# Exponential information measures on pairs of fuzzy sets

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**Abstract**

Exponential fuzzy entropy with a single fuzzy set has been considered. Here we consider exponential information theoretic measures with pair of fuzzy sets. This leads to results that parallel the results as Shannon's joint, conditional and mutual information measures between two fuzzy sets.
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## 1. Introduction

Fuzzy sets proposed by Zadeh [7] in 1965 have gained extensive applications in many areas such as engineering, artificial intelligence, medical science, signal processing, decision making and so on because of its capability to represent/ model non-statistical imprecision or vague concepts.

In Fuzzy set theory, the entropy is defined as a measure of fuzziness which expresses the amount of ambiguity or difficulty in making a decision whether an element belongs to a set or not. The first measure of fuzziness associated with a fuzzy set also mentioned by Zadeh [8] in 1968. In 1972, De Luca and Termini [2] formulated axioms for the entropy of fuzzy sets and defined the measure of fuzzy entropy based on Shannon's function [5].

In addition, Yager [6] defined a measure of fuzzy entropy in terms of a lack of distinction between fuzzy set and its complement. In 1989, Pal and Pal [3] proposed a new measure of fuzzy entropy based on exponential function called '*exponential fuzzy entropy'*. Recently, Verma and Sharma [4] have introduced a parametric generalized entropy measure for fuzzy sets called '*exponential fuzzy entropy of order-$\alpha$*'. In 2007, Ding et al. [1] extended the notion of fuzzy entropy to define conditional fuzzy entropy, joint fuzzy entropy and fuzzy mutual information corresponding to De Luca and Termini's fuzzy entropy and studied their relations also.

In this paper, we extend the idea of measure of exponential fuzzy entropy on pairs of fuzzy sets and propose some new exponential fuzzy entropy measures such as exponential fuzzy joint entropy, exponential fuzzy conditional entropies. Further, a measure of exponential fuzzy mutual information is defined here. Some relations among them are also studied.

This paper is organized as follows: In Section 2 basic definitions related to probability theory, fuzzy sets, and fuzzy entropy measures are briefly reviewed. In Section 3 exponential fuzzy joint entropy and exponential fuzzy conditional entropies are introduced and some of their properties are proved. In Section 4 the concept of exponential fuzzy mutual information measure is proposed and studied their properties. Our conclusions are presented in the final section.

## 2. Preliminaries

In this section we give some basic concepts and definitions related to probability theory, fuzzy sets, which will be used in the following analysis.

Let $\Delta_n = \left\{ P = (p_1, p_2, ..., p_n) : p_j \geq 0, \sum_{j=1}^{n} p_j = 1 \right\}$ , $n \geq 2$ be a set of $n$-complete probability distributions. For any probability distribution $P = (p_1, p_2, ..., p_n) \in \Delta_n$, Shannon's entropy [5], is defined as

$$H_S(P) = -\sum_{j=1}^{n} p_j \log p_j \tag{1}$$

After the pioneering work of De Luca and Termini [2], various measures of fuzzy entropy have been proposed by many researchers and developed their applications in different areas. In 1989, Pal and Pal [3] analyzed the classical Shannon's entropy and proposed a new measure of probabilistic entropy based on exponential function as follows

$$_eH(P) = \sum_{j=1}^{n} p_j \left( e^{1-p_j} - 1 \right) \tag{2}$$

Pal and Pal also pointed out that, the measure of exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $P = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, exponential entropy has a fixed upper bound

$$\lim {}_eH\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right) = (e-1) \qquad \text{as } n \to \infty. \tag{3}$$

which is not the case for Shannon's entropy.

**Definition 1. Fuzzy Set [5]:** A fuzzy set $A$ in a discrete universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is given by

$$A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \} , \tag{4}$$

where $\mu_A : X \to [0,1]$ is the membership function of $A$. The number $\mu_A(x)$ describes the degree of membership of $x \in X$ in $A$.

**Definition 2. Set Operations on Fuzzy Sets [7]:** Let $FS(X)$ denote the family of all FSs in $X$ and let $A, B \in FS(X)$ be given by

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$

$$B = \{\langle x, \mu_B(x)\rangle \mid x \in X\},$$

then set operations are defined as follows:

(i) **Containment:** $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x) \quad \forall\, x \in X$;

(ii) **Equality:** $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(iii) **Complement:** $A^C = \{ \langle x,\ 1-\mu_A(x)\rangle \mid x \in X\}$;

(iv) **Union:** $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$;

(v) **Intersection:** $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$.

**Definition 3. Sharpened Fuzzy Set**[2]**:** A fuzzy set $A^*$ is called a sharpened version of fuzzy set $A$ if the following conditions are satisfied:

$$\mu_{A^*}(x_j) \le \mu_A(x_j),\ \text{if}\ \mu_A(x_j) \le 0.5\quad \forall\ j$$

$$\mu_{A^*}(x_j) \ge \mu_A(x_j),\ \text{if}\ \mu_A(x_j) \ge 0.5\quad \forall\ j.$$

The first attempt to quantify the fuzziness was made in 1968 by Zadeh [8], who based on probabilistic framework, introduced the fuzzy entropy by combining probability and membership function of a fuzzy event as weighted Shannon entropy [5] given by

$$H_Z(A) = -\sum_{j=1}^{n} \mu_A(x_j)\ p_j\ \log p_j \tag{5}$$

In 1972, De Luca and Termini [2] defined the measure of fuzzy entropy for a fuzzy set $A$ corresponding (1) by

$$H_{DT}(A) = -\frac{1}{n}\sum_{j=1}^{n}\left[\mu_A(x_j)\ \log\left(\mu_A(x_j)\right) + \left(1-\mu_A(x_j)\right)\log\left(1-\mu_A(x_j)\right)\right]. \tag{6}$$

Fuzzy exponential entropy for fuzzy set $A$ corresponding to (2) has also been introduced by Pal and Pal [3] as

$${}_eH(A) = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\mu_A(x_j)\, e^{1-\mu_A(x_j)} + \left(1-\mu_A(x_j)\right) e^{\mu_A(x_j)} - 1\right]. \tag{7}$$

In the next section, we define exponential fuzzy joint and exponential fuzzy conditional entropies and study their properties.

## 3. Exponential fuzzy joint and exponential fuzzy conditional entropies

We proceed with the following formal definitions:

**Definition 4:** Let $A$ and $B$ be two fuzzy sets defined in $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_j)$ , $j = 1, 2, ..., n$, and $\mu_B(x_j)$ , $j = 1, 2, ..., n$, respectively.
Let

$$X^+ = \{x \mid x \in X,\ \mu_A(x_j) \ge \mu_B(x_j)\ \},$$

$$X^- = \{x \mid x \in X,\ \mu_A(x_j) < \mu_B(x_j)\}.$$

Based on the idea of Ding et al. [1], we propose the exponential fuzzy joint entropy and exponential fuzzy conditional entropies as follows:

***Exponential Fuzzy Joint Entropy (EFJE):***

$${}_eH(A \cup B) = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + \left(1-\mu_{A\cup B}(x_j)\right) e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$

$$(8)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^+}\left(\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-1\right) \\ +\sum_{x_j\in X^-}\left(\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}+\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-1\right)\end{array}\right].$$

***Exponential Fuzzy Conditional Entropies (EFCE):***

$${}_eH\left(A/B\right) = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^+}\left(\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-1\right) \\ -\sum_{x_j\in X^+}\left(\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}+\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-1\right)\end{array}\right]$$

$$(9)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^+}\left[\left(\begin{array}{c} \mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}-\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)} \\ +\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}\end{array}\right)\right],$$

$${}_eH\left(B/A\right) = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^-}\left(\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}+\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-1\right) \\ -\sum_{x_j\in X^-}\left(\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-1\right)\end{array}\right]$$

$$(10)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^-}\left[\left(\begin{array}{c} \mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}-\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)} \\ +\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}\end{array}\right)\right].$$

Some properties of these entropies are proved below:

**Theorem 1:** For $A, B \in FS\left(X\right)$,

(i) ${}_eH\left(A/B\right) \le {}_eH\left(A\right)$,

(ii) ${}_eH\left(B/A\right) \le {}_eH\left(B\right)$,

with equality if and only if $A = B$ i.e., $\mu_A\left(x_j\right) = \mu_B\left(x_j\right),\ \forall\ x_j \in X$.

***Proof:*** **(i).** Let us consider the expression

$$(11)\qquad {}_eH\left(A\right) - {}_eH\left(A/B\right)$$

$$\begin{aligned} &= \tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\mu_A\left(x_j\right)e^{1-\mu_A\left(x_j\right)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-1\right] \\ &\quad -\tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^+}\left[\left(\begin{array}{c} \mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}-\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)} \\ +\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}-\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}\end{array}\right)\right] \\ &= \tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^+}\left[\left(\mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}+\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}\right)-1\right] \\ &\qquad +\tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^-}\left[\left(\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}\right)-1\right] \\ &\ge\ 0. \end{aligned}$$

This completes the proof.

**(ii).** Let us consider the expression

$$(12)\qquad {}_eH\left(B\right) - {}_eH\left(B/A\right)$$

$$\begin{aligned} &= \tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\mu_B\left(x_j\right)e^{1-\mu_B\left(x_j\right)}+\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-1\right] \\ &\quad -\tfrac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^-}\left[\left(\begin{array}{c} \mu_B\left(x_j\right)e^{\left(1-\mu_B\left(x_j\right)\right)}-\mu_A\left(x_j\right)e^{\left(1-\mu_A\left(x_j\right)\right)} \\ +\left(1-\mu_B\left(x_j\right)\right)e^{\mu_B\left(x_j\right)}-\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A\left(x_j\right)}\end{array}\right)\right] \end{aligned}$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)}\right) - 1\right]$$
$$\geq \ 0.$$

This completes the proof.

***Remark 1:*** Note that ${}_eH\left(A/B\right) \neq {}_eH\left(B/A\right)$ in general. However ${}_eH\left(A\right) - {}_eH\left(A/B\right) = {}_eH\left(B\right) - {}_eH\left(B/A\right)$.

The naturalness of the definition of exponential fuzzy joint entropy and exponential fuzzy conditional entropy is exhibited by the fact that the fuzzy entropy of a pair of fuzzy sets is the fuzzy entropy of one plus the fuzzy conditional entropy of the other. This is proved in the following theorem.

**Theorem 2(*Chain rule*):** For $A, B \in FS\left(X\right)$,

(i) ${}_eH\left(A \cup B\right) = {}_eH\left(A\right) + {}_eH\left(B/A\right)$;

(ii) ${}_eH\left(A \cup B\right) = {}_eH\left(B\right) + {}_eH\left(A/B\right)$;

(iii) ${}_eH\left(A \cup B\right) = {}_eH\left(A\right) + {}_eH\left(B/A\right) = {}_eH\left(B\right) + {}_eH\left(A/B\right)$.

**Proof :(i)** Let us consider the expression

$$ {}_eH\left(A\right) + {}_eH\left(B/A\right) - {}_eH\left(A \cup B\right) \tag{13}$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_A\left(x_j\right) e^{1-\mu_A\left(x_j\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)} - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\begin{array}{c} \mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} - \mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} \\ + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)} - \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)} \end{array}\right)\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}\left(x_j\right) e^{\left(1-\mu_{A\cup B}\left(x_j\right)\right)} + \left(1-\mu_{A\cup B}\left(x_j\right)\right) e^{\mu_{A\cup B}\left(x_j\right)}\right) - 1\right]$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)}\right) - 1\right]$$

$$= 0.$$

This proves (i).

**(ii)** Let us consider the expression

$$ {}_eH\left(B\right) + {}_eH\left(A/B\right) - {}_eH\left(A \cup B\right) \tag{14}$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_B\left(x_j\right) e^{1-\mu_B\left(x_j\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)} - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\begin{array}{c} \mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} - \mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} \\ + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)} - \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)} \end{array}\right)\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}\left(x_j\right) e^{\left(1-\mu_{A\cup B}\left(x_j\right)\right)} + \left(1-\mu_{A\cup B}\left(x_j\right)\right) e^{\mu_{A\cup B}\left(x_j\right)}\right) - 1\right]$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A\left(x_j\right) e^{\left(1-\mu_A\left(x_j\right)\right)} + \left(1-\mu_A\left(x_j\right)\right) e^{\mu_A\left(x_j\right)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B\left(x_j\right) e^{\left(1-\mu_B\left(x_j\right)\right)} + \left(1-\mu_B\left(x_j\right)\right) e^{\mu_B\left(x_j\right)}\right) - 1\right]$$

$= 0.$

This proves (ii).

**(iii)** It obvious follows (i) and (ii).

This completes the proof.

In the Shannon's theory, another important concept is that of trans-information or mutual information. It is the measure of the amount of information contains one random variable about another. Based on the idea of fuzzy mutual information (FMI) [1], in the next section, we propose the concept of exponential fuzzy mutual information (EFMI) and study their properties.

## 4. Exponential fuzzy mutual information

**Definition 5:** Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universe of discourse, and $A, B \in FS(X)$, then the difference value, ${}_eH(A) - {}_eH(A/B)$, is called the EFMI between fuzzy set $A$ and $B$, denoted by ${}_eH(A \cap B)$ i.e.

$$
(15) \qquad {}_eH(A \cap B) = {}_eH(A) - {}_eH(A/B)
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right].
$$

Similarly, we define ${}_eH(B \cap A)$, given by

$$
(16) \qquad {}_eH(B \cap A) = {}_eH(B) - {}_eH(B/A)
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right].
$$

Some properties on lines parallel to Shannon's mutual information are proved below:

**Theorem 3:** For $A, B \in FS(X)$,

(i) ${}_eH(A \cap B) \geq 0$ and ${}_eH(B \cap A) \geq 0$;

(ii) ${}_eH(A \cap B) = {}_eH(B \cap A)$;

(iii) ${}_eH(A \cap B) = {}_eH(A) + {}_eH(B) - {}_eH(A \cup B)$;

(iv) ${}_eH(A \cap B) = {}_eH(A \cup B) - {}_eH(A/B) - {}_eH(B/A)$;

(v) ${}_eH(A \cap A) = {}_eH(A)$.

**Proof:** **(i)** It follows straight forwardly from Theorem 2.

**(ii)** It follows directly from Definition.

**(iii)** We consider the expression

$$
(17) \qquad {}_eH(A) + {}_eH(B) - {}_eH(A \cup B)
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_A(x_j)\, e^{1-\mu_A(x_j)} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)} - 1\right] + \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_B(x_j)\, e^{1-\mu_B(x_j)} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)} - 1\right] - \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j))\, e^{\mu_{A\cup B}(x_j)}\right) - 1\right]
$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_A(x_j)\, e^{1-\mu_A(x_j)} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)} - 1\right]$$
$$+\frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_B(x_j)\, e^{1-\mu_B(x_j)} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)} - 1\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$+\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$= {}_eH(A\cap B).$$

This completes the proof.

**(iv)**Let us consider the following expression

$$ {}_eH(A\cup B) - {}_eH(A/B) - {}_eH(B/A) \tag{18}$$
$$= \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j))\, e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\begin{pmatrix} \mu_A(x_j)\, e^{(1-\mu_A(x_j))} - \mu_B(x_j)\, e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j))\, e^{\mu_A(x_j)} - (1-\mu_B(x_j))\, e^{\mu_B(x_j)} \end{pmatrix}\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\begin{pmatrix} \mu_B(x_j)\, e^{(1-\mu_B(x_j))} - \mu_A(x_j)\, e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j))\, e^{\mu_B(x_j)} - (1-\mu_A(x_j))\, e^{\mu_A(x_j)} \end{pmatrix}\right]$$
$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\begin{pmatrix} \mu_A(x_j)\, e^{(1-\mu_A(x_j))} - \mu_B(x_j)\, e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j))\, e^{\mu_A(x_j)} - (1-\mu_B(x_j))\, e^{\mu_B(x_j)} \end{pmatrix}\right]$$
$$-\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\begin{pmatrix} \mu_B(x_j)\, e^{(1-\mu_B(x_j))} - \mu_A(x_j)\, e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j))\, e^{\mu_B(x_j)} - (1-\mu_A(x_j))\, e^{\mu_A(x_j)} \end{pmatrix}\right]$$
$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$+\frac{1}{n(\sqrt{e}-1)} \sum_{x_j\in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$= {}_eH(A\cap B).$$

This completes the proof.

**(v)** ${}_eH(A\cap A) = {}_eH(A) - {}_eH(A/A) = {}_eH(A)$**.**

Thus the mutual information of a fuzzy set with itself is the entropy of the fuzzy set.

## 5. Conclusions

This work introduces some new information measures on pair of fuzzy sets called exponential fuzzy joint entropy, exponential fuzzy conditional entropy, and exponential fuzzy mutual information measure in the setting of fuzzy set theory. Some properties and relations of these measures have been studied.

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