Risk Analysis of Portfolio Based on Kernel Density Estimation-Maximum Likelihood Method and Monte Carlo Simulation

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ABSTRACT.

Nowadays one of the most studied issues in economic or finance field is to get the best possible return with the minimum risk. Therefore, the objective of the paper is to select the optimal investment portfolio from SP500 stock market and CBOE Interest Rate 10-Year Bond to obtain the minimum risk in the financial market.

For this purpose, the paper consists of: 1) the marginal density distribution of the two financial assets is described with kernel density estimation to get the "high-picky and fat-tail" shape; 2) the relation structure of assets is studied with copula function to describe the correlation of financial assets in a nonlinear condition; 3) value at Risk (VaR) is computed through the combination of Copula method and Monte Carlo simulation to measure the possible maximum loss better.

Therefore, through the above three steps methodology, the risk of the portifolio is described more accuratly than the conventional method, which always underestimates the risk in the finicial market.

So it is necessary to pay attention to the happening of extreme cases like "Black Friday 2008" and appropriate investment allocation is a wise strategy to make diversification and spread risks in financial market.

uzzy regression model, fuzzy random variable, expected value, variance, confidence interval.

1 Introduction In finance market, with fierce volatility, the risk management has become a hot research issue in the study. Especially after the accident happened such as the closing down of Barings Bank and the bankrupt of Enron Corp, in the analysis of portfolio the emphasis has moved on the balance between profit and safety.

For the conventional methods, person coefficient is used to measure the correlation of variables and Risk metrics are common ways to calculate VaR. However, due to the assumption of the methods are based on normal distribution, the methods deviate from the real situation more or less.

Therefore, it is necessary to propose a new assets allocation method to evaluate the risk of portfolio in the financial market.

Firstly, according to Markowitz 1987[17]; Terrance. C. Mills 2002[18], the assumption that the distribution of assets return rate submits normal distribution always neglects the happening of extreme conditions, which results in lack of precaution and huge losses in the

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Table 1: Tabe l

	Distribution	Theory	Relation Structure	Risk at Value
Conventional	Normal Distribution	Central Limit Theorem	Person Coefficient	Risk Metrics
Burgeoning	High-picky; Fat-tail	Kernel Density Estimation	Copula Function	Monte Carlo S

end. Meanwhile, lots of experiments have indicated the return curve presents "high-picky" and "fat-tail". So it is necessary to estimate the probability distribution density of asset return with kernel smoothing under a wide precondition.

Secondly, from Embrechts 1999[7], based on figuring out the marginal density distribution of financial assets, the study of relation structure between two financial assets is an important step in the asset allocation and risk management. In the premise of normal distribution, Pearson correlation is a common option to describe the linear relationship. However, some defects such as restricted variance, and easy to be distorted show its bounded-ness in the nonlinear application.

Therefore, from Sklar1959 [30]; Nelsen 1999[19], Copula model is introduced and widely used as a link function $C(u_1, u_2, \cdots, u_N)$ to define the simultaneous distribution $F(x_1, x_2, \cdots, x_N)$ according to the marginal distribution $F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_N}(x_N)$ of random variables $X_1, X_2, \cdots, X_N$. Namely,

$$F(x_1, x_2, \cdots, x_N) = C[F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_N}(x_N)] \tag{1}$$

Copula function is not only the tool to build the joint probability of multi-dimensional random variables, but also the one to explore the relation structure among random variables.

Thirdly, after better fitting the joint distribution and describing the relation structure, we can obtain the value in risk of portfolio return more accurately, which has become main qualitative technology in risk degree.

From the definition of Philippe Jorion [13], Value at Risk (VaR) is aimed to compute the potential loss of financial assets using distribution function in a certain holding period and confidence level c. If z and VaR indicate the value of financial assets and the risk value respectively, then

$$P(z \leq VaR) = 1 - c \tag{2}$$

Here Monte Carlo simulation is applied to reckon the yield distribution of portfolio risk factors, hence the gains and losses could be constructed in the portfolio and the risk value is estimated in the light of given confidence level.

To sum up, the comparison of the conventional and burgeoning methodologies follows the next table:

Recently, from C.Perignon2010 [], D.Fantazzini2009 [] and J.Shin2009 [], the burgeoning methodology has an obvious effect on analyzing the risk of portfolio in the financial market.

The paper is organized as follows. Section 2 presents the kernel density estimation, the relation structure based on copula model and VaR calculation by Montel Carlo simulation. The combination of the three methods has an obvious advantage compared with the conventional one with linear premise. Section 3 discusses empirical results according to the past and present one, respectively. Section 4 discusses the empirical results. Section 5 is the conclusion.

2 Method

2.1 kernel density estimation (KDE) Experiences show that a large gap is formed between premises of the distribution of financial assets and the complexity in practice. So an approach like the kernel density estimator mitigates the rigidity of the function that belongs to a certain group and hence deserves to be applied in the financial issue.

Let $X_1, X_2, \cdots, X_n$ be independent samples obtained from an unknown density function $f(x)$. $f(x)$ is the formula of kernel density estimator (KDE) (M. Rosenblatt)[28]:

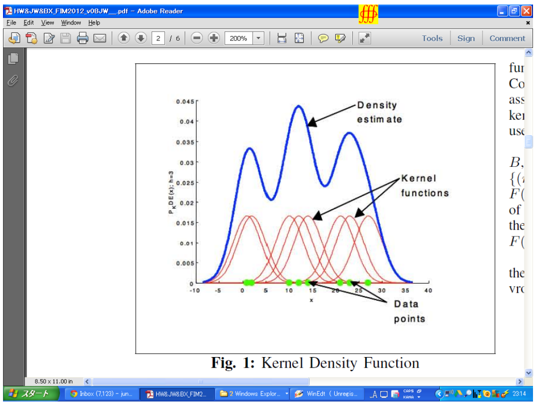

Figure 1: Kernel Density Function

Kernel Density Estimator:

$$P_{KDE}(x) : \hat{f}(x,h) = (nh)^{-1}\sum_{i=1}^{n} K(\frac{x-x_i}{h}) \quad (3)$$

K denotes kernel and h is bandwidth; The smooth kernel estimate is a sum of "bumps" and the kernel function K determines the shape of the bumps. Because of higher efficiency, Gaussian kernel

$K_G(u) = (\sqrt{2\pi})^{-1}\exp(-\frac{z^2}{2})$ is adopted; the parameter h, also called the "bandwidth," determines their width. (M.P.Wand; M.C.Johns)[31]

The bandwidth h plays the role of a scaling factor in determining the spread of the kernel. And it determines the amount of smoothing applied in estimating $f(x)$. The following is the "rule of thumb," which is the most widely used method. (Silveman)[29]

If $f(x)$ is a normal dessity function, then:

$$\int (f(x)^2 dx = \frac{3}{8}\pi^{-0.5}\sigma^{-5} \approx 0.212\sigma^{-5} \quad (4)$$

normal kernel

$$K(u) = (\sqrt{2\pi}^{-1}\exp(\frac{-u^2}{2}) \exp : h = 1.06\sigma n^{\frac{-1}{5}} \quad (5)$$

Hjort and Jones (1996)[10] proposed an improved rule obtained by using an Edgeworth expansion for $f(x)$ around the Gaussian density. Such a rule is given by:

$$\hat{h}^*_{opt} = h_{AMISE}(1 + \frac{35}{48}\hat{\gamma}_4 + \frac{35}{32}\hat{\gamma}_3 + \frac{385}{1024}\hat{\gamma}_4)^{\frac{-1}{5}} \quad (6)$$

2.2 Relation structure based on Copula function According to Sklar theorem, a multiple joint distribution function could be described with marginal distribution and Copula model. To portray the relation structure of financial assets, a kind of two phases method, which is named as kernel density estimation-maximum likelihood method, is used here. (Bouye 2000)[1]

When random variables are two financial assets A and B, whose observation series of return rate (r_A, r_B) is $\{(r_A^t, r_B^t)\}_{t=1}^T$, the simultaneous distribution function is $F(x, y)$ and the probability density and distribution function of r_A and r_B are $f_A(x)$, $F_A(x)$, and $g_B(x)$, $F_B(x)$, and the Copula $C : C_\alpha(u_t, v_t) = C(F_A(r'_A), F_B(r'_B)) = F(r'_A, r'_B)$.

1) Primarily, kernel density estimation is used to measure the unknown marginal density of the financial assets. (Devroye 1983[4]; Fan Yao 2003[32])

$$\begin{aligned} f_A(x) &= \frac{1}{Th_A}\sum_{t=1}^T K_A\left(\frac{x-r'_A}{h_A}\right); \\ g_B(x) &= \frac{1}{Th_B}\sum_{t=1}^T K_B\left(\frac{y-r'_B}{h_B}\right); \end{aligned} \tag{7}$$

When $K(\cdot)$ is the normal kernel:

$$\begin{aligned} u_i &= \frac{1}{T}\sum_{j=1}^T \phi\left(\frac{r_A^t-r_A^j}{h_A}\right); \\ v_i &= \frac{1}{T}\sum_{j=1}^T \phi\left(\frac{r_B^t-r_B^j}{h_B}\right); \end{aligned} \tag{8}$$

2) Next the unknown parameter a in Copula is estimated by maximum likelihood and examined by frequency histogram graph and Minimum Variance Test to choose a optimal copula function. (Genest, Rivest 1993)[23]

The partial derivative is taken to the two sides of formula 1

$$f(x, y) = c_\alpha(F_X(x;\theta_x), F_Y(y;\theta_y))f_X(x;\theta_x)f_Y(y;\theta_y), \tag{9}$$

$f_X(x;\theta_x)$ and $f_Y(y;\theta_y)$ are the marginal density function of $f(x, y)$, θ_x and θ_y are the parameters of marginal density $f_X(x)$ and $f_Y(y)$, α is the parameter of Copula, c_{alpha} is the density function of Copula: $c_{alpha}(u, v) = \dfrac{\partial^2 C(u, v)}{\partial u \partial v}$

Then the formula 8 is taken logarithm:

$$\begin{aligned} \ln L(\theta_x, \theta_y : \alpha) = \; & \ln c_\alpha(F_X, F_Y) + \ln f_X(x, \theta_x) + \\ & \ln f_Y(y;\theta_y) \end{aligned} \tag{10}$$

From maximum likelihood (ML) conception, the log-likelihood function is:

$$\begin{aligned} l(v) = \; & \sum_{t=1}^T \ln c(F_X(X_t;\theta_x), F_Y) + \ln f_x(X;\theta_x) \\ & + \sum_{t=1}^T \ln f_Y(Y_t;\theta_y) \end{aligned} \tag{11}$$

(V. Durrleman 2000[6]; Roberto De Matteis 2001[3]; Claudio Romano 2002[27])

Then, the parameter of Copula C is estimated with ML method:

$$\hat{\alpha} = arg\max \sum_{t=1}^T \ln c(u_t, v_t; \alpha), \tag{12}$$

$c(u, v)$ is the density of Copula,

To sum up, in the above two illustrated steps of the method, the density distribution of financial assets could be estimated in a wide postulated condition and a relation structure especially the tail dependence between them could be described effectively.

2.3 VaR Calculation Analytical Methods such as Variance-Covariance Approach offer an instinctive comprehension of the driving factors of risk in a portfolio, which derives from the risk metrics and obeys the normal distribution. When there are only two assets, the portfolio variance is: (Harry Markowitz, 1952[15]; Peter Zangari, 1996[33])

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{13}$$

And the portfolio VAR is then:

$$\begin{aligned} VaR_p &= \alpha\sigma_p W \\ &= \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}W \end{aligned} \tag{14}$$

where α is quantile of confidence, w weight, σ the variance of assets, ρ correlation coefficient, W the original value, respectively.

Withal, for Monte Carlo simulation based on Copula-VaR, on the one hand, Copula function has the advantage of depicting nonlinear and asymmetric and especially capturing the tail dependence; on the other hand, an abundance of random data that conform to historical distribution is generated to simulate the behavior of the return rate of financial assets by Monte Carlo method.

So the process of portfolio VaR of two assets X and Y based on Copula model and Monte Carlo simulation is followed: (Rank J, Siegl T, 2003[22]; Romano C, 2002[27])

1) The copula model is chosen to describe the marginal distribution of assets and related structure $C(*,*)$.

2) The parameter of Copula model is estimated according to the historical data of return rate of asset X and Y, and hence the distribution function of assets return $F(*)$, $G(*)$ and $C(u, v)$ that are to demonstrate the relation structure between assets could be confirmed. Thereinto, $u = F(Rx)$, $v = G(Ry)$, which submit to $(0, 1)$ even distribution.

3) Two independent random numbers u and v, which submit $(0, 1)$ even distribution, are generated. u is the first simulated pseudo random numbers (PRN). For another thing, $C_u(v) = w$, another PRN v could be calculated through the reversion function of $C_u(v)$: $v = C_u^{-1}(w)$.

4) The values of corresponding assets return $R_X = F^{-1}(u)$, $R_Y = G^{-1}(v)$ are obtained according to the distribution function of assets return $F(\cdot)$, $G(\cdot)$ and u, v;

5) The weight w is given in the portfolio and the return Z of portfolio is calculated: $z = wR_X + (1-w)R_Y$, which provides a possible perspective to the future yield of the portfolio.

6) (3)-(5) steps are repeated through K times, which means the k kinds of possible scenarios of the future yield of the portfolio are generated through simulation, which is amied to obtain the empirical distribution of the future return of the portfolio. For the given confidence $1-\alpha$, the VaR in the portfolio is confirmed from $P[Z < -VaR_\alpha] = \alpha$.

3 Numerical Experiment In the empirical experiment, it is assumed that the portfolio just includes stock and bond. The analyzed data of the two selected financial assets is from Standard&Poor's500 and CBOE Internet Rate 10-Year Bond (2008.7.1-2012.7.3), and the following is the graph of return rate r: $r_{At} = log[P_{At}/P_{At-1}]$

First, Kolmogorov-Smirnov test is used to make the test of normality in SPSS, which shows they don't satisfy normality; Augmented Dickey-Fuller (ADF) unit root test is aimed to demonstrate whether it is the stationary time series data, which demonstrates the time series are the stationary ones.

1) According to the formula 5 6, the bandwidths of SP500 and 10-year bond are 0.0012 and 0.0024, respectively. Through the optimal bandwidth and default Gaussian kernel

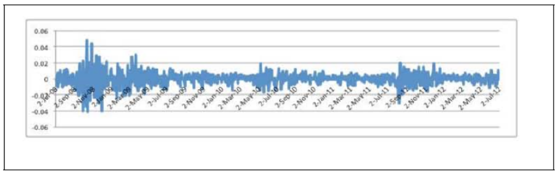

Figure 2: The time series of SP500 return rate

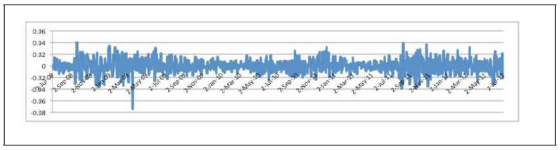

Figure 3: The time series of CBOE Internet Rate 10-Year Bond return rate

function, the density function and cumulative distribution function of the financial assets could be estimated through invoking KS density function in Matlab.

The following is the comparison of kernel density, frequency histogram and normal distribution density:

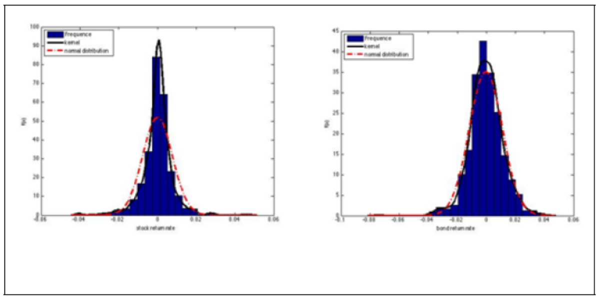

Figure 4: Frequency histogram, kernel density estimation and normal distribution density of the yield of SP500 stock and 10-year bond

The following is the comparison of the empirical, estimated and theoretical normal distribution function under the same conditions:

On the basis of the kernel density estimation to the unknown marginal density of the two financial assets, the parameter of copula model could be estimated.

2) The construction of the bi-variant copula model

Conventionally, Person correlation coefficient is written in the following:

$$\rho_{xy} = \frac{cov(x,y)}{(\sigma_x, \sigma_y)} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}} \tag{15}$$

[25]

It assumes the variables submit to the multi-variant normal distribution. Then, the correlation coefficient of SP500 and 10-year bond is 41.97%.

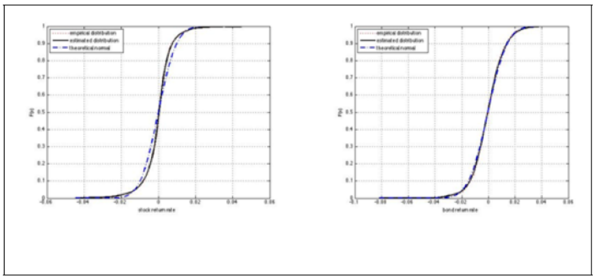

Figure 5: Empirical, estimated and theoretical normal distribution function graph of the return rate of SP500 and 10-year bond

According to the kernel density estimation-maximum likelihood method (8)-(11) and Minimum Variance Test Method

$$Var(\alpha) \cong \frac{4}{n}\alpha^{\frac{3}{2}}(1+\sqrt{\alpha})^2 \tag{16}$$

(Kendall and Stuart (1967)[14]; Mardia (1970)[16]), Gumbel and Clayton are adopted [19] and the corresponding Copula parameters are 1.4173 and 0.7515.

Then, the correlations of stock and bond could be obtained from function relationship between Kendall and Copula parameter: 29.44% and 27.3% respectively here, which is similar to 31.26% from Kendall rank correlation.

Then, through the parsing expression of the correlation coefficient in tail, the correlation coefficient in up-tail and low-tail could be measured according to Gumbel and Clayton function:

$$Gumbel : \lambda^{up} = 2 - 2^{\frac{1}{\alpha}} = 0.37 \tag{17}$$

$$Clayton : \lambda^{lo} = 2 - 2^{\frac{1}{\alpha}} = 0.40 \tag{18}$$

Fig 6 also shows the similar characteristic in the end of the diagonal.

Then, the VaR value could be computed by the combination of copula model and Montel Carlo simulation like the algorithm step (1)-(6) in 2.3.

3) VaR computation

For the analytical formula (13), the assumption is that $c = 95\%(a = 1.65)$ and the original value W is set to 1:

When $W_1 = W_2 = 0.5(c = 95\%, a = 1.65)$,

(19) `VaR value is equal to  0.01336;`

When VaR is minimum, the proportion of W_1 SP500 and W_2 10-year bond is respectively equal to 80.5% and 19.5%, and

(20) `VaR is 0.000055`

According to the Monte Carlo simulation (1)-(6), when $W_1 = W_2 = 0.5$, from Gumble or Clayton model:

(21) `VaR=0.0135;`

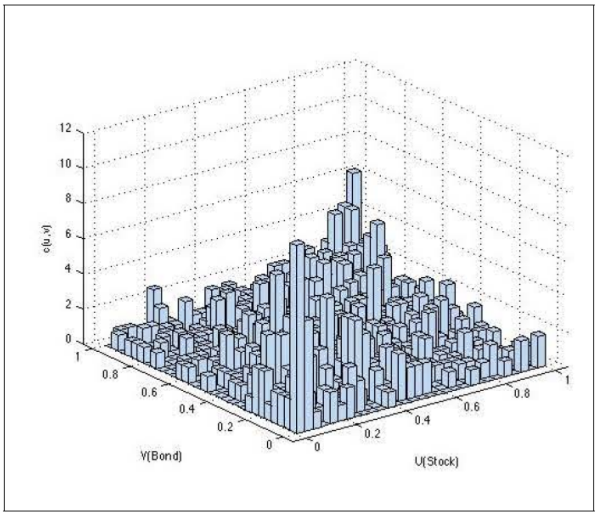

Figure 6: Bibariate frequency histogram

From the following graph, it is concluded that the ratio of stock and bond reaches 85% to 15%, the value at risk could be minimum,

(22) `which is about 0.0122;`

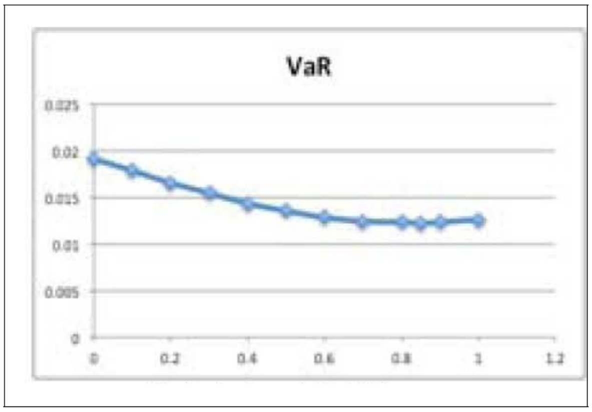

Figure 7: Stock weight-VaR

4 Discussions 1) From the time series graph Figs 2 and 3, the volatility of the two return rate series have the obvious "cluster" phenomenon, which means big fluctuations follow big ones and small fluctuation follow small ones, and there is a certain similarity between them, which shows some interaction exists in it.

From Figs 4 and 5, we can get the negative skewness and high kurtosis, which demonstrates falling days are less than rising days, but the falling average range is higher than

Table 2: VaR of different portions

W_1 (stock weight)	W_2 (bond weight)	VAR
0.00	1.00	0.0191
0.10	0.90	0.0178
0.20	0.80	0.0165
0.30	0.70	0.0154
0.40	0.60	0.0143
0.50	0.50	0.0135
0.60	0.40	0.0128
0.70	0.30	0.0124
0.80	0.20	0.0123
0.85	0.15	0.0122
0.90	0.10	0.0123
1.00	0.00	0.0126

the rising one and return rate happen near the separate average value. So compared with normal distribution, kernel density estimation is a better way to describe the feature of "fat tail and high picky" in the real situation.

2) Through the comparison between Person correlation coefficient and correlation coefficient from copula model, the value of Person one is higher than the one from copula model and Kendall correlation, which shows that the former overestimates the relation between stock market and bond

Contrary to the inability to capture the relevance in tail from linear perspective, the correlation coefficient in tail well describes the possibility of consistency in bond market when the exception situations happen in stock market such as boom or slump.

3) In the VaR comparsion part, it implies that 50% stock-50% bond portfolio has a 95% chance of losing the maximum value 0.01336 and 0.0135 under the above two methods when 1 is invested.

Through the contrast of the VaR results from analytical method and Monte Carlo simulation, it is found that the VaR value in assumption of the normal distribution is less than the one by Monte Carlo, which means the former underestimates the financial risk easily.

Meanwhile, to obtain the safest asset security, it is a wise strategy for a robust investor to allocate $80\%-85\%$ capital to stock market and $15\%-20\%$ one to 10-year bond theoretically according to results of the minimum VaR computation.

5 Conclusions In the analysis of portfolio, there is an importance in the study of relation structure between financial assets, which results in how to capture the principal of change between them especially in the tail with better correlation model.

In this paper, through kernel density estimation-maximum likelihood two steps, Gumbel and Clayton copula model are adopted to model the correlation between stock and bond. Then, VaR is analyzed based on it and the optimal allocation in the portfolio could be confirmed by Montel Carlo simulation.

By comparison between the present methods introduced in this paper and the conventional methods which is based on the normal distribution, it is concluded that the latter one always underestimate the happening of risk and the value of risk, which should be brought to the forefront.

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