{\ell_i} \diamond \delta_{\ell_j} = \delta_{\ell_i} \circ \delta_{\ell_j} \text{ when } \ell_j \neq \ell_i^*, \\ &\delta_{\ell_i} \diamond \delta_{\ell_i}^* = n_i^0 \omega(H) + \sum_{j=1}^{k} n_i^j \delta_{\ell_j} \end{aligned}$$

where $\delta_{\ell_i} \circ \delta_{\ell_i}^* = n_i^0 \delta_{\ell_0} + \sum_{j=1}^{k} n_i^j \delta_{\ell_j}$ and $\omega(H)$ is the normalized Haar measure of H.

3 Deformations of finite abelian groups

Let $K = \{c_0, c_1\}$ be a hypergroup of order two. Then the structure of K is determined by

$$\delta_{c_1} \circ \delta_{c_1} = q\delta_{c_0} + (1-q)\delta_{c_1}$$

where $0 < q \leq 1$. We denote it by $\mathbb{Z}_q(2)$ which is interpreted as a q-deformation of $\mathbb{Z}_2$. Stimulating by this fact, we have started to study q-deformations of finite groups.

3.1 Deformation $\mathbb{Z}_q(3)$ of $\mathbb{Z}_3$

First of all we discuss a q-deformation of $\mathbb{Z}_3$. It is easy to check the following proposition directly and this fact is also described in the paper ([19], [23] and [25]).

Proposition 3.1 Let $K = \{c_0, c_1, c_2\}$ be a hypergroup of order three. For each q $(0 < q \leq 1)$ there exists a unique hypergroup of order three such that $\delta_{c_1}^* = \delta_{c_2}$ and $\delta_{c_1} \circ \delta_{c_2} = q\delta_{c_0} + a_1\delta_{c_1} + a_2\delta_{c_2}$.

We denote the above K by $\mathbb{Z}_q(3)$, which is interpreted as a q-deformation of $\mathbb{Z}_3$. The structure equations of $\mathbb{Z}_q(3) = \{c_0, c_1, c_2\}$ $(0 < q \leq 1)$ are determined by

$$\begin{aligned}
\delta_{c_1} \circ \delta_{c_2} &= q\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}, \\
\delta_{c_1} \circ \delta_{c_1} &= \frac{1-q}{2}\delta_{c_1} + \frac{1+q}{2}\delta_{c_2}, \\
\delta_{c_2} \circ \delta_{c_2} &= \frac{1+q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}.
\end{aligned}$$

Put $\widehat{\mathbb{Z}_q(3)} = \{\chi_0, \chi_1, \chi_2\}$. Then the character table of $\mathbb{Z}_q(3)$ is

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	ω_q	$\overline{\omega_q}$
χ_2	1	$\overline{\omega_q}$	ω_q

where $\omega_q = \frac{-q+i\sqrt{q^2+2q}}{2}$.

By the symmetry of the character table we see that $\widehat{\mathbb{Z}_q(3)} \cong \mathbb{Z}_q(3)$.

3.2 Deformation $\mathbb{Z}_{(p,q)}(4)$ of $\mathbb{Z}_4$

We investigated several kinds of extension problem in the category of commutative hypergroups, refer to [6], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The cyclic group $\mathbb{Z}_4$ of order four is a non-splitting extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Then one can consider a non-splitting extension $\mathbb{Z}_{(p,q)}(4)$ $(0 < p \leq 1,\ 0 < q \leq 1)$ of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2)$ as follows.

Proposition 3.2 (Example 4.2 in [14]) For (p, q) $(0 < p \leq 1,\ 0 < q \leq 1)$ there exists a unique hypergroup $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ of order four, which is an extension hypergroup of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2) = \{c_0, c_2\}$ such that $c_1^* = c_3$.

The structure of $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ $(0 < p \le 1, 0 < q \le 1)$ is given by

$$\begin{aligned}
\delta_{c_1} \circ \delta_{c_1} &= \delta_{c_3} \circ \delta_{c_3} = \frac{1-q}{2}\delta_{c_1} + q\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_2} &= p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_1} \circ \delta_{c_2} = \frac{1-p}{2}\delta_{c_1} + \frac{1+p}{2}\delta_{c_3}, \\
\delta_{c_1} \circ \delta_{c_3} &= \frac{2pq}{1+p}\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{q-pq}{1+p}\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_3} &= \frac{1+p}{2}\delta_{c_1} + \frac{1-p}{2}\delta_{c_3}.
\end{aligned}$$

Put $\widehat{\mathbb{Z}_{(p,q)}(4)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then the character table of $\mathbb{Z}_{(p,q)}(4)$ is

	c_0	c_1	c_2	c_3
χ_0	1	1	1	1
χ_1	1	$i\sqrt{pq}$	$-p$	$-i\sqrt{pq}$
χ_2	1	$-q$	1	$-q$
χ_3	1	$-i\sqrt{pq}$	$-p$	$i\sqrt{pq}$

It is easy to see that $\mathbb{Z}_{(p,q)}(4)$ is interpreted as a (p,q)-deformation of $\mathbb{Z}_4$ and $\widehat{\mathbb{Z}_{(p,q)}(4)} \cong \mathbb{Z}_{(q,p)}(4)$.

4 Deformations of non-abelian finite groups

Let α be an action of a finite group G on a finite hypergroup $H = (H, M^b(H), \circ, *)$. Then a semi-direct product hypergroup $S := H \rtimes_\alpha G$ is introduced in [5]. A convolution $\circ_\alpha$ in $M^b(S)$ is defined by

$$(\varepsilon_{h_1} \otimes \delta_{g_1}) \circ_\alpha (\varepsilon_{h_2} \otimes \delta_{g_2}) := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)} \otimes \delta_{g_1 g_2}),$$

where ε and δ stand for Dirac measures in $M^b(H)$ and $M^b(G)$ respectively. Unit element is $\varepsilon_e \otimes \delta_e$. An involution $^-$ is

$$(\mu \otimes \delta_g)^- := \alpha_g^{-1}(\mu^*) \otimes \delta_{g^{-1}}$$

for all $\mu \in M^b(H)$ and $g \in G$.

4.1 Deformation $S_q(3)$ of the symmetric group S_3

The symmetric group S_3 is a semi-direct product $\mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ where α is an action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$.

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on a hypergroup $\mathbb{Z}_q(3) = \{h_0, h_1, h_2\}$ $(0 < q \le 1)$ such that

$$\alpha_g(h_1) = h_2, \quad \alpha_g(h_2) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$S_q(3) := \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$$

which is a q-deformation of the symmetric group $S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$.

4.2 Deformation $D_{(p,q)}(4)$ of the dihedral group D_4

The dihedral group D_4 is written by a semi-direct product $\mathbb{Z}_4 \rtimes_\alpha \mathbb{Z}_2$.

Let $H = \mathbb{Z}_{(p,q)}(4) = \{h_0, h_1, h_2, h_3\}$ $(0 < p \leq 1, 0 < q \leq 1)$ be the (p,q)-deformation of $\mathbb{Z}_4$ and α an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_{(p,q)}(4)$ given by

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$D_{(p,q)}(4) := \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2.$$

Hence, we obtain a (p,q)-deformation $D_{(p,q)}(4)$ of the dihedral group D_4.

4.3 Another deformation $W_q(4)$ of the dihedral group D_4

The dihedral group D_4 is also written by a semi-direct product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2$ where β is a flip action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2) = \{(h_0, h_0), (h_0, h_1), (h_1, h_0), (h_1, h_1)\ ;\ h_0, h_1 \in \mathbb{Z}_q(2)\}$ be a q-deformation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let β be a flip action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ given by

$$\beta_g((h_i, h_j)) = (h_j, h_i) \quad (i, j = 0 \text{ or } 1).$$

Then we obtain a semi-direct product hypergroup

$$W_q(4) := (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2.$$

The hypergroup $W_q(4)$ is another q-deformation of D_4.

4.4 Deformation $Q_q(4)$ of the quaternion group Q_4

The structure of the quaternion group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is determined by

$$i^2 = j^2 = k^2 = -1, \quad ij = k.$$

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_4 = \{h_0, h_1, h_2, h_3\}$ such that

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Let c be a $\mathbb{Z}_4$-valued 2-cocycle of $\mathbb{Z}_2$ which is also given by

$$c(e,e) = c(e,g) = c(g,e) = h_0 \ \text{ and } \ c(g,g) = h_2.$$

Then a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$ is defined by the product

$$(h, g)(h', g') = (h\alpha_g(h')c(g, g'), gg')$$

for $h, h' \in \mathbb{Z}_4$ and $g, g' \in \mathbb{Z}_2$. The quaternion group Q_4 is isomorphic to $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$. Hence we interpret Q_4 as a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$.

Let $\mathbb{Z}_{(1,q)}(4) = \{h_0, h_1, h_2, h_3\}$ be a q-deformation of $\mathbb{Z}_4$ with a subgroup $\{h_0, h_2\}$ and c a $\mathbb{Z}_{(1,q)}(4)$-valued 2-cocycle which is also given by

$$c(e,e) = c(e,g) = c(g,e) = h_0 \ \text{ and } \ c(g,g) = h_2.$$

Then, we obtain a twisted semi-direct product hypergroup

$$Q_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2.$$

The hypergroup $Q_q(4)$ is a q-deformation of the quaternion group $Q_4 = \mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$.

5 Deformations of finite hypergroups

In this section we discuss q-deformations of several kinds of finite hypergroups in a similar way to the case of finite groups.

5.1 Deformations of orbital hypergroups

Given an action α of a finite group G on a commutative hypergroup H, we obtain a orbit $O = \{\alpha_g(h) \ ; \ g \in G\}$ of $h \in H$ under the action α. Let $\{O_0, O_1, \cdots, O_m\}$ be the set of all orbits in H. We denote an element c_j which is corresponding to each orbit O_j and put $H^\alpha = \{c_0, c_1, \cdots, c_m\}$. Let $M^b(H)^\alpha$ denote the fixed point algebra of $M^b(H)$ under the action α, namely

$$M^b(H)^\alpha = \{\mu \in M^b(H) \ ; \ \alpha_g(\mu) = \mu \text{ for all } g \in G\}.$$

We note that $M^b(H)^\alpha$ is a $*$-subalgebra of $M^b(H)$. For $c_j \in H^\alpha$, put

$$\delta_{c_j} = \frac{1}{|O_j|} \sum_{h \in O_j} \delta_h = \frac{1}{|G|} \sum_{g \in G} \alpha_g(\delta_h).$$

Then $\delta_{c_j} \in M^b(H)^\alpha \cap M^1(H)$. $\mathcal{K}^\alpha(H) = (H^\alpha, M^b(H)^\alpha, \circ, *)$ becomes a hypergroup which is called an orbital hypergroup of H by the action α.

Example 1 The orbital hypergroup $\mathcal{K}^\alpha(\mathbb{Z}_q(3)) = \{c_0, c_1\}$ is a q-deformation of $\mathcal{K}^\alpha(\mathbb{Z}_3)$.

The structure equations are

$$\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + \left(1 - \frac{q}{2}\right)\delta_{c_1}.$$

Remark $\mathcal{K}^\alpha(\mathbb{Z}_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2)$.

Example 2 The orbital hypergroup $\mathcal{K}^\alpha(\mathbb{Z}_{(p,q)}(4)) = \{c_0, c_1, c_2\}$ is a q-deformation of $\mathcal{K}^\alpha(\mathbb{Z}_4)$.

The structure equations are

$$\delta_{c_1} \circ \delta_{c_1} = p\delta_{c_0} + (1-p)\,\delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_2} = \delta_{c_2},$$
$$\delta_{c_2} \circ \delta_{c_2} = \frac{pq}{1+p}\delta_{c_0} + \frac{q}{1+p}\delta_{c_1} + (1-q)\delta_{c_2}.$$

Remark $\mathcal{K}^\alpha(\mathbb{Z}_{(p,q)}(4)) = \mathbb{Z}_p(2) \vee \mathbb{Z}_q(2)$.

5.2 Deformations of character hypergroups of semi-direct product hypergroups

Let $S = H \rtimes_\alpha G$ be a semi-direct product hypergroup defined by an action α of a finite abelian group G on a finite commutative hypergroup H (Refer to [5]). $\hat{S} = \widehat{H \rtimes_\alpha G}$ is the set of all equivalence classes of irreducible representations of S. For $(\pi, \mathcal{H}(\pi)) \in \hat{S}$, the character $ch(\pi)$ of π is defined by

$$ch(\pi)((h,g)) = \frac{1}{\dim \pi}\mathrm{tr}(\pi(h,g))$$

where $(h,g) \in H \rtimes_\alpha G$ and tr is the trace of $B(\mathcal{H}(\pi))$. Put $\mathcal{K}(\hat{S}) = \{ch(\pi) \ ; \ \pi \in \hat{S}\}$.

Proposition 5.1 ([5] and [7]) If the action α satisfies the regularity condition, then $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ becomes a commutative hypergroup by the product of functions on $S = H \rtimes_\alpha G$.

This hypergroup is called a character hypergroup of the semi-direct product hypergroup $S = H \rtimes_\alpha G$.

Example 3 The character hypergroup $\mathcal{K}(\widehat{S_q(3)})$ of $S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$ is a q-deformation of $\mathcal{K}(\widehat{S_3})$.

$\widehat{S_q(3)} = \widehat{H \rtimes_\alpha G} = \{\chi_0 \odot \tau_0, \chi_0 \odot \tau_1, \pi\}$, where π is a two-dimensional irreducible representation of $S_q(3)$. $\mathcal{K}(\widehat{S_q(3)}) = \{ch(\chi_0 \odot \tau_0), ch(\chi_0 \odot \tau_1), ch(\pi)\}$. The character table is

	(h_0, e)	(h_1, e)	(h_2, e)	(h_0, g)	(h_1, g)	(h_2, g)
$\gamma_0 = ch(\chi_0 \odot \tau_0)$	1	1	1	1	1	1
$\gamma_1 = ch(\chi_0 \odot \tau_1)$	1	1	1	-1	-1	-1
$\gamma_2 = ch(\pi)$	1	$-\frac{q}{2}$	$-\frac{q}{2}$	0	0	0

and the structure equations of $\mathcal{K}(\widehat{S_q(3)})$ are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{q}{4}\gamma_0 + \frac{q}{4}\gamma_1 + \left(1 - \frac{q}{2}\right)\gamma_2, \quad \gamma_1\gamma_2 = \gamma_2.$$

Example 4 The character hypergroup $\mathcal{K}(\widehat{D_{(p,q)}(4)})$ of $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2$ is a (p, q)-deformation of $\mathcal{K}(\widehat{D_4})$.

The structure equations of $\mathcal{K}(\widehat{D_{(p,q)}(4)}) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ are

$$\begin{aligned}
&\gamma_1\gamma_1 = \gamma_0, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_2,\\
&\gamma_2\gamma_2 = \gamma_3\gamma_3 = q\gamma_0 + (1-q)\gamma_2, \quad \gamma_2\gamma_3 = q\gamma_1 + (1-q)\gamma_3,\\
&\gamma_4\gamma_4 = \frac{pq}{2(1+q)}\gamma_0 + \frac{pq}{2(1+q)}\gamma_1 + \frac{p}{2(1+q)}\gamma_2 + \frac{p}{2(1+q)}\gamma_3 + (1-p)\gamma_4,\\
&\gamma_1\gamma_4 = \gamma_4, \quad \gamma_2\gamma_4 = \gamma_4, \quad \gamma_3\gamma_4 = \gamma_4.
\end{aligned}$$

Example 5 The character hypergroup $\mathcal{K}(\widehat{Q_q(4)})$ of $Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2$ is a q-deformation of $\mathcal{K}(\widehat{D_4})$.

The structure equations of $\mathcal{K}(\widehat{Q_q(4)}) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ are

$$\begin{aligned}
&\gamma_1\gamma_1 = \gamma_0, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_2,\\
&\gamma_2\gamma_2 = \gamma_3\gamma_3 = q\gamma_0 + (1-q)\gamma_2, \quad \gamma_2\gamma_3 = q\gamma_1 + (1-q)\gamma_3,\\
&\gamma_4\gamma_4 = \frac{q}{2(1+q)}\gamma_0 + \frac{q}{2(1+q)}\gamma_1 + \frac{1}{2(1+q)}\gamma_2 + \frac{1}{2(1+q)}\gamma_3,\\
&\gamma_1\gamma_4 = \gamma_4, \quad \gamma_2\gamma_4 = \gamma_4, \quad \gamma_3\gamma_4 = \gamma_4.
\end{aligned}$$

5.3 Deformations of generalized conjugacy class hypergroups

Let $S = H \rtimes_\alpha G$ be a semi-direct product hypergroup. Then there exists the canonical conditional expectation E from $M^b(S)$ onto the center $Z(M^b(S))$ of $M^b(S)$. Put

$$\mathcal{K}(H \rtimes_\alpha G) := \{E(\delta_{(h,g)}) \ ; \ (h, g) \in H \rtimes_\alpha G\}.$$

Proposition 5.2 ([6]) If the action α satisfies the regularity condition, then $\mathcal{K}(H \rtimes_\alpha G)$ becomes a commutative hypergroup with the convolution in the center $Z(M^b(S))$. Moreover $\hat{\mathcal{K}}(H \rtimes_\alpha G) \cong \mathcal{K}(\widehat{H \rtimes_\alpha G})$ holds.

We call $\mathcal{K}(H \rtimes_\alpha G)$ a generalized conjugacy class hypergroup of $H \rtimes_\alpha G$.

Example 6 The generalized conjugacy class hypergroup $\mathcal{K}(S_q(3))$ of $S_q(3)$ is a q-deformation of $\mathcal{K}(S_3)$.

The structure equations of $\mathcal{K}(S_q(3)) = \{c_0, c_1, c_2\}$ are

$$\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + \left(1 - \frac{q}{2}\right)\delta_{c_1}, \quad \delta_{c_2} \circ \delta_{c_2} = \frac{q}{q+2}\delta_{c_0} + \frac{2}{q+2}\delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_2} = \delta_{c_2}.$$

Example 7 The generalized conjugacy class hypergroup $\mathcal{K}(D_{(p,q)}(4))$ of $D_{(p,q)}(4)$ is a (p,q)-deformation of $\mathcal{K}(D_4)$.

The structure equations of $\mathcal{K}(D_{(p,q)}(4)) = \{c_0, c_1, c_2, c_3, c_4\}$ are

$$\begin{aligned}
&\delta_{c_1} \circ \delta_{c_1} = \delta_{c_4} \circ \delta_{c_4} = \frac{pq}{1+p}\delta_{c_0} + (1-q)\delta_{c_1} + \frac{q}{1+p}\delta_{c_2}, \\
&\delta_{c_2} \circ \delta_{c_2} = p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_3} \circ \delta_{c_3} = \frac{p}{1+p}\delta_{c_0} + \frac{1}{1+p}\delta_{c_2}, \\
&\delta_{c_1} \circ \delta_{c_2} = \delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_3} = \delta_{c_4}, \quad \delta_{c_1} \circ \delta_{c_4} = q\delta_{c_3} + (1-q)\delta_{c_4}, \\
&\delta_{c_2} \circ \delta_{c_4} = \delta_{c_4}, \quad \delta_{c_2} \circ \delta_{c_3} = \delta_{c_3}, \quad \delta_{c_3} \circ \delta_{c_4} = \delta_{c_1}.
\end{aligned}$$

Example 8 The generalized conjugacy class hypergroup $\mathcal{K}(Q_q(4))$ of $Q_q(4)$ is a q-deformation of $\mathcal{K}(Q_4)$.

The structure equations of $\mathcal{K}(Q_q(4)) = \{c_0, c_1, c_2, c_3, c_4\}$ are

$$\begin{aligned}
&\delta_{c_1} \circ \delta_{c_1} = \delta_{c_4} \circ \delta_{c_4} = \frac{q}{2}\delta_{c_0} + (1-q)\delta_{c_1} + \frac{q}{2}\delta_{c_2}, \\
&\delta_{c_2} \circ \delta_{c_2} = \delta_{c_0}, \quad \delta_{c_3} \circ \delta_{c_3} = \frac{1}{2}\delta_{c_0} + \frac{1}{2}\delta_{c_2}, \\
&\delta_{c_1} \circ \delta_{c_2} = \delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_3} = \delta_{c_4}, \quad \delta_{c_1} \circ \delta_{c_4} = q\delta_{c_3} + (1-q)\delta_{c_4}, \\
&\delta_{c_2} \circ \delta_{c_4} = \delta_{c_4}, \quad \delta_{c_2} \circ \delta_{c_3} = \delta_{c_3}, \quad \delta_{c_3} \circ \delta_{c_4} = \delta_{c_1}.
\end{aligned}$$

By the above structure equations, we have the following theorem.

Theorem There are deformations $S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$ of the symmetric group S_3, $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2$ and $W_q(4) = (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2$ of the dihedral group D_4 and $Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes^c_\alpha \mathbb{Z}_2$ of the quaternion group Q_4 in the category of hypergroups. These deformations have the following properties.

(1) $\mathcal{K}(\widehat{S_q(3)}) = \mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$ and $\mathcal{K}(S_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2) \vee \mathbb{Z}_2$.

(2) $\mathcal{K}(\widehat{D_{(p,q)}(4)})$ is a (q,p)-deformation of $\mathcal{K}(\widehat{D_4})$ and $\mathcal{K}(D_{(p,q)}(4))$ is a (p,q)-deformation of $\mathcal{K}(D_4)$. $\mathcal{K}(\widehat{Q_q(4)})$ is a q-deformation of $\mathcal{K}(\widehat{Q_4})$ and $\mathcal{K}(Q_q(4))$ is a q-deformation of $\mathcal{K}(Q_4)$. Moreover $\mathcal{K}(\widehat{D_{(1,q)}(4)}) \cong \mathcal{K}(\widehat{Q_q(4)})$ and $\mathcal{K}(D_{(1,q)}(4)) \cong \mathcal{K}(Q_q(4))$ although $D_{(1,q)}(4)$ is not isomorphic to $Q_q(4)$.

(3) $\mathcal{K}(\widehat{W_q(4)})$ is not a hypergroup when $q \neq 1$.

Proof (1) We put $\mathbb{Z}_2 = \{b_0, b_1\}$ and $\mathbb{Z}_{\frac{q}{2}}(2) = \{c_0, c_1\}$, where $\delta_{b_1} \circ \delta_{b_1} = \delta_{b_0}$ and $\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + (1 - \frac{q}{2})\delta_{c_1}$ $(0 < q \leq 1)$. The structure of $\mathcal{K}(\widehat{S_q(3)})$ in Example 3 is the same of the hypergroup join $\mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$. Hence $\mathcal{K}(\widehat{S_q(3)}) = \mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$. In a similar way we get $\mathcal{K}(S_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2) \vee \mathbb{Z}_2$ as in Example 6.

(2) The former properties follow directly from above examples 4, 7, 5 and 8. Both of $D_{(1,q)}(4)$ and $Q_q(4)$ are extension hypergroups of $\mathbb{Z}_2$ by $\mathbb{Z}_{(1,q)}(4)$. However $D_{(1,q)}(4)$ is of splitting type but $Q_q(4)$ is of non-splitting type. Hence $D_{(1,q)}(4)$ is not isomorphic to $Q_q(4)$.

(3) We put $\widehat{\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ and $\widehat{\mathbb{Z}_2} = \{\tau_0, \tau_1\}$. Then

$$\widehat{W_q(4)} = \{\chi_0 \odot \tau_0, \chi_0 \odot \tau_1, \chi_3 \odot \tau_0, \chi_3 \odot \tau_1, \pi\},$$

where π is the two-dimensional irreducible representation of $W_q(4)$ given by

$$\pi = \mathrm{ind}_{\mathbb{Z}_q(2)\times\mathbb{Z}_q(2)}^{W_q(4)}(\chi_1 \odot \tau_0).$$

Hence,

$$\mathcal{K}(\widehat{W_q(4)}) = \{ch(\chi_0 \odot \tau_0), ch(\chi_0 \odot \tau_1), ch(\chi_3 \odot \tau_0), ch(\chi_3 \odot \tau_1), ch(\pi)\}.$$

Assume that $\mathcal{K}(\widehat{W_q(4)})$ is a hypergroup for $q \neq 1$. Then

$$ch(\chi_3\odot\tau_0)ch(\chi_3\odot\tau_0) = a_0 ch(\chi_0\odot\tau_0)+a_1 ch(\chi_0\odot\tau_1)+a_2 ch(\chi_3\odot\tau_0)+a_3 ch(\chi_3\odot\tau_1)+a_4 ch(\pi),$$

where $\sum_{j=0}^{4} a_j = 1$ and $a_j \geq 0$ $(j = 0, 1, 2, 3, 4)$. Since

$$ch(\chi_0 \odot \tau_1)(h_0, g) = -1, \quad ch(\chi_3 \odot \tau_1)(h_0, g) = -1,$$

$$ch(\pi)(h_0, g) = 0 \ \text{ and } \ ch(\chi_3 \odot \tau_0)ch(\chi_3 \odot \tau_0)(h_0, g) = 1$$

where h_0 is the unit of $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ and $\mathbb{Z}_2 = \{e, g\}$, $g^2 = e$, we see that

$$a_0 - a_1 + a_2 - a_3 = 1.$$

This implies that $a_1 = 0, a_3 = 0, a_4 = 0$. Hence, we get

$$ch(\chi_3 \odot \tau_0)ch(\chi_3 \odot \tau_0) = a_0 ch(\chi_0 \odot \tau_0) + a_2 ch(\chi_3 \odot \tau_0).$$

Restricting this equality to $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$, we obtain

$$\chi_3\chi_3 = a_0\chi_0 + a_2\chi_3.$$

This contradicts with the fact :

$$\chi_3\chi_3 = q^2\chi_0 + q(1-q)\chi_1 + q(1-q)\chi_2 + (1-q)^2\chi_3.$$

Hence, $\mathcal{K}(\widehat{W_q(4)})$ is not a hypergroup when $q \neq 1$. □

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