FUZZY LINEAR PROGRAMS WITH OCTAGONAL FUZZY NUMBERS

Felbin C. Kennedy *and S.U. Malini †

Abstract. Zimmermann [9] developed the decision making concept in a fuzzy environment which was proposed and analysed by Bellman and Zadeh [3] in 1970. Its application in fuzzy linear programming was well handled by Tanaka et al. [7] and by Maleki et.al in [6] . Later several kinds of fuzzy linear programming problems have been dealt with and various methodologies have been adopted to solve such problems using trapezoidal fuzzy numbers for example as in [4, 8]. The concept of octagonal fuzzy numbers was introduced by the authors in an earlier paper [5]. In this paper the octagonal fuzzy numbers are used to solve fuzzy linear programming problems (FLP) involving simplex method. A method for solving FLP involving symmetric octagonal fuzzy numbers is developed and it may be noted that it is solved without converting to crisp linear programming problem. The process is illustrated with a numerical example involving a real life problem.

The distinguishing factor which is innovative in the present study is the use of a new arithmetic on symmetrical octagonal fuzzy numbers. On this class is introduced a binary operation of multiplication denoted by $*$ defined in Definition 1.2 that is more natural having the desired property $\widetilde{A}*\widetilde{B} \approx -(-\widetilde{A})*\widetilde{B}$ and such a property is absent in the multiplication introduced by earlier authors in [4].

Keywords Fuzzy linear programming, symmetric octagonal fuzzy numbers, ranking.

1 Introduction We adhere to the concepts, notions and notations in [5]. Here we consider a subclass of octagonal fuzzy numbers called symmetrical octagonal fuzzy numbers using which a method for solving fuzzy linear programming problems without converting them to crisp linear programming problem has been discussed. The $*$ multiplication defined in this paper is more natural as it coincides with multiplication of real numbers in crisp case.

In section 1 octagonal fuzzy numbers that are symmetrical is considered and fuzzy arithmetic on this class and fuzzy measure of octagonal fuzzy numbers are defined. In section 2, a general fuzzy linear programming problem is cited and the theory related to simplex algorithm for solving FLP is dealt with. The same is illustrated by using a numerical example in section 3.

Definition 1.1. *A fuzzy number $\tilde{A}$ is called a symmetric octagonal fuzzy number if there exist real numbers $a_1, a_2, a_1 < a_2$ and $h > s > g > 0$ such that*

$$\mu_{\tilde{A}}(x) = \begin{cases} k\left[\frac{x}{h-s} + \frac{h-a_1}{h-s}\right], & x \in [a_1 - h, a_1 - s] \\ k, & x \in [a_1 - s, a_1 - g] \\ k + (1-k)\left[\frac{x}{g} + \frac{g-a_1}{g}\right], & x \in [a_1 - g, a_1] \\ 1, & x \in [a_1, a_2] \\ k + (1-k)\left[\frac{a_2+g}{g} - \frac{x}{g}\right], & x \in [a_2, a_2 + g] \\ k, & x \in [a_2 + g, a_2 + s] \\ k\left[\frac{a_2+h}{h-s} - \frac{x}{h-s}\right], & x \in [a_2 + s, a_2 + h] \\ 0, & otherwise \end{cases} \tag{1.1}$$

We denote it by $\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$.When $h = s = g = 0$; $\tilde{A} \approx (a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2; k, 1)$ reduces to a trapezoidal fuzzy number. The set of all symmetric octagonal fuzzy numbers is denoted by $\mathcal{F}(S_O)$.

*Felbin C. Kennedy is an .Associate Professor, Department of Mathematics, Stella Maris College (Autonomous), Chennai, Tamilnadu, India. (felbinckennedy@gmail.com),

†S.U Malini is a Research Scholar, Department of Mathematics, Stella Maris College & Faculty, D.G. Vaishnav College, Chennai,Tamilnadu, India. (malinisu14@gmail.com)

Definition 1.2.

If $\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$ and $\tilde{B} \approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$ are two symmetric octagonal fuzzy numbers. Then

(i) Addition:

$$\begin{aligned} \tilde{A} + \widetilde{B} \approx \; & (a_1 + b_1 - (h + m), a_1 + b_1 - (s + l), a_1 + b_1 - (g + f), a_1 + b_1, \\ & a_2 + b_2, a_2 + b_2 + (g + f), a_2 + b_2 + (s + l), a_2 + b_2 + (h + m); k, 1) \end{aligned}$$

(ii)Subtraction:

$$\begin{aligned} \tilde{A} - \widetilde{B} \approx \; & (a_1 - b_2 - (h + m), a_1 - b_2 - (s + l), a_1 - b_2 - (g + f), a_1 - b_2, \\ & a_2 - b_1, a_2 - b_1 + (g + f), a_2 - b_1 + (s + l), a_2 - b_1 + (h + m); k, 1) \end{aligned}$$

(iii) Multiplication:

$$\begin{aligned} \widetilde{A} * \widetilde{B} \approx \; & \left(\left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - p \right) - \left(\left| \frac{a_1 + a_2}{2} \right| m + \left| \frac{b_1 + b_2}{2} \right| h \right), \right. \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - p \right) - \left(\left| \frac{a_1 + a_2}{2} \right| l + \left| \frac{b_1 + b_2}{2} \right| s \right), \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - p \right) - \left(\left| \frac{a_1 + a_2}{2} \right| f + \left| \frac{b_1 + b_2}{2} \right| g \right), \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - p \right), \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p \right) \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p \right) + \left(\left| \frac{a_1 + a_2}{2} \right| f + \left| \frac{b_1 + b_2}{2} \right| g \right), \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p \right) + \left(\left| \frac{a_1 + a_2}{2} \right| l + \left| \frac{b_1 + b_2}{2} \right| s \right), \\ & \left. \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p \right) + \left(\left| \frac{a_1 + a_2}{2} \right| m + \left| \frac{b_1 + b_2}{2} \right| h \right); k, 1 \right) \end{aligned}$$

where $p = \left(\frac{\beta - \alpha}{2} \right)$, $\alpha = \min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$, $\beta = \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$

Also, it is clear from (iii) that for any real λ

$$\lambda \tilde{A} \approx \begin{cases} (\lambda(a_1 - h), \lambda(a_1 - s), \lambda(a_1 - g), \lambda a_1, \lambda a_2, \lambda(a_2 + g), \\ \qquad \lambda(a_2 + s), \lambda(a_2 + h); k, 1), \text{for } \lambda \geq 0 \\ (\lambda(a_2 + h), \lambda(a_2 + s), \lambda(a_2 + g), \lambda a_2, \lambda a_1, \lambda(a_1 - g), \\ \qquad \lambda(a_1 - s), \lambda(a_1 - h); k, 1), \text{for } \lambda < 0 \end{cases}$$

Remark 1.3. *Any real number $r \in \mathbb{R}$ can be expressed as $(r, r, r, r, r, r, r, r; k, 1)$. Continuing this view point consider two real numbers $r, s \in \mathbb{R}$ expressed as symmetric octagonal fuzzy numbers $(r, r, r, r, r, r, r, r; k, 1)$ $(s, s, s, s, s, s, s, s; k, 1)$. Using the Definition 1.2 we obtain its product as $(rs, rs, rs, rs, rs, rs, rs, rs; k, 1)$.*

Definition 1.4. *For any symmetric octagonal fuzzy number $\tilde{x}$, let us define $\tilde{x} \succcurlyeq \tilde{0}$ if there exist $a \geq 0$ and $h \geq s \geq g \geq 0$ such that*

$\tilde{x} \succcurlyeq (-(a + h), -(a + s), -(a + g), -a, a, (a + g), (a + s), (a + h); k, 1)$. Note that $(-(a + h), -(a + s), -(a + g), -a, a, (a + g), (a + s), (a + h); k, 1)$ is equivalent to $\tilde{0}$.

Remark 1.5. *$\tilde{x}$ is called zero symmetric octagonal fuzzy number if $\tilde{x} \approx \widetilde{0}$. $\tilde{x}$ is said to be a non-zero symmetric octagonal fuzzy number, if $\tilde{x} \not\approx \widetilde{0}$. If $\tilde{x}$ is a non-negative symmetric octagonal fuzzy number and $\tilde{x}$ is not equivalent to $\widetilde{0}$, then $\tilde{x}$ is called a positive symmetric octagonal fuzzy number denoted $\tilde{x} \succ \widetilde{0}$. If $\tilde{x}$ is a non-positive symmetric octagonal fuzzy number and is not equivalent to $\widetilde{0}$, then $\tilde{x}$ is called a negative symmetric octagonal fuzzy number denoted $\tilde{x} \prec \widetilde{0}$.*

Definition 1.6. *Two symmetrical octagonal fuzzy numbers $\tilde{A}$, $\widetilde{B}$ are called equivalent denoted, $\tilde{A} \approx \widetilde{B}$ if and only if for*

$\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$ and
$\tilde{B} \approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$,
we have

$$\begin{aligned} & (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1) \\ & -(b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1) \\ \approx\ & \left(-\alpha - (h + m), -\alpha - (s + l), -\alpha - (g + f), -\alpha, \alpha, \alpha + (g + f),\right. \\ & \left.\alpha + (s + l), \alpha + (h + m); k, 1\right) \\ \approx\ & \widetilde{0} \end{aligned}$$

i.e.$(a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$
$\approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$
Note that this is possible even if $a_1 \neq b_1$ and $a_2 \neq b_2$.

Remark 1.7. *A particular case of Definition 1.2 taking $h = s = g$ and $m = l = f$ resulting in symmetric trapezoidal fuzzy numbers whose $*$ multiplication will be as follows:*

For $\tilde{A} \approx (a_1 - g, a_1, a_2, a_2 + g)$, $\quad \tilde{B} \approx (b_1 - f, b_1, b_2, b_2 + f)$
Then

$$\begin{aligned} \widetilde{A} * \widetilde{B} \ =\ & \left(\left(\left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) - p\right) - \left(\left|\frac{a_1 + a_2}{2}\right| f + \left|\frac{b_1 + b_2}{2}\right| g\right),\right. \\ & \left(\left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) - p\right), \left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) + p, \\ & \left.\left(\left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) + p\right) + \left(\left|\frac{a_1 + a_2}{2}\right| f + \left|\frac{b_1 + b_2}{2}\right| g\right)\right) \end{aligned}$$

where $p = \left(\frac{\beta - \alpha}{2}\right)$, $\alpha = \min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$, $\beta = \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$

In this case, we note that $\tilde{A} * \tilde{B} \approx -(-\tilde{A}) * \tilde{B}$ is satisfied which falls in line with the classical problems in the crisp case.

Remark 1.8.

(i) Symmetric Octagonal fuzzy numbers satisfy the distributive property i.e. Let $\tilde{a}, \tilde{b}$ and $\tilde{c}$ be any three symmetric octagonal fuzzy numbers, then
$\tilde{c} * (\tilde{a} + \tilde{b}) \approx (\tilde{c} * \tilde{a} + \tilde{c} * \tilde{b})$ and $\tilde{c} * (\tilde{a} - \tilde{b}) \approx (\tilde{c} * \tilde{a} - \tilde{c} * \tilde{b})$,
where addition, subtraction and multiplication is defined by Definition 1.2

(ii) Multiplication operation given by Definition 1.2 asserts that product of two symmetrical octagonal fuzzy numbers is a symmetrical octagonal fuzzy number.

(iii) Insistence on a symmetric product is easier to handle for computational purposes.

Remark 1.9. *In the literature the following fuzzy numbers are considered - triangular fuzzy numbers, trapezoidal fuzzy numbers and hexagonal fuzzy numbers. In this concept we have used octagonal fuzzy numbers recently. The class of such numbers form a tower of subclasses. The last one namely octagonal fuzzy numbers forming the largest subclass. Therefore operations defined for the last class of octagonal fuzzy numbers evidently apply to the smaller classes.*

Definition 1.10. *[5] Let $\tilde{A}$ be a normal octagonal fuzzy number. The value $M_0^{Oct}(\tilde{A})$, called the measure of $\tilde{A}$ is calculated as follows:*

$$\begin{aligned} M_0^{Oct}(\tilde{A}) &= \frac{1}{2}\int_0^k (l_1(r)+l_2(r))dr + \frac{1}{2}\int_k^1 (s_1(t)+s_2(t))dt \quad where \quad 0<k<1 \\ &= \frac{1}{4}[k(a_1+a_2+a_7+a_8)+(1-k)(a_3+a_4+a_5+a_6)] \end{aligned}$$

Remark 1.11. *[5]*

1) If $a_1+a_2+a_7+a_8 = a_3+a_4+a_5+a_6$ we would get the measure of an octagonal number same for any value of k. $(0<k<1)$

2) In case of symmetric octagonal fuzzy numbers, the condition $a_1+a_2+a_7+a_8 = a_3+a_4+a_5+a_6$ holds and hence the measure is independent of the choice of k.

3) If $\tilde{A}$ and $\tilde{B}$ are two normal octagonal fuzzy numbers, then as in [5] we adhere to the following definitions:

i.If $M_0^{Oct}(\tilde{A}) \leq M_0^{Oct}(\tilde{B})$ then $\tilde{A} \preccurlyeq \tilde{B}$

ii. If $M_0^{Oct}(\tilde{A}) = M_0^{Oct}(\tilde{B})$ then $\tilde{A} \approx \tilde{B}$

iii. If $M_0^{Oct}(\tilde{A}) \geq M_0^{Oct}(\tilde{B})$ then $\tilde{A} \succcurlyeq \tilde{B}$

4) Also $\tilde{A} \preccurlyeq \tilde{B}$ and $\tilde{B} \preccurlyeq \tilde{A} \not\Rightarrow \tilde{A} \approx \tilde{B}$

2 FUZZY LINEAR PROGRAM

The mathematical model

$$(2.1) \quad \left. \begin{array}{c} \min \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j \\ \text{Subject to constraints} \\ \sum_{j=1}^{n} a_{ij}\tilde{x}_j \preccurlyeq \tilde{b}_i, \ i=1,2,\ldots,m_0 \\ \sum_{j=1}^{n} a_{ij}\tilde{x}_j \succeq \tilde{b}_i, \quad i=m_0+1, m_0+2,\ldots,m \\ \text{and } \tilde{x}_j \succcurlyeq \tilde{0} \text{ for all } j=1,2,\ldots,n \end{array} \right\}$$

where $a_{ij} \in \mathbb{R}$, $\tilde{c}_j$, $\tilde{x}_j$, $\tilde{b}_i \in \mathcal{F}(S_O)\, i=1,2,\ldots,m, j=1,2,\ldots,n$ and $\mathcal{F}(S_O)$ the set of all symmetric octagonal fuzzy numbers, is called a fuzzy linear programming problem.

Definition 2.1. *Any $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n) \in \mathcal{F}^n(S_O)(= \mathcal{F}(S_O) \times \mathcal{F}(S_O) \times \ldots \times \mathcal{F}(S_O)$: (n fold)), where each $\tilde{x}_i \in \mathcal{F}(S_O)$, which satisfies 2.1 is said to be a fuzzy feasible solution to equation 2.1.*

Definition 2.2. *A fuzzy feasible solution is called a fuzzy optimum solution to equation 2.1, denoted $(\tilde{x}_1^o, \tilde{x}_2^o, \ldots, \tilde{x}_n^o) \in Q$ if $\sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j^o \preccurlyeq \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j \forall$ elements of Q ,where Q is the set of all fuzzy feasible solutions of equation 2.1 .*

Definition 2.3. *If $\tilde{x}_j \approx (-(\alpha_j+h_j), -(\alpha_j+s_j), -(\alpha_j+g_j), -\alpha_j, \alpha_j, (\alpha_j+g_j), (\alpha_j+s_j), (\alpha_j+h_j); k, 1)$ for some $\alpha_j \geq 0$ and $h_j \geq s_j \geq g_j \geq 0$, then $\tilde{x}$ is said to be a fuzzy basic solution, where $\tilde{x}$ solves $A\tilde{x} \approx \tilde{b}$, A being the appropriate matrix (a_{ij}). If $\tilde{x}_j \not\approx (-(\alpha_j+h_j), -(\alpha_j+s_j), -(\alpha_j+g_j), -\alpha_j, \alpha_j, (\alpha_j+g_j), (\alpha_j+s_j), (\alpha_j+h_j); k, 1)$ for all $\alpha_j \geq 0$ and $h_j \geq s_j \geq g_j \geq 0$, then $\tilde{x}$ has some non-zero components which can be reordered if required, say $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_t, 1 \leq t \leq n$.Then $A\tilde{x} \approx \tilde{b}$ becomes*

$$
\begin{aligned}
& a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_t\tilde{x}_t + a_{t+1}[(-(\alpha_{t+1}+h_{t+1}), -(\alpha_{t+1}+s_{t+1}), \\
& -(\alpha_{t+1}+g_{t+1}), -\alpha_{t+1}, \alpha_{t+1}, (\alpha_{t+1}+g_{t+1}), (\alpha_{t+1}+s_{t+1}), \\
& (\alpha_{t+1}+h_{t+1}); k, 1] + a_{t+2}[(-(\alpha_{t+2}+h_{t+2}), -(\alpha_{t+2}+s_{t+2}), \\
& -(\alpha_{t+2}+g_{t+2}), -\alpha_{t+2}, \alpha_{t+2}, (\alpha_{t+2}+g_{t+2}), (\alpha_{t+2}+s_{t+2}), (\alpha_{t+2}+h_{t+2}); k, 1] \\
& + \cdots + a_n[(-(\alpha_n+h_n), -(\alpha_n+s_n), -(\alpha_n+g_n), -\alpha_n, \\
& \alpha_n, (\alpha_n+g_n), (\alpha_n+s_n), (\alpha_n+h_n); k, 1] \\
\approx\ & \tilde{b}
\end{aligned}
$$

And $\tilde{x}$ will become a fuzzy basic solution if the columns $a_1, a_2, \ldots, a_t$ corresponding to these non-zero components $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_t$ are linearly independent.

Remark 2.4. *Given a system of m simultaneous fuzzy linear equations involving symmetric octagonal fuzzy numbers in n unknowns $(m \leq n)$ $A\tilde{x} \approx \tilde{b}$; $\tilde{b} \in \mathcal{F}^m(S_O)$ where A is a $(m \times n)$ real matrix and rank of A is m. Let B be any $(m \times m)$ matrix formed by m linearly independent columns of A. Then the fuzzy basic solution is $\tilde{x}_B = B^{-1}\tilde{b}$, where $\tilde{x}_B \in \mathcal{F}^m(S_O)$. We will eventually prove that, if $\tilde{x}_B$ is a basic solution for the fuzzy linear programming problem equation 2.1, then a solution to the given system is $[\tilde{x}_B, \tilde{0}]$ where $\tilde{0} \in \mathcal{F}^{n-m}(S_O)$*

i.e.$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k, \tilde{0}, \tilde{0}, \ldots \tilde{0})$. In this case we also say that $\tilde{x}_B$ is a fuzzy basic solution.

We shall now give the fuzzy analogues of some important linear programming results.

The standard form of any fuzzy linear programming problem is given by:

$$
\left.
\begin{array}{rl}
 & \min \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j \\
\text{Subject to} & A\tilde{x} \approx \tilde{b} \ \text{ and } \tilde{x} \succcurlyeq \tilde{0}
\end{array}
\right\} \tag{2.2}
$$

where $A = (a_{ij})$ is an $(m \times n)$ real matrix, $\tilde{b}$, $\tilde{c}$, $\tilde{x}$ are $(m \times 1)$, $(1 \times n)$, $(n \times 1)$ fuzzy matrices consisting of symmetric octagonal fuzzy numbers.

Definition 2.5. *We say that a fuzzy vector $\tilde{x} \in \mathcal{F}(\mathbb{R})^n$ is a fuzzy feasible solution to the problem given by equation 2.2 if $\tilde{x}$ satisfies the constraints of the problem.*

Definition 2.6. *A fuzzy feasible solution $\tilde{x}_* \in \mathcal{F}(\mathbb{R})^n$ is a fuzzy optimal solution for equation 2.2, if for all fuzzy feasible solution $\tilde{x}$ for equation 2.2, we have $\tilde{c}\tilde{x} \preccurlyeq \tilde{c}\tilde{x}_*$*

Improving a fuzzy basic feasible solution

Let the basis for the columns of A be $B = (b_1, b_2, \ldots, b_m)$. Let a fuzzy basic feasible solution be $\tilde{x}_B \approx B^{-1}\tilde{b}$ and the fuzzy value of $\tilde{z}$ is given by $\tilde{z}_0 \approx \tilde{c}_B * \tilde{x}_B$, where

$\tilde{c}_B = (\tilde{c}_{B_1}, \tilde{c}_{B_2}, \tilde{c}_{B_3}, \ldots \tilde{c}_{B_m})$ is the corresponding cost vector of $\tilde{x}_B$.

Suppose that

$$
a_j = \sum_{i=1}^{m} y_{ij} b_i = y_j B
$$

and the symmetric octagonal fuzzy number $\tilde{z}_j = \sum_{i=1}^{m} \tilde{c}_{B_i} y_{ij} = \tilde{c}_B y_j$ are known for every column vector a_j in A, which is not in B. Let us now examine the possibility of finding another fuzzy feasible solution which will improve the fuzzy value of $\tilde{z}$, by replacing one of the columns in B by a_j.

Theorem 2.7. *Let $\tilde{x}_B$ be a fuzzy basic feasible solution of equation 2.2 such that $\tilde{x}_B \approx B^{-1}\tilde{b}$. If the condition $(\tilde{z}_j - \tilde{c}_j) \succ \tilde{0}$ hold for any column a_j in A which is not in B and $y_{ij} > 0$ for some i, $i \in \{1, 2, 3, \ldots, m\}$ then we can obtain a new fuzzy basic feasible solution by replacing one of the columns in B by a_j.*

Proof. Let $\tilde{x}_B \approx (\tilde{x}_{B_1}, \tilde{x}_{B_2}, \ldots, \tilde{x}_{B_m})$ be a fuzzy basic feasible solution with t positive components such that $B\tilde{x}_B = \tilde{b}$ or $\tilde{x}_B = B^{-1}\tilde{b}$ where $\tilde{x}_{B_i} \approx ((\alpha_i - h_i), (\alpha_i - s_i), (\alpha_i - g_i), \alpha_i, \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1), \alpha_i \leq \beta_i, h_i \geq s_i \geq g_i \geq 0, 0 < k < 1$and

$M_0^{Oct}(\tilde{x}_{B_i}) > 0$ for $i = 1, 2, \ldots, t$ and $M_0^{Oct}(\tilde{x}_{B_i}) = 0$ for $i = t+1, t+2, \ldots, m$

i.e., $\tilde{x}_{B_i} \succ \tilde{0}$ for $i = 1, 2, \ldots, t$ and

$\tilde{x}_{B_i} \approx (-(\beta_i + h_i), -(\beta_i + s_i), -(\beta_i + g_i), -\beta_i, \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1)$ for $i = t+1, t+2, \ldots, m; 0 < k < 1$ Then $B\tilde{x}_B \approx \tilde{b}$ is written as

$$\begin{aligned}
& \sum_{i=1}^{t} \tilde{x}_{B_i} b_i + (-(\beta_{t+1} + h_{t+1}), -(\beta_{t+1} + s_{t+1}), -(\beta_{t+1} + g_{t+1}), -\beta_{t+1}, \\
& \beta_{t+1}, (\beta_{t+1} + g_{t+1}), (\beta_{t+1} + s_{t+1}), (\beta_{t+1} + h_{t+1}); k, 1) b_{t+1} + \\
& (-(\beta_{t+2} + h_{t+2}), -(\beta_{t+2} + s_{t+2}), -(\beta_{t+2} + g_{t+2}), -\beta_{t+2}, \beta_{t+2}, (\beta_{t+2} + g_{t+2}), \\
& (\beta_{t+2} + s_{t+2}), (\beta_{t+2} + h_{t+2}); k, 1) b_{t+2} + \cdots \\
& +(-(\beta_m + h_m), -(\beta_m + s_m), -(\beta_m + g_m), -\beta_m, \beta_m, \\
& (\beta_m + g_m), (\beta_m + s_m), (\beta_m + h_m); k, 1) b_m \\
\approx\ & \tilde{b}
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \sum_{i=1}^{t} \tilde{x}_{B_i} b_i + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), -(\beta_i + g_i), -\beta_i, \\
& \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ & \tilde{b} \qquad (2.3)
\end{aligned}$$

Now any column a_j of A not in B can be written as

$a_j = \sum\limits_{i=1}^{m} y_{ij} b_i = y_{1j} b_1 + y_{2j} b_2 + \cdots + y_{rj} b_r + \cdots + y_{mj} b_m = \mathbf{y}_j B.$

Also if the basis vector b_r for which $y_{rj} \neq 0$ is replaced by a_j of A, then $(b_1, b_2, \ldots, b_{r-1}, a_j, b_{r+1}, \ldots, b_m)$ still forms a basis.

Now for $y_{rj} \neq 0$ and $r \leq t$, we can write

$b_r = \frac{a_j}{y_{rj}} - \sum\limits_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}} b_i = \frac{a_j}{y_{rj}} - \sum\limits_{\substack{i=1 \\ i \neq r}}^{t} \frac{y_{ij}}{y_{rj}} b_i = \frac{a_j}{y_{rj}} - \sum\limits_{i=t+1}^{m} \frac{y_{ij}}{y_{rj}} b_i$

Equation 2.3 becomes

$$\begin{aligned}
& \sum_{\substack{i=1 \\ i \neq r}}^{t} \tilde{x}_{B_i} b_i + \tilde{x}_{B_r} b_r + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), -(\beta_i + g_i), -\beta_i, \\
& \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ & \tilde{b}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & \sum_{\substack{i=1 \\ i \neq r}}^{t} \tilde{x}_{B_i} b_i + \frac{\tilde{x}_{B_r}}{y_{rj}} a_j - \frac{\tilde{x}_{B_r}}{y_{rj}} \sum_{\substack{i=1 \\ i \neq r}}^{t} y_{ij} b_i - \frac{\tilde{x}_{B_r}}{y_{rj}} \sum_{i=t+1}^{m} y_{ij} b_i + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), \\
& -(\beta_i + g_i), -\beta_i, \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ & \tilde{b}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad & \sum_{\substack{i=1\\ i\neq r}}^{t}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j+\sum_{i=t+1}^{m}(-(\beta_i+h_i),-(\beta_i+s_i),-(\beta_i+g_i),-\beta_i,\\
& \beta_i,(\beta_i+g_i),(\beta_i+s_i),(\beta_i+h_i);k,1)-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij})b_i\\
\approx \quad & \tilde{b}
\end{aligned}
$$

Since $\tilde{x}_{B_i}\approx(-(\beta_i+h_i),-(\beta_i+s_i),-(\beta_i+g_i),-\beta_i,\beta_i,(\beta_i+g_i),(\beta_i+s_i),(\beta_i+h_i);k,1)$ for $i=t+1,t+2,\ldots,m$, we have

$$
\sum_{\substack{i=1\\ i\neq r}}^{t}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j+\sum_{i=t+1}^{m}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i\approx\tilde{b}
$$

$$
\Rightarrow\sum_{\substack{i=1\\ i\neq r}}^{m}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j\approx\tilde{b}
$$

$$
\Rightarrow\sum_{\substack{i=1\\ i\neq r}}^{m}\hat{\tilde{x}}_{B_i}b_i+\hat{\tilde{x}}_{B_r}a_j\approx\tilde{b}
$$

where $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right),i\neq r$ and $\hat{\tilde{x}}_{Br}=\frac{\tilde{x}_{B_r}}{y_{rj}}$ which is a improved fuzzy basic solution to $A\tilde{x}\approx\tilde{b}$.

We shall now prove that the improved solution is also feasible. That is to prove that $\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)\succcurlyeq\tilde{0},i\neq r$ and $\frac{\tilde{x}_{B_r}}{y_{rj}}\succcurlyeq\tilde{0}$. To this end, select $y_{rj}>0$ such that $\frac{\tilde{x}_{B_r}}{y_{rj}}\approx min\left\{\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}>0\right\}$. Then $\frac{\tilde{x}_{B_r}}{y_{rj}}\preccurlyeq\frac{\tilde{x}_{B_i}}{y_{ij}}$

$$
\begin{aligned}
\Rightarrow \quad & \left(\frac{\alpha_r-h_r}{y_{rj}},\frac{\alpha_r-s_r}{y_{rj}},\frac{\alpha_r-g_r}{y_{rj}},\frac{\alpha_r}{y_{rj}},\frac{\beta_r}{y_{rj}},\frac{\beta_r+g_r}{y_{rj}},\frac{\beta_r+s_r}{y_{rj}},\frac{\beta_r+h_r}{y_{rj}}\right)\\
\preccurlyeq \quad & \left(\frac{\alpha_i-h_i}{y_{ij}},\frac{\alpha_i-s_i}{y_{ij}},\frac{\alpha_i-g_i}{y_{ij}},\frac{\alpha_i}{y_{ij}},\frac{\beta_i}{y_{ij}},\frac{\beta_i+g_i}{y_{ij}},\frac{\beta_i+s_i}{y_{ij}},\frac{\beta_i+h_i}{y_{ij}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \left(\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{h_r}{y_{rj}}+\frac{h_i}{y_{ij}}\right),\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{s_r}{y_{rj}}+\frac{s_i}{y_{ij}}\right),\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{g_r}{y_{rj}}+\frac{g_i}{y_{ij}}\right),\right.\\
& \frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}},\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}},\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{g_r}{y_{rj}}+\frac{g_i}{y_{ij}}\right),\\
& \left.\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{s_r}{y_{rj}}+\frac{s_i}{y_{ij}}\right),\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{h_r}{y_{rj}}+\frac{h_i}{y_{ij}}\right)\right)\\
\succcurlyeq \quad & \tilde{0}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}},\frac{\alpha_i-\beta_r}{y_{ij}},\right.\\
& \left.\frac{\beta_i-\alpha_r}{y_{ij}},\frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)\\
\succcurlyeq \quad & \tilde{0}
\end{aligned}
$$

$\Rightarrow$

$$M_0^{Oct}\left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}}, \frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}}, \frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}}, \frac{\alpha_i-\beta_r}{y_{ij}},\right.$$
$$\left.\frac{\beta_i-\alpha_r}{y_{ij}}, \frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}}, \frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}}, \frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)$$
$$\geq \ \tilde{0}$$

$$\Rightarrow \ \frac{1}{4}\left(k\left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}}+\frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}}+\frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}}+\frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)\right.$$
$$\left.+\ (1-k)\left(\frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}}+\frac{\alpha_i-\beta_r}{y_{ij}}+\frac{\beta_i-\alpha_r}{y_{ij}}+\frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}}\right)\right.$$
$$\geq \ 0$$

$$\Rightarrow \frac{1}{4}\left(\frac{\alpha_i-\beta_r}{y_{ij}}+\frac{\beta_i-\alpha_r}{y_{ij}}\right)\geq 0$$

$$\Rightarrow \left(\frac{\alpha_i+\beta_i}{y_{ij}}\right)-\left(\frac{\alpha_r+\beta_r}{y_{ij}}\right)\geq 0 \Rightarrow \left(\frac{\tilde{x}_{B_i}}{y_{ij}}-\frac{\tilde{x}_{B_r}}{y_{rj}}\right)\succcurlyeq \tilde{0}$$

and hence the improved solution is a fuzzy basic feasible solution and the theorem is proved. □

Remark 2.8. *The new basis matrix obtained after replacing the basis vectors is $\hat{B}=(\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_m)$, where $\hat{b}_i=b_i$ for $i\neq r$ and $\hat{b}_r=a_j$. The new fuzzy basic feasible solution is $\hat{\tilde{x}}_B$, where $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right), i\neq r$ and $\hat{\tilde{x}}_{B_r}=\frac{\tilde{x}_{B_r}}{y_{rj}}$ are the basic variables.*

Theorem 2.9. *If $\tilde{x}_B$ and $\hat{\tilde{x}}_B$ are the fuzzy basic feasible solutions of 2.2 having their objective values $\tilde{z}_0\approx\tilde{c}_B*\tilde{x}_B$ and $\hat{\tilde{z}}\approx\hat{\tilde{c}}_B*\hat{\tilde{x}}_B$ respectively and if $\hat{\tilde{x}}_B$ was the value obtained after admitting a_j in the basis, it being a non basic column vector and also for which $(\tilde{z}_j-\tilde{c}_j)\succ\tilde{0}$ and $y_{ij}>0$ for some $i, i\in\{1,2,3,\ldots,m\}$, then $\hat{\tilde{z}}\preccurlyeq\tilde{z}_0$.*

Proof. Given $\tilde{x}_B$ be a fuzzy basic feasible solution and $\tilde{z}_0\approx\tilde{c}_B\tilde{x}_B$. Let b_r be the column vector removed from the basis in place of which a_j is introduced. Also given that $\hat{\tilde{x}}_B$ is the new fuzzy basic feasible solution, then $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right), i\neq r$ and $\hat{\tilde{x}}_{B_r}=\frac{\tilde{x}_{B_r}}{y_{rj}}$.

Since $\hat{\tilde{c}}_{B_i}\approx\tilde{c}_{B_i}, i\neq r$ and $\hat{\tilde{c}}_{B_r}=\tilde{c}_j$, the modified fuzzy value of the objective function is

$$\hat{\tilde{z}}\approx\hat{\tilde{c}}_B\hat{\tilde{x}}_B\approx\sum_{i=1}^{m}\hat{\tilde{c}}_{B_i}\hat{\tilde{x}}_{B_i}\approx\sum_{\substack{i=1\\i\neq r}}^{m}\hat{\tilde{c}}_{B_i}\hat{\tilde{x}}_{B_i}+\hat{\tilde{c}}_{B_r}\hat{\tilde{x}}_{B_r}$$
$$\approx\sum_{\substack{i=1\\i\neq r}}^{m}\tilde{c}_{B_i}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\sum_{i=1}^{m}\tilde{c}_{B_i}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\sum_{i=1}^{m}\tilde{c}_{B_i}\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}\sum_{i=1}^{m}\tilde{c}_{B_i}y_{ij}+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\tilde{z}_0-\frac{\tilde{x}_{B_r}}{y_{rj}}\tilde{z}_j+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$

$$\approx \tilde{z}_0 - \frac{\tilde{x}_{B_r}}{y_{rj}}(\tilde{z}_j - \tilde{c}_j) \quad (2.4)$$

Since $y_{rj} > 0$, $(\tilde{z}_j - \tilde{c}_j) \succ \tilde{0}$ and $\frac{\tilde{x}_{B_r}}{y_{rj}} \succcurlyeq \tilde{0}$, hence $\frac{\tilde{x}_{B_r}}{y_{rj}}(\tilde{z}_j - \tilde{c}_j) \succeq \tilde{0}$.

So Equation 2.4 implies $\hat{\tilde{z}} \preccurlyeq \tilde{z}_0$. Hence the new fuzzy basic feasible solution gives the improved fuzzy value of the objective function. □

Condition of optimality

Similar to classical linear programming problem, we can prove that the process of inserting and removing vectors from the basis matrix will lead to the following situations

i) unbounded solution

ii) infeasible solution

iii) optimal solution. In which case, $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$.

Hence we have the following theorem whose proof is immediate.

Theorem 2.10. *If $\tilde{x}_B = B^{-1}\tilde{b}$ is a fuzzy basic feasible solution of 2.2 and if $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$ for every column a_j of A, then $\tilde{x}_B$ is a fuzzy optimal solution to 2.2.*

Remark 2.11. *Here we solve a fuzzy linear programming problem whose optimal function is to be minimised and whose variables are non-negative. Hence we arrive at two situations and they are i) we obtain the optimal solution or ii)we obtain an infeasible solution in sense the optimality will not be reached inspite of repeated iterations.*

Remark 2.12. *Theorem2.9 and Theorem2.10 gives sufficient condition for existence of optimal solution. It is to be seen whether this sufficient condition ensures convergence of the iteration problem. Condition for convergence of iteration process needs to be studied separately.*

Remark 2.13. *If we consider the maximisation problem given by*

$$\max \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j$$
$$\textit{Subject to} \quad A\tilde{x} \approx \tilde{b} \quad \textit{and} \quad \tilde{x} \succcurlyeq \tilde{\mathbf{0}}$$

then this problem can be converted into a minimisation problem given by

$$max\tilde{z} \approx -min(-\tilde{z}) \approx \sum_{j=1}^{n}(-\tilde{c}_j) * \tilde{x}_j$$
$$\text{Subject to} \quad A\tilde{x} \approx \tilde{b} \quad \text{and} \quad \tilde{x} \succcurlyeq \tilde{\mathbf{0}}$$

and solved as above.

3 Numerical Example

Food A contains 20 units of Proteins and 40 units of minerals per gram. Food B contains 30 units each of Proteins and minerals. The daily minimum human requirements of Protein and Mineral are 900 units and 1200 units respectively. How many grams of each type of food should be consumed so as to minimize the cost, if food A costs $Rs.6$ per gram and food B costs $Rs.8$ per gram.

Note that the daily minimum human requirements of proteins and minerals may vary from individual to individual. Also the cost of food may vary depending on the market condition. Due to these uncertain variations the problem is modelled as a fuzzy linear programming problem and symmetrical octagonal fuzzy numbers are used to describe these uncertain values.

Cost of Food A 6 is modeled as $(2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)$and the same is done for the other parameters also . Hence the mathmatical formulation of the above problem is given by fuzzy linear programming problem as

$\min \tilde{z} \approx (2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)\tilde{x}_1 + (3, 4, 5, 7, 9, 11, 12, 13; 0.3, 1)\tilde{x}_2$

Subject to $20\tilde{x}_1 + 30\tilde{x}_2 \succcurlyeq (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$

$40\tilde{x}_1 + 30\tilde{x}_3 \succcurlyeq (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$

$\tilde{x}_1 \succcurlyeq \tilde{0}, \tilde{x}_2 \succcurlyeq \tilde{0}.$

Hence the fuzzy linear programming problem in standard form is

$min\tilde{z} \approx (2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)\tilde{x}_1 + (3, 4, 5, 7, 9, 11, 12, 13; 0.3, 1)\tilde{x}_2$

Subject to $20\tilde{x}_1 + 30\tilde{x}_2 - \tilde{S}_1 + \tilde{A}_1 \approx (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$

$40\tilde{x}_1 + 30\tilde{x}_3 - \tilde{S}_2 + \tilde{A}_2 \approx (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$

$\tilde{x}_1, \tilde{x}_2, , \tilde{S}_1, \tilde{S}_2, \tilde{A}_1, \tilde{A}_2 \succcurlyeq \tilde{0}$

where $\tilde{S}_1, \tilde{S}_2,$ are the surplus fuzzy variables and $\tilde{A}_1, \tilde{A}_2,$ are artificial variables. That is $min\ \tilde{z} \approx \tilde{c}\tilde{x}$ subject to $A\tilde{x} \approx \tilde{b}$ and $\tilde{x} \succcurlyeq \tilde{0}$, where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 20 & 30 & -1 & 0 & 1 & 0 \\ 40 & 30 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$\tilde{b} \approx \begin{pmatrix} (885,886,888,890,910,912,914,915;0.3,1) \\ (1190,1191,1193,1195,1205,1207,1209,1210;0.3,1) \end{pmatrix}, \tilde{x} \approx (\tilde{x}_1 \quad \tilde{x}_2 \quad \tilde{S}_1 \quad \tilde{S}_2 \quad \tilde{A}_1 \quad \tilde{A}_2)$

And $\tilde{c} \approx ((885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1),$
$(1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1), \tilde{0}, \tilde{0}, \tilde{M}, \tilde{M})$

Initial Iteration: The initial fuzzy basic feasible solution is given by $\tilde{x}_B \approx B^{-1}\tilde{b}$, where

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{x} = \begin{matrix} \tilde{A}_1 \\ \tilde{A}_2 \end{matrix}, \tilde{b} \approx \begin{pmatrix} (885,886,888,890,910,912,914,915;0.3,1) \\ (1190,1191,1193,1195,1205,1207,1209,1210;0.3,1) \end{pmatrix}$ and

$\tilde{x}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{x}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{A}_1 \approx (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$,
$\tilde{A}_2 \approx (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$

and the fuzzy objective value is

$\tilde{z} \approx (2075M, 2077M, 2081M, 2085M, 2115M, 2119M, 2123M, 2125M; 0.3, 1)$.

Now $(\tilde{z}_1 - \tilde{c}_1) \succ \tilde{0}$ is one among the highest positive value among all $\tilde{z}_j - \tilde{c}_j$

First Iteration: By Theorem 2.7 and Theorem 2.9, we get a new fuzzy basic feasible solution
$\tilde{x}_1 \approx \left(\frac{1190}{40}, \frac{1191}{40}, \frac{1193}{40}, \frac{1195}{40}, \frac{1205}{40}, \frac{1207}{40}, \frac{1209}{40}, \frac{1210}{40}; 0.3, 1\right)$, $\tilde{x}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$
$\tilde{A}_1 \approx \left(\frac{560}{2}, \frac{563}{2}, \frac{569}{2}, \frac{575}{2}, \frac{625}{2}, \frac{631}{2}, \frac{637}{2}, \frac{640}{2}; 0.3, 1\right)$, $\tilde{A}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$

with the improved fuzzy objective value

$$\tilde{z} \approx \Big(\frac{11200M+2320}{40}, \frac{11260M+3532}{40}, \frac{11380M+4751}{40}, \frac{11500M+5970}{40}, \frac{12500M+8430}{40}, \frac{12620M+9649}{40}, \frac{12740M+10868}{40}, \frac{12800M+12080}{40}; 0.3, 1).$$

Here $(\tilde{z}_2 - \tilde{c}_2) \succ \tilde{0}$ is the highest positive value among all $\tilde{z}_j - \tilde{c}_j$

Second Iteration: Proceeding in a similar way, we get a new fuzzy basic feasible solution
$\tilde{x}_1 \approx \left(\frac{550}{40}, \frac{554}{40}, \frac{562}{40}, \frac{570}{40}, \frac{630}{40}, \frac{638}{40}, \frac{646}{40}, \frac{650}{40}; 0.3, 1\right)$,
$\tilde{x}_2 \approx \left(\frac{560}{30}, \frac{563}{30}, \frac{569}{30}, \frac{575}{30}, \frac{625}{30}, \frac{631}{30}, \frac{637}{30}, \frac{640}{30}; 0.3, 1\right)$,
$\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1), \tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{A}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1), \tilde{A}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$

with the improved fuzzy objective value,

$\tilde{z} \approx \left(\frac{7830}{120}, \frac{12412}{120}, \frac{17186}{120}, \frac{24460}{120}, \frac{35540}{120}, \frac{42814}{120}, \frac{47588}{120}, \frac{52170}{120}; 0.3, 1\right)$.

Here $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$ for all j. Here by Theorem 2.10, the feasible solution obtained now is a fuzzy optimal solution.

Remark 3.1. *The parameter k is involved in calculating the ratio between the minimum requirements and the selected y_{ij} column elements.*

Remark 3.2. *The measure M_0^{oct} considered (as in Definition 1.10) is used in comparing the ratios mentioned in Remark.3.1*

Remark 3.3. *The choice of trapezoidal, hexagonal or octagonal fuzzy numbers for minimum feasible solution range seems to be dependent on the parameter k. Details of these investigations will be published after completion.*

Remark 3.4. *In [1] the authors have proved the following theorem, For a fixed partition P_m, the set $\mathcal{F}_{c,m}(\mathbb{R})$ is isomorphic to the convex closed cone $C = \left\{z \in \mathbb{R}^{2(m+1)} : Bz \geqslant 0\right\}$ with*

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & ... & 0 \\ 0 & -1 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & 1 & ... & 0 \\ . & & & & . & \\ . & & & & . & \\ . & & & & . & \\ 0 & 0 & ... & 0 & -1 & 1 \end{bmatrix}_{(2m+1)x(2m+2)}$$

where $\mathcal{F}_{c,m}(\mathbb{R})$ is the space of real fuzzy subsets of $\mathbb{R}$ with level sets belonging to the space of compact convex subsets of $\mathbb{R}$ endowed with suitable Hausdorff metric.This implies that trapezoidal fuzzy numbers correspond to the case $m = 4$ and octagonal fuzzy numbers correspond to the case $m = 8$ in the mentioned theorem. As these cones are distinct it is clear that these numbers also have distinct properties. Also trapezoidal fuzzy numbers can be viewed as special case of octagonal fuzzy numbers.

4 Conclusion In this paper FLP is solved without converting it to crisp linear programming problem. If this problem is solved using several steps with trapezoidal fuzzy numbers, then there is a sizeable difference in the spread of the solution when compared to the solution obtained for octagonal fuzzy number problem and this diference is dependent on the choice of k. The concept of octagonal fuzzy numbers enables using more data relating to the problem and obtaining similarly more data about the solution. *Also the $*$ multiplication defined in this paper is more natural.* Approximation of any fuzzy number by trapezoidal fuzzy numbers was considered by Ban et.al. and symmetric trapezoidal approximation is cited as future work [2].We may have to consider whether the data given in our problem can be approximated to trapezoidal fuzzy numbers given in [2] which would give a solution, which is nearer to the solution given using octagonal fuzzy numbers as such. Octagonal Approximations in a fuzzy environment may be considered for future work.

Acknowledgement The authors wish to thank Dr. M.S. Rangachari and Dr. P.V. Subrahmanyam for their useful discussions and reviewers for their comments which were useful in shaping the paper.

References

[1] Báez-Sáncheza A.D., A.C.Morettib, M.A.Rojas-Medar, "On polygonal fuzzy sets and numbers", Fuzzy Sets and Systems, 209 (2012) 54–65.

[2] A. I.Ban, L. Coroianu, "Existence, uniqueness and continuity of trapezoidal approximations of fuzzy numbers under a general condition", Fuzzy Sets and Systems, (2013), http://dx.doi.org/10.1016/j.fss.2013.07.004

[3] Bellman, R.E. and L.A. Zadeh, "Decision making in fuzzy environment", Management Science 17(1970), 141-164.

[4] K. Ganesan and P. Veeramani, "Fuzzy Linear Programs with trapezoidal fuzzy numbers", Ann. Oper. Res 143 (2006) 305-315.

[5] S.U. Malini, Felbin C Kennedy, "An Approach for Solving Fuzzy Transportation Problem Using Octagonal Fuzzy Numbers", Applied Mathematical Sciences, 7(2013), no.54, 2661-2673.

[6] Maleki, H.R.., M.Tata and M. Mashinchi, "Linear Programming with Fuzzy variables", Fuzzy Sets and Systems 109(2000), 21-33.

[7] Tanaka H., T. Okuda and K.Asai, "On Fuzzy Mathematical Programming" Journal of Cybernetics, 3 (1974) 37-46.

[8] Tanaka, H. and K. Asai , "Fuzzy Linear Programming Problems with Fuzzy Numbers", Fuzzy Sets and Systems, 13(1984) 1-10.

[9] Zimmermann, H.J., "Optimization in Fuzzy Environment, presented at XXI Int. TIMES and 46th ORSA Conference, San Juan, Puerto Rioco,1974.