ON THE SPACE OF FUZZY NUMBERS

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1 Introduction Ever since Frechet introduced them in 1928, metric spaces have come to stay as a basic aspect of abstract analysis. Several important classes of functions and their modes of convergence typify metric spaces and supplement our understanding of these classes. Zadeh[22], propounded the theory of fuzzy sets and fuzzy logic, in a path breaking publication in 1965 to study quantitatively problems involving uncertainty due to subjective considerations. Since then a number of attempts have been made to endow fuzzy sets with interesting metrics. A metric being a non-negative real-valued function it is natural to explore if it could take values in the set of fuzzy real numbers. Notable contributions along this line are due to Kaleva and Seikkala [12] followed by Felbin [7]. Kaleva [9] had also shown that a fuzzy metric space (in the sense of Kaleva and Seikkala [12]) has a completion unique up to isometry. In another direction Kramosil and Michalek [13] defined a fuzzy metric space in analogy with and equivalent to a statistical metric space as defined by Menger [14]. Inspired by an intermediate function considered by Hausdorff in defining the Hausdorff distance between closed and bounded subsets of a metric space, Erceg [6] defined a pseudo quasimetric as a map satisfying some natural conditions from $L^X \times L^X$ into $[0, \infty]$, L^X being the set of all maps from a set X into L, a completely distributive lattice with order-preserving involution. For fuzzy points, a pseudo metric was defined and studied by Deng [1]. Subsequently Peng Yu Wei [15] simplified the concept of Erceg's pseudo quasi metric and also related his concept and results to Erceg's theory. Later Rodabaugh [17] and subsequently Jian-Zhong Xiao and Xing-hua Zhu [20] examined $L-$ fuzzy real line for a completely distributive lattice L, vis-a-vis Erceg's pseudo metric.

Dubois and Prade [4] defined a fuzzy real number as a continuous function $\mu : \mathbb{R} \to [0, 1]$ vanishing outside a compact interval $[c, d]$ of real numbers such that for some real numbers a and b with $c \leq a \leq b \leq d$, μ increases on $[c, a]$ and decreases on $[b, d]$ and $\mu(x)$ is 1 on $[a, b]$. Goetschel and Voxman [8] modified the assumption of continuity in the definition of Dubois and Prade to upper semicontinuity to avoid any inconsistency, while including the characteristic functions of singleton real numbers. More importantly they defined a metric for this set of fuzzy real numbers, based on the Hausdorff distance between closed and bounded subsets. This metric has found applications in the study of fuzzy random variables (see Puri and Ralescue [16]), fuzzy differential equations (Kaleva [10]) and the calculus of fuzzy real variables (Kaleva [11]) and has been extensively studied by Diamond and Kloeden in their monograph [3]. Besides this metric, other metrics on fuzzy real numbers have also been studied by Voxman [18] (using reducing functions), Yang and Zhang [21] (endograph metric) and Wu Congxion, Hongliang and Xuekun [19] (sendograph metric). Diamond and Kloeden ([3], [2]) may be consulted for further details.

The purpose of this paper is to consider a wider class of fuzzy subsets of real numbers that can be topologized by a family (gauge) of pseudometrics and study its properties. These fuzzy numbers need not have bounded supports, though their supports intersect a

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fixed closed set. In this way this class of fuzzy numbers serves to supplement the existing theory of fuzzy real numbers.

2 The space $F_U(\mathbb{R})$

We recall the following

Definition 2.1. *A fuzzy subset u of a topological space (X,τ) is called upper semi-continuous if $u : X \to [0,1]$ is a mapping such that $[u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ is a closed subset X for each $\alpha \in [0,1]$. A fuzzy subset $u : X \to [0,1]$ is called normal if $\{x : u(x) = 1\} = \{x : u(x) \geq 1\}$ is nonempty.*

We denote the set of all normal upper semi-continuous fuzzy subsets of X by $F_U(X)$. In particular $F_U(\mathbb{R})$ is the set of all normal upper semi-continuous mappings of $\mathbb{R}$ (with the normal topology) into $[0,1]$.

We now prove a representation theorem for members of $F_U(X)$, X being a topological space.

Theorem 2.1. *Let (X,τ) be a topological space and $u \in F_U(X)$, the set of all normal upper semi-continuous fuzzy subsets of X. For each $\alpha \in I = [0,1]$, let $C_\alpha = [u]^\alpha = \{x \in X : u(x) \geq \alpha\}$. Then*

(i) for each $\alpha \in I$, C_α is a nonempty closed subset of X;

(ii) $C_\beta \subseteq C_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$;

(iii) $C_\alpha = \bigcap_{i=1}^{\infty} C_{\alpha_i}$, for each sequence α_i increasing to α in I.

Conversely, if in a topological space (X,τ), there is a family of nonempty closed subsets $\{C_\alpha : \alpha \in I = [0,1]\}$ satisfying properties (i), (ii) and (iii) above, then there is a unique $u \in F_U(X)$ such that $[u]^\alpha = C_\alpha$ for each $\alpha \in [0,1]$.

Proof. Since $u \in F_U(X)$ in an upper semi-continuous map of X into $[0,1]$, $C_\alpha = [u]^\alpha$ in a closed subset of X for each $\alpha \in [0,1]$. Since $C_1 = \{x : u(x) \geq 1\}$ is nonempty, $C_\alpha (\supseteq C_1)$ is nonempty for each $\alpha \in [0,1]$. For $0 \leq \alpha \leq \beta \leq 1$, $C_\beta \subseteq C_\alpha$ is obvious. Thus for $u \in F_U(X)$, (i), (ii) and (iii) are true.

Conversely, suppose $\{C_\alpha : \alpha \in I = [0,1]\}$ is a family of subsets of X satisfying (i)-(iii). Define $u : X \to [0,1]$ by

$$u(x) = sup\{\alpha \in I : x \in C_\alpha\}$$

Clearly u is a well-defined map of X into $[0,1]$, since $C_0 = X$. Since $C_1 \neq \emptyset$, $u(x) = 1$ for some $x \in X$ and so u is normal. For $\alpha \in I$, if $x \in [u]^\alpha$, then $u(x) \geq \alpha$. Let $I_x = \{\beta \in I : x \in C_\beta\}$ and $\alpha^{'} = sup\ I_x$, so that $\alpha^{'} = u(x)$. Clearly $\alpha^{'}\ (= u(x)) \geq \alpha$ and by hypothesis $x \in C_{\alpha'} \subseteq C_\alpha$. So $[u]^\alpha \subseteq C_\alpha$. On the other hand if $x \in C_\alpha$, then $u(x) = sup\ I_x = \alpha^{'} \geq \alpha$ and consequently $x \in [u^\alpha]$, so that $C_\alpha \subseteq [u]^\alpha$. Thus each $[u]^\alpha = C_\alpha$ for $\alpha \in I$ and hence u is an upper semi-continuous function. □

For topological spaces which are sums of an increasing family of proper closed subsets this representation theorem can be stated in a different form. For this we need the following

Definition 2.2. *A topological space (X,τ) is called $F-$ summable if $X = \bigcup\{F_t : t \in P\}$ satisfying the following conditions:*

(i) $(P, \leq)$ is a totally ordered set with a least element $\widehat{o}$;

(ii) every nonempty subset of P has a greatest lower bound in P;

(iii) *each F_t is a nonempty proper closed subset of X and $F_t \geq F_s$ for $t \geq s$, $t, s \in P$. Further $F_t \neq F_s$ for $t > s$.*

Theorem 2.2. *Let X be an $F-$ summable topological space as in Definition 2.2 and $u \in F_U(X)$. Then for each $t \in P$ and $\alpha \in I = [0,1]$, the sets $C_{\alpha,t} = u^{[\alpha]} \cap F_t$ satisfy the following:*

(i) *$C_{\alpha,t}$ is a nonempty closed subset of X for all $t \geq t_0 \in P$ for all $\alpha \in [0,1]$;*

(ii) *$C_{\beta,t} \subseteq C_{\alpha,t}$ for all $0 \leq \alpha \leq \beta \leq 1$ for all $t \in P$;*

(iii) *If $C_{\alpha,t} \neq \emptyset$ and $\alpha_i(\in [0,1]) \uparrow \alpha$ then $C_{\alpha,t'} = \cap_{i=1}^{\infty} C_{\alpha_i,t'}$ for all $t^{'} \geq t$;*

(iv) *$[u]^\alpha = \cup_{t \in P} C_{\alpha,t}$ is closed for each $\alpha \in I$.*

Conversely, if X is an $F-$ summable topological space (as in Definition 2.2) and $C_{\alpha,t}$, $\alpha \in [0,1]$, $t \in P$ is a family of closed subsets of X satisfying $(i) - (iv)$ above. Then there exists a unique $u \in F_U(X)$ such that for each $\alpha \in I$ and $t \in p$, $[\ u\ ]^\alpha \cap F_t = C_{\alpha,t}$.

Proof. While the proof of necessity part of the theorem is straight-forward, for proving the sufficiency part, define $u : X \to [0,1]$ by $u(x) = sup\{\alpha \in [0,1] :\ x \in C_{\alpha,t}\ for\ least\ t \in P\}$. Since $x \in X = \cup_{t \in P} F_t$, $x \in F_t$ for smallest $t \in P$ and $1 \geq u(x) \geq 0$. Let $\alpha_0 = u(x)$. Then $x \in C_{\alpha_0,t_0}$ clearly $[u]^\alpha \cap F_{t_0} = C_{\alpha_0,t_0}$. Further $[u]^\alpha = \cup_{t \geq t_0} C_{\alpha,t}$ is closed, by (iv). Thus u is upper semi-continuous. Since $C_{1,t}$ is a nonempty closed subset of X for some t_0, $[u]^1 = \cup_{t \geq t_0} C_{1,t}$ is a closed set by (iv) and u is normal. Thus $u \in F_U(X)$. ☐

3 A topology on a subspace of $F_U(\mathbb{R})$ Let (X, d) be a metric space and $F_U(X)$ the set of all normal upper semi-continuous fuzzy subsets of X. For a fixed element a of X, let B_n denote the closed ball in X centered at a and radius r_n and H_n be the Hausdorff metric on the nonempty closed subsets of B_n for each $n \in \mathbb{N}$. As B_n is bounded, the Hausdorff distance H_n induced by d is well-defined on the family of nonempty closed subsets of B_n. We recall the following

Definition 3.1. *Let d_λ be a pseudometric on a nonempty set X for each $\lambda \in \Lambda$. The family $D = \{d_\lambda : \lambda \in \Lambda\}$ is called separating if for $x, y \in X$ with $x \neq y$, there exists $\lambda_0 \in \Lambda$ such that $d_{\lambda_0}(x, y) > 0$. The topology $\tau(D)$ with the subbase $\{B(x; d_\lambda, t) :\ x \in X, \lambda \in \Lambda$ and $\epsilon > 0\}$ is called the topology on X induced by the family D. D is called a gauge and a topological space whose topology admits a gauge structure is called a gauge space.*

Definition 3.2. *Let (X, D) be a gauge space and (x_n), a sequence in D is called Cauchy if $lim_{n,m \to \infty} d_\lambda(x_n, x_m) = 0$ for each $d_\lambda \in D$. If every Cauchy sequence in (X, D) converges to a limit, (X, D) is called sequentially complete.*

We also recall the following

Theorem 3.1. *(see Dugundji [5]) A topological space is a gauge space if and only if it is completely regular (or Tychonoff). A gauge space is metrizable if and only if it has a countable gauge.*

With these preliminaries, we can provide a metric topology on $CL_1(X)$ for any metric space (X, d), that have a non-void intersection with $B(a; r_1)$.

Theorem 3.2. *Let (X, d) be a metric space. Then H_n is a pseudo-metric on $CL_1(X)$, the set of all nonvoid closed subsets of X, that have a non-void intersection with $B(a; r_1)$ for each $n \in \mathbb{N}$, where $H_n(A, B) = H(A \cap B_n,\ B \cap B_n)$ for $A, B \in CL_1(X)$, H being the Hausdorff distance induced by d. Then $CL_1(X)$ is a gauge space with the gauge $\{H_n : x \in \mathbb{N}\}$ and is metrizable assuming that $lt_{n\to\infty} r_n = \infty$ and $\sum \frac{r_n}{2^n}$ converges. If X is complete then $CL_1(X)$ is also complete.*

Proof. Since H_n is the Hausdorff metric on $CL(B_n)$, H_n is a pseudo-metric on $CL(X)$ for each $n \in \mathbb{N}$. For $A \neq B$ in $CL(X)$, $A \cap B_n \neq B \cap B_n$ for some $n = n_0$. So $H_{n_0}(A, B) = r > 0$. Thus $\{H_n : n \in \mathbb{N}\}$ is a countable separating family of pseudometrics on $CL_1(X)$. Thus $CL_1(X)$ is a metrizable gauge space, in view of Theorem 3.1. Since $H_n(A, B) \leq r_n$ for all $A, B \in CL_1(X)$ and $n \in \mathbb{N}$, and $\sum_1^\infty \frac{r_n}{2^n}$ converges. $H(A, B) = \sum_1^\infty \frac{H_n(A,B)}{2^n}$ defines a metric on $CL_1(X)$. Further, this metric topology is the same as the gauge topology (we take $H_n(A, B) = 0$ whenever $F_n \cap A$ or $F_n \cap B = \phi$).

Let $\{C_n\}$ be a Cauchy sequence in $CL_1(X)$. Without loss of generality, we can assume that $\sum_1^\infty H(C_i, C_{i+1}) < \infty$. For C_1 and C_2 we can find n_1 so that $C_1 \cap F_{n_1}$ and $C_2 \cap F_{n_1} \neq \phi$. So for $x_1 \in C_1 \cap F_{n_1}$, noting that $\frac{H_{n_1}}{2^{n_1}}$ is the Hausdorff metric on F_{n_1} induced by $\frac{d}{2^{n_1}}$, we can find $x_2 \in C_2 \cap F_{n_1}$ such that

$$\tfrac{d(x_1,x_2)}{2^{n_1}} < \tfrac{1}{2^{n_1}}(H_{n_1}(C_1, C_2) + 1).$$

For this n_1, we can find $n_2 > n_1$ so that $C_2 \cap F_{n_2}$ and $x_3 \in C_3 \cap F_{n_2} \neq \phi$. Since $\frac{1}{2^{n_2}} d$ induces $\frac{1}{2^{n_2}} H_{n_2}$, a Hausdorff metric on F_{n_2}, we can find $C_3 \cap F_{n_3}$ so that

$$\tfrac{d(x_2,x_3)}{2^{n_2}} < \tfrac{1}{2^{n_2}}(H_{n_2}(C_2, C_3) + \tfrac{1}{2}).$$

Thus proceeding we get a sequence of elements $(x_k) \in C_n \cap F_{n_k}$ so that

$$\frac{d(x_k, x_{k+1})}{2^{n_k}} < \frac{1}{2^{n_k}}(H_{n_k}(C_k, C_{k+1}) + \frac{1}{2^k}). \tag{1}$$

Since $\sum_{i=1}^\infty H(C_i, C_{i+1})$ is convergent,

$$\textstyle\sum_{k=1}^\infty H_{n_k}(C_{n_k}, C_{n_{k+1}}) + \frac{1}{2^k}$$

converges. From (1) it follows that $d(x_k, x_{k+1}) < H_{n_k}(C_{n_k}, C_{n_{k+1}}) + \frac{1}{2^k}$ and so $\sum_{k=1}^\infty d(x_k, x_{k+1})$ is finite. Hence $\{x_n\}$ is a Cauchy sequence that converges to some element x by the completeness of X. Since $x_k \in C_k$ for each k, $x_n \in \cup_{n\geq k} C_n$ for all $n \geq k$. So $x^* = lim_{n>k} x_n$, $x^* \in \overline{\bigcup_{k\geq n} C_k}$ for all k. Thus $x^* \in \cap_{n=1}^\infty (\overline{\bigcup_{k\geq n}^\infty C_k})$.

Define $C = \cap_{n=1}^\infty (\overline{\bigcup_{k\geq n}^\infty C_k})$. Then C is nonempty and closed and is the closure of the set of all limit points of $\{x_n\}$. We now show that $H(C, C_n) \to 0$ as $n \to \infty$. For any $\epsilon > 0$ given, let $n = N(\epsilon)$ be chosen so that $\sum_{n=N(\epsilon)}^\infty [H(C_n, C_{n+1}) + \frac{1}{2^n}] < \frac{\epsilon}{2}$. Let $x^* \in C$ and x_0 be the limit of sequence (x_n) so that $d(x^*, x_0) < \frac{\epsilon}{2}$. Then the distance of x^* from $C_k = d(x^*, C_k)$ is

$$\begin{aligned} &\leq d(x^*, x_0) + \textstyle\sum_{n=k}^\infty d(x_n, x_{n+1}) \\ &< d(x^*, x_0) + \textstyle\sum_{n=k}^\infty [H(C_n, C_{n+1}) + \frac{1}{2^n}] \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \text{ for } k \geq N(\epsilon). \end{aligned}$$

Since any $x_k \in C_k$ can be the starting point of such a convergent sequence (x_n) converging to x_0,

$$d(x_k, x_0) \leq \textstyle\sum_{n\geq k}^\infty d(x_n, x_{n+1}) < \sum_{n\geq k}^\infty [H(C_n, C_{n+1}) + \frac{1}{2^n}]$$

$$< \frac{\epsilon}{2}, \text{ for all } k \geq N(\epsilon).$$

So it follows that $H(C, C_k) = max\{sup_{x^* \in C} d(x^*, C_k), sup_{x_k \in C_k} d(x_k, C)\} < \epsilon$ for $k \geq N(\epsilon)$. Thus $lim_{k\to\infty} C_k = C$ in $CL_1(X)$. Thus for a complete metric space (X, d) that is the countable union of closed spheres $B(a; r_n)$ where (r_n) increases to ∞ with $\sum \frac{r_n}{2^n} < \infty$, the set of all non-void closed subsets of X that intersect $B(a; r_1)$ (and hence $B(a; r_n)$ for all n) can be given a complete metric using the Hausdorff metric on $\subset B(a; r_n)$. Consider a subclass $F_U^1(\mathbb{R})$ of $F_U(\mathbb{R})$ comprising upper semicontinuous functions $u : \mathbb{R} \to [0, 1]$ such that $[u]^1 \subseteq [-r_1, r_1]$ where $\mathbb{R} = \cup_{n=1}^{\infty}[-r_n, r_n]$, $0 < r_n$, $lim_{n\to\infty} r_n = \infty$ and $\sum \frac{r_n}{2^n} < +\infty$. Clearly for such functions the level sets need not be compact nor convex. Although for such functions, the level of normality has to lie in $[-r_1, r_1]$, by choosing r_1 sufficiently large many fuzzy numbers with compact support can be found in $F_U^1(\mathbb{R})$. The following theorem shows that $F_U^1(\mathbb{R})$ and more generally $F_U^1(X)$ admits a complete metric so that analysis can be carried out in $F_U^1(X)$. □

Theorem 3.3. *Let (X, d) be a complete metric space. Suppose $X = \cup_{n=1}^{\infty} B(a; r_n)$ where $B(a; r_n)$ is the closed sphere centered at a and radius r_n with $lim_{n\to\infty} r_n = +\infty$ and $\sum_1^{\infty} \frac{r_n}{2^n} < \infty$. Let $F_U^1(X)$ be the set u of all normal upper semicontinuous fuzzy subsets of X, so that $[u]^1 \cap B(a, r_1) \neq \emptyset$. Then $F_U^1(X)$ is a complete metric space under the metric $\triangle$ defined by $\triangle(u, v) = sup_{0\leq\alpha\leq1} H([u]^\alpha, [v]^\alpha)$ where $H(A, B) = \sum_{n=1}^{\infty} \frac{H_n(A,B)}{2^n}$ (as defined in Theorem 3.2), for $A, B \in CL_1(X)$.*

Proof. Clearly $F_U^1(X)$ is nonempty, as the characteristic function of $B(a, r_1)$ is in $F_U^1(X)$. For $u \in F_U^1(X)$, for all $\alpha \in [0, 1]$, the closed sets $[u]^\alpha \supseteq [u]^1$ and the nonempty set $[u]^1 \subseteq B(a, r_1)$. So for $u, v \in F_U^1(X)$, for $0 \leq \alpha \leq 1$,

$$H([u]^\alpha, [v]^\alpha) = \sum_{n=1}^{\infty} \frac{H_n([u]^\alpha, [v]^\alpha)}{2^n} \leq \sum_{n=1}^{\infty} \frac{r_n}{2^n} = k < \infty,$$

for $sup_{0\leq\alpha\leq1} H([u]^\alpha, [v]^\alpha) = \triangle(u, v) \leq k$ is well-defined. Also for $0 \leq \alpha \leq 1$, $u, v, w \in F_U^1(X)$

$$H([u]^\alpha, [v]^\alpha) \leq H([u]^\alpha, [w]^\alpha) + H([w]^\alpha, [v]^\alpha)$$

and so $\triangle(u, v) \leq \triangle(u, w) + \triangle(w, v)$. Thus $(F_U^1(X), \triangle)$ is a metric space.

For proving the completeness of $F_U^1(X)$ under $\triangle$, consider a Cauchy sequence u_n in $F_U^1(X)$. So given $\epsilon > 0$, we can find $M(\epsilon) \in \mathbb{N}$ such that $\triangle(u_k, u_m) < \epsilon$ for all $k, m \geq M(\epsilon)$. Let $H_n^1(u, v) = sup_{0\leq\alpha\leq1} H_n([u]^\alpha, [v]^\alpha)$ for each $n \in \mathbb{N}$. Since the gauge $\{H_n^1 : n \in \mathbb{N}\}$ generates $\triangle$ and $\{u_n\}$ is Cauchy with respect to $\{H_n^1 : n \in \mathbb{N}\}$, it follows that $\{[u]_n^\alpha \cap B_n\}$ is uniformly Cauchy in α for a fixed n and being a Cauchy sequence of closed sets in the complete space $CL(B_n)$, $[u]_n^\alpha \cap B_n$ converges to $C^\alpha \cap B_n$ for each n uniformly in α in $CL(B_n)$. Clearly the family of closed sets $\{C^\alpha \cap B_n : \alpha \in [0, 1], n \in \mathbb{N}\}$ satisfies the conditions of Theorem 2.2 and so there exists a function u in $F_U^1(X)$ for which $[u]^\alpha = \cup_{n=1}^{\infty} C^\alpha \cap B_n = C^\alpha$ is closed for $0 \leq \alpha \leq 1$. Further $H_n^1(u, u_m) \to 0$ as $m \to \infty$ for each $n \in \mathbb{N}$. Thus $F_U^1(X)$ is complete. □

Remark 3.1. *If we specialise X to $\mathbb{R}$ or $\mathbb{R}^n$ $(n > 1)$, the $F_U^1(X)$ is a special space of fuzzy numbers whose support can be unbounded. It will also contain all fuzzy numbers with support lying in a prescribed interval. In a sense this can supplement the space (E^n, d_∞) considered notably by Kaleva [10] and Kloeden and Diamond [3].*

4 An alternative approach While the space $F_U^1(X)$ complements E^1 or E^n with d_∞ for $X = \mathbb{R}^1$ *or* $\mathbb{R}^n$ respectively, $F_U^1(\mathbb{R})$ or $F_U^1(\mathbb{R}^n)$ does not contain E^1 or E^n. However, this situation can be remedied in the following manner: for a metric space (X, d), the metric topology induced is the same as the topology induced by the bounded metric d^* defined by $d^* = min\{1, d(x, y)\}$ for $x, y \in X$ so that (X, d^*) is complete whenever (X, d) is complete. The following theorem is easy to prove.

Theorem 4.1. *Let (X, d) be a metric space and d^* be defined by $d^*(x, y) = min\{1, d(x, y)\}$ for $x, y \in X$. Then (X, d^*) is a metric space and H^* be the Hausdorff metric induced by d^* on $CL(X)$, the set of all non-void closed subsets of X. If d is complete, then d^* is complete, Further $(CL(X), H^*)$ is also complete.*

This enables us to define a metric on $F_U(X)$, the space of normal upper semi-continuous fuzzy subsets of metric space (X, d) into $[0, 1]$. Again the proof of the following theorem is straight forward.

Theorem 4.2. *Let (X, d) be a complete metric space. Then $F_U(X)$, the space of all normal upper semi-continuous fuzzy subsets of X is a complete metric space with the metric D^* defined by $D^* = sup_{0 \leq \alpha \leq 1} H^*([u]^\alpha, [v]^\alpha)$ for $u, v \in F_U(X)$, H^* being the Hausdorff metric on $CL(X)$ induced by $d^* = min\{1, d\}$.*

Remark 4.1. *Besides D^*, other bounded metrics homeomorphic to d can be used to generate Hausdorff metrics on $CL(X)$. This, in turn can be used to metrize $F_U(X)$, the space of normal upper semi-continuous fuzzy subsets of X.*

Remark 4.2. *If (X, d) is a real normed linear space, then taking $r_n = n \in \mathbb{N}$ and $a = 0$, the zero vector, it can be seen that the maps ϕ_t defined by*

$$\phi_t(x) = \begin{cases} 1 & x = 1, \\ t & x \in [0, 1), \\ 0 & otherwise \end{cases} \tag{2}$$

are in $F_U^1(\mathbb{R})$ and $\triangle(\phi_t, \phi_s) \geq \frac{1}{2}$. Consequently $F_U^1(X)$ containing an isometric copy of $F_U^1(\mathbb{R})$ is not separable.

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