Hesitant Fuzzy Geometric Heronian Mean Operators and Their Application to Multi-Criteria Decision Making

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ABSTRACT. Aggregation is the process of fusing a large data in one representative value. This is done in different ways, through what may be called 'operators', every operator having special characteristics. Expanding study of vague phenomena, through hesitant fuzzy information of hesitant fuzzy set (HFS) theory and their applications has attracted useful aggregation techniques. Paper explores the geometric Heronian mean (GHM) under hesitant fuzzy environment and defines some new geometric Heronian mean operators such as the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and the weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM) operator. Further, we give definition of hesitant fuzzy geometric Heronian element (HFGHE), which is a basic calculation unit in HFGGHM and reflects the conjunction between two aggregated arguments. Properties of the new aggregation operators are reported and their special cases are considered. Furthermore, based on the WHFGGHM operator, an approach to deal with multi-criteria decision-making problems under hesitant fuzzy environment is developed. Finally, a practical example is provided to illustrate the multi-criteria decision-making process.
Keywords: fuzzy sets; fuzzy multi-sets; intuitionistic fuzzy set; hesitant fuzzy sets.

1. Introduction

Mathematics is known for its quantitative and logically sound foundations. It started with study of deterministic phenomena. However, the wider world phenomena, all the more those in man-made world, are not deterministic in nature. Ingenuity of mathematicians expanded mathematical study to a class of in-deterministic/uncertain phenomena that are statistical/probabilistic nature. Without sacrificing its quantitative and logically sound basis, a vast discipline of statistics developed. Moving thus a major step forward in the study of uncertain phenomena, it was observed that there are uncertain phenomena that are not statistically stable in which chances of happening of an event can be quantified in terms of probabilities and distribution-patterns. This presented mathematicians with a challenge to define phenomena that are uncertain in non-statistical ways. In general these may be called vague or imprecise. Zadeh [44] was the first to capture this idea in defining fuzzy sets. Several extensions and generalizations of Zadeh' fuzzy-sets have since been made as intuitionistic fuzzy sets [1, 2], interval-valued fuzzy sets [10, 19], type-2 fuzzy sets [45], type-n fuzzy sets [45], fuzzy multisets [6, 35], vague sets [9], and hesitant fuzzy sets [17, 18], etc. In a rather natural way, set operations were defined and it was found that these present a panorama of laws as the defining terms in these sets involve functions, which was not the case with theory of crisp sets. These studies enriched areas of applications in different ways [4, 5, 7, 8, 11-16, 20-34, 38-41, 45-50].

The vagueness/fuzziness that appeared to be diluting/loosing precise quantitative tenor of things in the process, Zadeh and thereafter others defined measures of fuzziness of various shades over family of fuzzy-sets. These measures of fuzziness are quantitative in nature and follow the pattern of measures defined in place Shannon's probabilistic information theory.

Another age old idea is that of 'aggregation,' a process of meaningfully fusing a collection of values into one representative value. It is, in fact, a multi-faceted avtar of the simple idea of arithmetical and other means/averages of a given set of numbers. In probabilistic-statistics, one encounters it at several places – 'statistical expectations,' correlation and regression analysis, etc.

Shannon's entropy of a probability distribution being average of self-information arguments of its elements is, generally speaking, an aggregation of self-information elements. With this background, information aggregation in hesitant fuzzy set theory has been studied with quite some interest by researchers and practitioners in recent years. Xia and Xu [27] developed some arithmetic and geometric aggregation operators under hesitant fuzzy environment, investigated the connections of these operators and applied them to multi-criteria decision making. To aggregate the hesitant fuzzy information under confidence levels, Xia et al. [26] developed a series of confidence-induced hesitant fuzzy aggregations operators. Xu et al. [30] developed several series of aggregation operators for hesitant fuzzy information using the quasi-arithmetic means. Gu et al. [11] utilized the hesitant fuzzy weighted average (HFWA) operator to investigate the evaluation model for risk investment with hesitant fuzzy information. Based on the prioritized weighted average (PWA) operator [37, 38], Yu [40] proposed the hesitant fuzzy prioritized weighted average (HFPWA) operator and the hesitant fuzzy prioritized weighted geometric (HFPWA) operator to aggregate the hesitant fuzzy information. Wei [22] also developed some prioritized aggregation operators for aggregating hesitant fuzzy information and then applied them to develop models for hesitant fuzzy multiple attribute decision making.

Reflecting on the concept of aggregation, it may be noted that the above discussed aggregation operators with hesitant fuzzy information are based on the assumption that all aggregating arguments are independent However, in real world situations there are always some degrees of interrelationships between arguments. To deal with this issue, Yu et al. [39] and Wei et al. [21] developed some hesitant fuzzy correlative operators, such as the hesitant fuzzy Choquet integral (HFCI) operator, the hesitant fuzzy Choquet ordered average (HFCOA) operator, the hesitant fuzzy Choquet ordered geometric (HFCOG) operator, the generalized hesitant fuzzy Choquet ordered average (GHFCOA) operator and the generalized hesitant fuzzy Choquet ordered geometric (GHFCOG) operator and found their application to multiple attribute decision making. Motivated by the idea of power average (PA) operator [36], Zhang [48] developed some hesitant fuzzy power average (HFPA) operators and hesitant fuzzy power geometric (HFPG) operators for aggregating hesitant fuzzy correlative information. Further, Zhu et al. [51] and Zhu and Hu [52] extended the Bonferroni mean (BM) to hesitant fuzzy environment and introduced some hesitant fuzzy Bonferroni means such as the hesitant fuzzy Bonferroni mean (HFBM), the weighted hesitant fuzzy Bonferroni mean (WHFBM), the hesitant fuzzy geometric Bonferroni mean (HFGBM), the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM).

The Heronian mean (HM) is another aggregation technique, which is better suited to aggregate the exact numerical values [3]. A prominent characteristic of HM is its capability to capture interrelationships between input arguments. This makes HM useful in various application fields, such as decision making, information retrieval, pattern recognition, and data mining etc. The HM is different from power average or Choquet integral. The HM operator focuses on the aggregated arguments while the Choquet integral or power average on changing the weight vector of the aggregation operators. Based on HM operator, Yu [43] defined some generalized HM operators such as generalized geometric Heronian mean (GGHM), the generalized geometric intuitionistic fuzzy Heronian mean (GGIFHM) and the weighted generalized geometric intuitionistic fuzzy Heronian mean (WGGIFHM).

In this paper, we extend the idea of generalized geometric Heronian mean operator to hesitant fuzzy environment. In order to do so, we propose the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and the weighted hesitant fuzzy generalized geometric Heronian mean (WHFGWBM) operator for aggregating the hesitant fuzzy correlative information. We study their properties and discuss special cases. We show that several aggregation operators on hesitant fuzzy sets studied earlier are special cases of our generalized operator. Also, there are others interesting particular cases that as well arise from it. Further, we develop an approach for multi-criteria decision making under hesitant fuzzy information environment.

The paper is organized as follows: In Section 2 some basic concepts related to fuzzy sets, hesitant fuzzy sets and Heronian mean operators are briefly given. In Section 3 we propose the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and study some of their properties. Some special cases of HFGGHM are also discussed in this section. In Section 4 we introduce the weighted hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and develop an approach for solving multi-criteria decision making under hesitant fuzzy environment. In Section 5 finally, a numerical example is presented to illustrate the proposed approach to multi-criteria decision-making and our conclusions are presented in Section 6.

2. Preliminaries

Definition 1. ***Fuzzy set*** [44]**:** A fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$A = \{\langle x, \mu_A(x)\rangle \, | x \in X\}, \tag{1}$$

where $\mu_A(x) : X \to [0, 1]$ is the membership function of A and the number $\mu_A(x)$ describing the degree of membership of $x \in X$ in the set A.
An step further, the concept of hesitant fuzzy sets (HFSs) was introduced by Torra and Narukawa [17] and Torra [18]. An HFS permits the membership degree of an element to be a set of several possible membership values between 0 and 1. This better describes the situations where a set of people have hesitancy in providing their preferences over objects in the process of decision making.

Definition 2. ***Hesitant Fuzzy Set***[18]**:** Let $X = \{x_1, x_2, ..., x_n\}$ be a reference set, a set E defined in X given by

$$E = \{\langle x, h_E(x)\rangle \, | x \in X\} \tag{2}$$

where $h_E(x)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E, is called a hesitant fuzzy set.

Further, Torra [18] defined the 'empty hesitant fuzzy set' and the 'full hesitant fuzzy set' as follows:

$$E^{\circ} = \{\langle x, h_{E^{\circ}}(x)\rangle \, | x \in X\}, \text{ where } h_{E^{\circ}}(x) = \{0\} \;\; \forall x \in X,$$
$$E^{*} = \{\langle x, h_{E^{*}}(x)\rangle \, | x \in X\}, \text{ where } h_{E^{*}}(x) = \{1\} \;\; \forall x \in X.$$

For convenience, Xia and Xu [27] named the set $h = h_E(x)$ as the hesitant fuzzy element (HFE) and let $HFE(X)$ represent the family of all hesitant fuzzy elements defined in X.

Definition 3.*Algebraic Operations on HFEs*: Let $h, h_1, h_2 \in HFE(X)$, Xia and Xu [26] defined the following operations:

1. $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$, $\lambda > 0$;
2. $\lambda h = \bigcup_{\gamma \in h} \{1 - (1-\gamma)^\lambda\}$, $\lambda > 0$;
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$.

Definition 4. *Score Function* [27]: Let h be a hesitant fuzzy element, the score function S of an HFE is defined as follows:

$$S(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma, \tag{3}$$

where $\#h$ is the number of elements in h.

To ranking any two h_i, $i = 1, 2$, we shall use the following definition of Xia & Xu [27]:

Definition 5: Let $h_1$and h_2 be two hesitant fuzzy elements with their respective scores $S(h_1)$ and $S(h_2)$, then

1. h_1 is larger than h_2, denoted by $h_1 > h_2$ if $S(h_1) > S(h_2)$.
2. $h_1 = h_2$, if $S(h_1) = S(h_2)$.

Heronian mean (HM), which is one of the aggregation methods, is characterized by the ability to capture the relevance between the input arguments. The definition of HM is as follows:

Definition 6. *Heronian Mean*[3]: For a collection a_i, $i = 1, 2, \cdots, n$, of nonnegative real numbers, their Heronian mean (HM) is defined as:

$$HM(a_1, a_2, \cdots, a_n) = \frac{2}{n(n+1)} \sum_{i,j=1}^{n} \sqrt{a_i a_j} \tag{4}$$

Based on Definition 6, Yu [42] proposed the geometric Heronian mean (GHM) as follows:

Definition 7. ***Geometric Heronian Mean*[43]:** For a collection a_i, $i = 1, 2, \cdots, n$, of nonnegative real numbers, their the geometric Heronian mean (GHM) is defined by:

$$GHM(a_1, a_2, \cdots, a_n) = \prod_{i,j=1}^{n} \left(\frac{a_i + a_j}{2}\right)^{\frac{2}{n(n+1)}} \tag{5}$$

Further, using the idea of geometric Bonferroni mean [31], Yu [43] also proposed the generalized geometric Heronian mean (GGHM) as follows:

Definition 8. ***Generalized Geometric Heronian Mean*** [43]**:** Let $p, q \geq 0$, p, q do not take the value 0 simultaneously and let a_i, $i = 1, 2, \cdots, n$, be a collection of nonnegative real numbers, then generalized geometric Heronian mean (GGHM) is given by:

$$GGHM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} (pa_i + qa_j)^{\frac{2}{n(n+1)}} \tag{6}$$

It may be noted that $GGHM^{p,q}$ have the following properties.

1. $GGHM^{p,q}(0, 0, \cdots, 0) = 0$ and $GGHM^{p,q}(1, 1, \cdots, 1) = 1$;
2. $GGHM^{p,q}(a_1, a_2, \cdots, a_n) = a$ if $a_i = a$, $\forall i$;
3. If $a_i \leq b_i$ $\forall i$, then $GGHM^{p,q}(a_1, a_2, \cdots, a_n) \leq GGHM^{p,q}(b_1, b_2, \cdots, b_n)$ i.e., $GGHM^{p,q}$ is monotonic;
4. $\min_i \{a_i\} \leq GGHM^{p,q}(a_1, a_2, \cdots, a_n) \leq \max_i \{a_i\}$.

In the next section, in respect of hesitant fuzzy environment, we extend the GGHM to hesitant fuzzy environment and propose:

(i) The hesitant fuzzy generalized geometric Heronian mean (HFGGHM);
(ii) The weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM).

3. Hesitant Fuzzy Generalized Geometric Heronian Means

We propose the following definition:

Definition 9.***Hesitant Fuzzy Generalized Geometric Heronian Mean***: Let $p, q > 0$ and h_i $i = 1, 2, \cdots, n$ be a collection of HFEs, the hesitant fuzzy generalized geometric Heronian mean ($HFGGHM^{p,q}$) is given by:

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2}{n(n+1)}} \tag{7}$$

Next, based on the operational laws of HFEs, we have the following theorem:

Theorem 1: Let $p, q > 0$ and h_i $i = 1, 2, \cdots, n$ be a collection of hesitant fuzzy elements, then the aggregated value by using the $HFGGHM^{p,q}$ operator is also a hesitant fuzzy element, and

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\eta_{i,j} \in \sigma_{i,j}; i \leq j} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \leq j}}^{n} (\eta_{i,j})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\} \tag{8}$$

where $\sigma_{i,j;\,i\leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ reflects the interrelationship between h_i and h_j, $i, j = 1,\, 2, \dots,\, n$.

Proof: Since

$$\sigma_{i,j;\,i\leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)) = \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\leq j}} \{\eta_{i,j}\} \tag{9}$$

which is also a HFE, then Equation (8) can be written as:

$$HFGGHM^{p,q}\,(h_1, h_2,\, \dots,\, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\,\leq j}}^{n} \left(\sigma_{i,j}\right)^{\frac{2}{n(n+1)}} \tag{10}$$

Furthermore, we have

$$\bigotimes_{\substack{i,j=1\\ i\,\leq j}}^{n} \left(\sigma_{i,j}\right)^{\frac{2}{n(n+1)}} = \left(\bigotimes_{\substack{i,j=1\\ i\,\leq j}}^{n} \left(\sigma_{i,j}\right)\right)^{\frac{2}{n(n+1)}}$$

$$= \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\leq j}} \left\{\left(\prod_{\substack{i,j=1\\ i\leq j}}^{n} \eta_{i,j}\right)^{\frac{2}{n(n+1)}}\right\}$$

$$= \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\leq j}} \left\{\left(\prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\eta_{i,j}\right)^{\frac{2}{n(n+1)}}\right)\right\}$$

and then

$$\frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(\sigma_{i,j}\right)^{\frac{2}{n(n+1)}} = \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\leq j}} \left\{1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\eta_{i,j}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}. \tag{11}$$

This completes the proof of the Theorem 1.

It is noted that, in Theorem 1, $\sigma_{i,j}$ is a basic element in (8), which we call a hesitant fuzzy geometric Heronian element (HFGHE). Apparently, $\sigma_{i,j}$ represents the interrelationship between the HFEs h_i and h_j by two types of conjunction calculations, i.e., "$\oplus$" and "$\otimes$".

Further, we discuss some properties of the $HFGGHM^{p,q}$:

1. Let h_i, $i = 1, 2, \cdots, n$, be collection of HFEs. If $h_i = h$ for all i, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q}((p+q)h)^2. \tag{12}$$

Proof: Since $h_i = h$ for all i, we have

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h, h, \cdots, h)$$

$$= \frac{1}{p+q} \underset{\substack{i,j=1 \\ i \le j}}{\overset{n}{\otimes}} ((ph \oplus qh) \otimes (ph \oplus qh))^{\frac{2}{n(n+1)}}$$

$$= \frac{1}{p+q}((ph \oplus qh) \otimes (ph \oplus qh))$$

$$= \frac{1}{p+q}((p \oplus q)h)^2. \tag{13}$$

This proves the property.

Corollary 1: If h_i, $i = 1, 2, \cdots, n$, is a collection of the empty HFEs, i.e., $h_i = h^\circ = \{0\}$, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h^\circ, h^\circ, \cdots, h^\circ) = \{0\}. \tag{14}$$

Corollary 2: If h_i, $i = 1, 2, \cdots, n$ is a collection of the full HFEs, i.e., $h_i = h^* = \{1\}$, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h^*, h^*, \cdots, h^*) = \{1\}. \tag{15}$$

2. (Monotonicity). Let $h_\alpha = (h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_n})$ and $h_\beta = (h_{\beta_1}, h_{\beta_2}, \cdots, h_{\beta_n})$ be two collections of HFEs, $\sigma_{\alpha_{i,j}} = ((ph_{\alpha_i} \oplus qh_{\alpha_j}) \otimes (ph_{\alpha_j} \oplus qh_{\alpha_i}))$ and $\sigma_{\beta_{i,j}} = ((ph_{\beta_i} \oplus qh_{\beta_j}) \otimes (ph_{\beta_j} \oplus qh_{\beta_i}))$, if for any $\gamma_{\alpha_i} \in h_{\alpha_i}$, $\gamma_{\beta_i} \in h_{\beta_i}$, we have $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \le \gamma_{\beta_j}$ for all $i, j = 1, 2, \ldots, n$, then

$$HFGGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) \le HFGGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_n}). \tag{16}$$

Proof: Since $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \le \gamma_{\beta_j}$ for all $i, j = 1, 2, \ldots, n$, we have

$$(1 - (1 - \gamma_{\alpha_i})^p (1 - \gamma_{\alpha_j})^q) \le (1 - (1 - \gamma_{\beta_i})^p (1 - \gamma_{\beta_j})^q), \tag{17}$$

$$(1 - (1 - \gamma_{\alpha_j})^p (1 - \gamma_{\alpha_i})^q) \le (1 - (1 - \gamma_{\beta_j})^p (1 - \gamma_{\beta_i})^q). \tag{18}$$

Additionally, we obtain

$$\sigma_{i,j;\, i \le j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$$

(19)

$$= \Big(\bigcup_{\gamma_{\alpha_i} \in h_{\alpha_i}, \gamma_{\alpha_j} \in h_{\alpha_j}} \{1 - (1 - \gamma_{\alpha_i})^p + 1 - (1 - \gamma_{\alpha_j})^q - (1 - (1 - \gamma_{\alpha_i})^p)(1 - (1 - \gamma_{\alpha_j})^q)\} \Big)$$

$$\otimes \Big(\bigcup_{\gamma_{\alpha_i} \in h_{\alpha_i}, \gamma_{\alpha_j} \in h_{\alpha_j}} \{1 - (1 - \gamma_{\alpha_j})^p + 1 - (1 - \gamma_{\alpha_i})^q - (1 - (1 - \gamma_{\alpha_j})^p)(1 - (1 - \gamma_{\alpha_i})^q)\} \Big)$$

Let $\eta_{\alpha_{i,j}} \in \sigma_{\alpha_{i,j;i\leq j}}$ and $\eta_{\beta_{i,j}} \in \sigma_{\beta_{i,j;i\leq j}}$, for all $i, j = 1, 2 \ldots, n$; $i \leq j$, then from Equations (17)-(19), we have

(20)

$$\eta_{\alpha_{i,j}} = \Big(\Big(\bigcup_{\gamma_{\alpha_i}\in h_{\alpha_i},\gamma_{\alpha_j}\in h_{\alpha_j}} \{1-(1-\gamma_{\alpha_i})^p+1-(1-\gamma_{\alpha_j})^q-(1-(1-\gamma_{\alpha_i})^p)(1-(1-\gamma_{\alpha_j})^q)\}\Big)$$
$$\otimes\Big(\bigcup_{\gamma_{\alpha_j}\in h_{\alpha_j},\gamma_{\alpha_i}\in h_{\alpha_i}} \{1-(1-\gamma_{\alpha_j})^p+1-(1-\gamma_{\alpha_i})^q-(1-(1-\gamma_{\alpha_j})^p)(1-(1-\gamma_{\alpha_i})^q)\}\Big)\Big)$$
$$\leq \eta_{\beta_{i,j}} = \Big(\Big(\bigcup_{\gamma_{\beta_i}\in h_{\beta_i},\gamma_{\beta_j}\in h_{\beta_j}} \{1-(1-\gamma_{\beta_i})^p+1-(1-\gamma_{\beta_j})^q-(1-(1-\gamma_{\beta_i})^p)(1-(1-\gamma_{\beta_j})^q)\}\Big)$$
$$\otimes\Big(\bigcup_{\gamma_{\beta_j}\in h_{\beta_j},\gamma_{\beta_i}\in h_{\beta_i}} \{1-(1-\gamma_{\beta_j})^p+1-(1-\gamma_{\beta_i})^q-(1-(1-\gamma_{\beta_j})^p)(1-(1-\gamma_{\beta_i})^q)\}\Big)\Big)$$

thus

$$\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \geq \left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}. \tag{21}$$

According to Definition 9 and Equation (21), we get

$$\begin{aligned}
HFGGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) &= \frac{1}{p+q}\bigotimes_{\substack{i,j=1\\ i\leq j}}^{n}\left(\sigma_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\\
&= \bigcup_{\eta_{\alpha_{i,j}}\in\sigma_{\alpha_{i,j;i\leq j}}}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}\\
&\leq \bigcup_{\eta_{\beta_{i,j}}\in\sigma_{\beta_{i,j,i\leq j}}}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}\\
&= \frac{1}{p+q}\bigotimes_{\substack{i,j=1\\ i\leq j}}^{n}\left(\sigma_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\\
&= HFGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_n}). \qquad (22)
\end{aligned}$$

This proves the property.

3. (Commutativity). Let h_i, $i = 1, 2, \ldots, n$, be collection of HFEs, and $(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n)$ be any permutation of $(h_1, h_2, \ldots, h_n)$, then

$$HFGGHM^{p,q}(h_1, h_2, \ldots, h_n) \leq HFGGHM^{p,q}(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n). \tag{23}$$

Proof: Since $(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n)$ is a permutation of $(h_1, h_2, \ldots, h_n)$, then

$$\begin{aligned} HFGGHM^{p,q}\left(h_{\alpha_1}, h_{\alpha_2}, ..., h_{\alpha_n}\right) &= \frac{1}{p+q} \underset{\substack{i,j=1 \\ i \leq j}}{\overset{n}{\otimes}} \left(\sigma_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}} \\ &= \frac{1}{p+q} \underset{\substack{i,j=1 \\ i \leq j}}{\overset{n}{\otimes}} \left(\dot{\sigma}_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}} \\ &= HFGGHM^{p,q}(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n), \end{aligned} \tag{24}$$

where $\sigma_{i,j;\, i \leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ and $\dot{\sigma}_{i,j;\, i \leq j} = (p\dot{h}_i \oplus q\dot{h}_j) \otimes (p\dot{h}_j \oplus q\dot{h}_i)$, $i, j = 1, 2, \ldots, n$.

This proves the property.

4. (Boundedness). Let h_i, $i = 1, 2, \ldots, n$ be collection of HFEs, $h_i^+ = \bigcup_{\gamma_i \in h_i} \max\{\gamma_i\}$, $h_i^- = \bigcup_{\gamma_i \in h_i} \min\{\gamma_i\}$, $\gamma^+ \in h_i^+$, $\gamma^- \in h_i^-$, and $\sigma_{i,j} = (ph_i \oplus qh_j) = \bigcup_{\eta_{i,j} \in \sigma_{i.j}} \{\eta_{i.j}\} = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j} \{1 - (1-\gamma_i)^p (1-\gamma_j)^q\}$, then

$$\begin{aligned} \bigcup_{\gamma^- \in h_i^-} \left\{1 - \left(1 - \left(1 - \left(1 - \gamma^-\right)^{p+q}\right)^2\right)^{\frac{1}{p+q}}\right\} \\ \leq HFGGHM^{p,q}(h_1, h_2, \ldots, h_n) \\ \leq \bigcup_{\gamma^- \in h_i^-} \left\{1 - \left(1 - \left(1 - \left(1 - \gamma^+\right)^{p+q}\right)^2\right)^{\frac{1}{p+q}}\right\}. \end{aligned} \tag{25}$$

Proof: Since $\gamma^- \leq \gamma_i \leq \gamma^+$ and $\gamma^- \leq \gamma_j \leq \gamma^+$ $\forall\, i, j = 1, 2, \ldots, n$, then

$$1 - \left(1 - \gamma^-\right)^{p+q} \leq 1 - (1-\gamma_i)^p (1-\gamma_j)^q \leq 1 - \left(1 - \gamma^+\right)^{p+q} \tag{26}$$

$$1 - \left(1 - \gamma^-\right)^{p+q} \leq 1 - (1-\gamma_j)^p (1-\gamma_i)^q \leq 1 - \left(1 - \gamma^+\right)^{p+q} \tag{27}$$

and

$$\begin{aligned} &\sigma_{i,j;\, i \leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)) \\ &= \left(\bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \{1 - (1-\gamma_i)^p (1-\gamma_j)^q\}\right) \otimes \left(\bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \{1 - (1-\gamma_j)^p (1-\gamma_i)^q\}\right) \\ &\geq \left(\bigcup_{\gamma^- \in h_i^-} \left\{\left(1 - \left(1 - \gamma^-\right)^{p+q}\right)^2\right\}\right). \end{aligned} \tag{28}$$

Similarly, we have

$$\sigma_{i,j;\,i\leq j} \leq \left(\bigcup_{\gamma^{+}\in h_i^{+}} \left\{ \left(1-\left(1-\gamma^{+}\right)^{p+q}\right)^{2} \right\} \right). \tag{29}$$

According to Definition 9, Equations (28) and (29), we obtain

$$\bigcup_{\gamma^{-}\in h_i^{-}} \left\{ 1-\left(1-\left(1-\left(1-\gamma^{-}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}} \right\} \leq HFGGHM^{p,q}\left(h_1,h_2,\ldots,h_n\right)$$

$$\leq \bigcup_{\gamma^{+}\in h_i^{+}} \left\{ 1-\left(1-\left(1-\left(1-\gamma^{+}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}} \right\}. \tag{30}$$

This proves the property.

Some special cases of $HFGGHM^{p,q}$ for different values of parameters p and q.

(i) If $q \to 0$ (or $p \to 0$), then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\to 0} HFGHM^{p,q}\left(h_1,h_2,\ldots,h_n\right) = \frac{1}{p} \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(ph_i \otimes ph_j\right)^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\gamma_i\in h_i,\gamma_j\in h_i} \left\{ 1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\left(1-\left(1-\gamma_i\right)^{p}\right)\left(1-\left(1-\gamma_j\right)^{p}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p}} \right\}, \tag{31}$$

which we call the generalized hesitant fuzzy geometric Heronian mean (GHFGHM).

(ii) If $p = 1$ and $q \to 0$, then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\to 0} HFGHM^{1,q}(h_1,h_2,\ldots,h_n) = \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(h_i \otimes h_j\right)^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\gamma_i\in h_i,\gamma_j\in h_i} \left\{ \prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\left(1-\left(1-\gamma_i\right)\right)\left(1-\left(1-\gamma_j\right)\right)\right)^{\frac{2}{n(n+1)}} \right\}, \tag{32}$$

which we call the hesitant fuzzy geometric Heronian mean (GHFGHM).

(iii) If $p = 2$ and $q \to 0$, then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\to 0} HFGHM^{2,q}\left(h_1, h_2, \ldots, h_n\right) = \frac{1}{2} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} \left(2h_i \otimes 2h_j\right)^{\frac{2}{n(n+1)}},$$

(33)
$$= \bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \le j}}^{n} \left(\left(1-(1-\gamma_i)^2\right)\left(1-(1-\gamma_j)^2\right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\},$$

which we call the hesitant fuzzy square geometric Heronian mean (HFSGHM).

(iv) If $p = q = 1$, let $\sigma^{1,1}_{i,j;\,i\le j} = ((h_i \oplus h_j) \otimes (h_j \oplus h_i)) = \bigcup_{\varepsilon_{i,j} \in \sigma^{1,1}_{i,j;i\le j}} \{\varepsilon_{i,j}\}$, the $HFGGHM^{p,q}$ reduces to

$$HFGHM^{1,1}\left(h_1, h_2, \ldots, h_n\right) = \frac{1}{2} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} \left((h_i \oplus h_j) \otimes (h_j \oplus h_i)\right)^{\frac{2}{n(n+1)}}$$

(34)
$$= \bigcup_{\varepsilon_{i,j} \in \sigma^{1,1}_{i,j;i\le j}} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \le j}}^{n} \left(\varepsilon_{i,j}\right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\}$$

which we call the hesitant fuzzy interrelated square geometric Heronian mean (HFISGHM).

Further, to consider the importance of aggregated arguments, we define a weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM) operator as follows:

Definition 10: Let $h_i, i = 1, 2, \ldots, n$, be a collection of HFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_i where w_i indicates the importance degree of h_i, satisfying $w_i \ge 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. For any $p, q > 0$, the weighted hesitant fuzzy generalized geometric Heronian mean ($WHFGGHM^{p,q}$) is given by:

(35)
$$WHFGGHM^{p,q}(h_1, h_2, \ldots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} \left((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)\right)^{\frac{2w_i w_j}{(1+w_i)}}.$$

In view of Equation (35), we prove a result in the following theorem:

Theorem 2. Let $p, q > 0$, and $h_i, i = 1, 2, \ldots, n$ be a collection of HFEs with weight vector $w = (w_1, w_2, \ldots, w_n)^T$ satisfying $w_i \ge 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated value using the WHFGGHM is also an HFE, and

$$WHFGGHM^{p,q}\left(h_1, h_2, \ldots, h_n\right) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} \left((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)\right)^{\frac{2w_i w_j}{(1+w_i)}},$$

$$(36) \qquad = \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\leq j}} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i\leq j}}^{n} (\eta_{i,j})^{\frac{2w_i w_j}{(1+w_i)}} \right)^{\frac{1}{p+q}} \right\}$$

and $\sigma_{i,j;\,i\leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ reflects the interrelationship between h_i and h_j, $i, j = 1, 2, \ldots, n$.

Proof: This theorem is easy to prove on lines similar to that of Theorem 1.

Note: If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then $WHFGGHM^{p,q}$ in (36) reduces to $HFGGHM$ in (8).

In the following section, we suggest application of the proposed $WHFGGHM^{p,q}$ operator to multi criteria decision making problems with hesitant fuzzy information and give an illustrative numerical example.

4. An Approach to Multi Criteria Decision Making under Hesitant Fuzzy Environment

For a multi criteria decision making problem, let $A = (A_1, A_2, \ldots, A_m)$ be a set of m alternatives and $C = (C_1, C_2, \ldots, C_n)$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0,\, 1]$ and $\sum_{j=1}^{n} w_i = 1$. The decision makers provide all the possible values that the alternative A_i satisfies the criterion C_j represented by HFEs $h_{ij} = \bigcup_{\gamma_{ij}\in h_{ij}} \{\gamma_{ij}\}$, and all $h_{ij}, i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$, construct the hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$:

Table 1: Hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$

	C_1	C_2	$\ldots$	C_n
A_1	h_{11}	h_{12}	$\ldots$	h_{1n}
A_2	h_{21}	h_{21}	$\ldots$	h_{2n}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
A_m	h_{m1}	h_{m2}	$\ldots$	h_{mn}

To harmonize the data,first step is to look at the criteria.These in general can be of different types. If all the criteria $C = (C_1, C_2, \ldots, C_n)$ are of the same type, then the criteria values do not need harmonization. However if these involve different scales and /or units, there is need to be convert them all to the same scale and/or unit. Just to make this point clear, let us consider two types of criteria, namely, (i) cost type and the (ii) benefit type. Considering their natures, a benefit criterion (the bigger the values better is it) and cost criterion (the smaller the values the better is it) are of rather opposite type. In such cases, we need to first transform the criteria values of cost type into the criteria values of benefit type. So, transform the hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$ into the normalized hesitant fuzzy decision matrix $B = [b_{ij}]_{m\times n}$ by the method given by Zhu and Xu [52], where

$$(37) \qquad b_{ij} = \begin{cases} h_{ij} & \text{for benefit criterion } C_j \\ h_{ij}^c, & \text{for cost criterion } C_j \end{cases}, \qquad i = 1,\, 2,\, ...,\, m\,;\;\; j = 1,\, 2,\, ...,\, n,$$

where $h_{ij}^c = \bigcup_{\gamma_{ij}\in h_{ij}} \{1 - \gamma_{ij}\}$ is the complement of h_{ij}.

With criteria harmonized and using the $WHFGGHM$ operator, we now formulate an algorithm to solve multi criteria decision making problem with hesitant fuzzy information:

Algorithm:

Step 1: Use the $WHFGGHM$ operator to aggregate all the performance values $b_{ij}, j = 1, 2, \dots, n$, of the i^{th} row, and get the overall performance value b_i corresponding to the alternative $A_i, i = 1, 2, \dots, m$:

$$b_i = WHFGGHM_w^{p,q}(h_{i1}, h_{i2}, \dots, h_{in}). \tag{38}$$

Step 2: By Definition 3, calculate the scores $S(b_i)$ of b_i and rank the overall performance values $b_i, i = 1, 2, \dots, m$.

Step 3: Rank the alternatives$A_i, i = 1, 2, \dots, m$, in accordance with $b_i, i = 1, 2, \dots, m$, in descending order and select the most desirable alternative(s).

We demonstrate the above proposed algorithm to a real life multi-criteria decision making through following illustrative example.

Example[52]**:** Consider a factory site selection problem for new buildings. After pre-elimination process, only three alternatives $A_i, i = 1, 2, 3$, are being considered for further evaluation and selection. The decision makers take into account three criteria to decide the best site: C_1: price, C_2: environment, and C_3: location. The weights of criteria are $w = (0.5, 0.3, 0.2)^T$. Next let the characteristics of the alternative $A_i, i = 1, 2, 3$, with respect to the criteria $C_j, j = 1, 2, 3$, be represented by the HFEs $h_{ij} = \bigcup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\}$, where γ_{ij} indicates that the alternative A_i satisfies the criterion C_j. All $h_{ij}, i, j = 1, 2, 3$, are contained in shown in the following hesitant fuzzy decision matrix $H = [h_{ij}]_{3\times3}$:

Table 2: Hesitant fuzzy decision matrix $H = [h_{ij}]_{3\times3}$

	C_1	C_2	C_3
A_1	$\{0.6, 0.7, 0.8\}$	$\{0.25\}$	$\{0.4, 0.5\}$
A_2	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.3, 0.55, 0.6\}$
A_3	$\{0.2, 0.4\}$	$\{0.6, 0.5\}$	$\{0.7, 0.5\}$

Considering that all the criteria $C_j, j = 1, 2, 3$, are of the benefit type, then the preference values of the alternatives $A_i, i = 1, 2, 3$, do not need harmonization, therefore, $B = [b_{ij}]_{m\times n} = [h_{ij}]_{3\times3}$.

Step 1: Using the WHFGGHM operator (here, we take $p = q = 1$) to aggregate all the preference values $b_{ij}, j = 1, 2, 3$ of the i^{th} row and get the overall performance values b_i corresponding to the alternative A_i as

$$b_1 = \{0.2795, 0.2836, \dots, 0.3803, 0.3849\},$$

$$b_2 = \{0.2052, 0.2145, \dots, 0.2902, 0.2913\},$$

$$b_3 = \{0.2066, 0.2045, \dots, 0.2941, 0.2883\}.$$

Step 2: We calculate the scores of all the alternatives according to $b_i, i = 1, 2, 3$:

$$S(b_1) = 0.3325, S(b_2) = 0.2542, S(b_3) = 0.2538.$$

Step 3: Since $S(b_1) > S(b_2) > S(b_3)$, by Definition 4, the ranking of the HFEs $b_i, i = 1, 2, 3$, that is, $b_1 > b_2 > b_3$, and thus, the ranking of the alternatives $A_i, i = 1, 2, 3$, is $A_1 > A_2 > A_3$. Hence A_1 is the best alternative.

Next, if we take $p = 1$ and $q = 3$ in WHFGGHM operator, then

$$b_1 = \{0.2805, 0.2911, \ldots, 0.3687, 0.3722\},$$

$$b_2 = \{0.2805, 0.2911, \ldots, 0.3644, 0.3644\},$$

$$b_3 = \{0.2262, 0.2254, \ldots, 0.3552, 0.3512\}.$$

and the scores of all the alternatives are

$$S(b_1) = 0.3449, S(b_2) = 0.3292, S(b_3) = 0.2907.$$

Thus, the ranking of the alternatives $A_i, i = 1, 2, 3$, now is $A_1 > A_2 > A_3$. Hence A_1 is still the best alternative.

4. Conclusions

In this paper, we extended the idea of aggregation and considering a wider range of aggregating operators, introduced Hesitant Fuzzy Generalized Geometric Heronian Mean (HFGGHM) operator and also that of Weighted Hesitant Fuzzy Generalized Geometric Heronian Mean (WHFGGHM) operator. Properties of the proposed operators are studied and their special cases are examined. Furthermore, we have applied the WHFGGHM operator to multi criteria decision making with hesitant fuzzy numbers. Finally, an illustrative example is given to verify the developed method and to demonstrate its practicality and effectiveness. The work has scope for extensive further application and results on these new measures.

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