Criteria for the $\tilde{C}$-integral

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ABSTRACT. The C-integral was introduced by B. Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. B. Bongiorno, Di Piazza and Preiss gave some criteria for the C-integrability. The $\tilde{C}$-integral was introduced by D. Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the improper Newton integral. She gave some criteria for the $\tilde{C}$-integrability. On the other hand, Nakanishi gave some criteria for the restricted Denjoy integrability. Kawasaki and Suzuki gave new criteria for the C-integrability in the style of Nakanishi. In this paper we will give new criteria for the $\tilde{C}$-integrability in the style of Nakanishi.

1 Introduction Throughout this paper we denote by $\mathbf{(L)}(S)$ and $\mathbf{(D^*)}(S)$ the class of all Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a,b]$ into $(0,\infty)$ and a δ-fine McShane partition is a collection $\{(I_k,x_k) \mid k=1,\dots,k_0\}$ of non-overlapping intervals $I_k \subset [a,b]$ satisfying $I_k \subset (x_k-\delta(x_k), x_k+\delta(x_k))$ and $\sum_{k=1}^{k_0}|I_k| = b-a$. If $\sum_{k=1}^{k_0}|I_k| \le b-a$, then we say that the collection is a δ-fine partial McShane partition. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b]$. We denote by $\mathbf{(N)}([a,b])$ the class of all Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [3] B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. Furthermore in [1–3] B. Bongiorno et al. gave some criteria for the C-integrability. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is improper Newton integrable if there exist a countable subset $N \subset [a,b]$ and a function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b] \setminus N$. We denote by $\mathbf{(N^*)}([a,b])$ the class of all improper Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [4] D. Bongiorno gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the improper Newton integral. It is given as follows:

Definition 1.1. A function f from an inteval $[a,b]$ into $\mathbb{R}$ is said to be $\tilde{C}$-integrable if there exist a countable subset $N \subset [a,b]$ and a number A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k,x_k) \mid k=1,\dots,k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$$

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and $x_k \in N$ implies $x_k \in I_k$. The constant A is denoted by

$$A = (\tilde{C}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{C})([a,b])$ the class of all $\tilde{C}$-integrable functions from $[a,b]$ into $\mathbb{R}$.

Furthermore in [4] D. Bongiorno gave some criteria for the $\tilde{C}$-integrability. On the other hand, in [9, 12] Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, new criteria were considered in [8] for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$. In this paper, motivated by the results above, we will give new criteria for the $\tilde{C}$-integrability in the style of Nakanishi.

2 Preliminaries In [4] D. Bongiorno gave a criterion for the $\tilde{C}$-integrability. By the theorem $(\tilde{C})([a,b])$ is the minimal class which contains $(\mathbf{L})([a,b])$ and $(\mathbf{N}^*)([a,b])$. Moreover it is contained in $(\mathbf{D}^*)([a,b])$. In this paper we refer to the following theorems given by D. Bongiorno [4].

Theorem 2.1. *Let $f \in (\tilde{C})([a,b])$. Then there exists a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (\tilde{C}) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$$

and $x_k \in N$ implies $x_k \in I_k$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a,b]$ is an interval function on $[a,b]$. If an interval function F on $[a,b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a,b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive.

Definition 2.1. Let F be an interval function on $[a,b]$. Then F is said to be $\tilde{C}$-absolutely continuous on $E \subset [a,b]$ if for any positive number ε there exist a countable subset $N \subset E$, a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in N$ implies $x_k \in I_k$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_{\tilde{C}}(E)$ the class of all $\tilde{C}$-absolutely continuous interval functions on E. Moreover F is said to be $\tilde{C}$-generalized absolutely continuous on $[a,b]$ if there exists a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a,b]$ and $F \in \mathbf{AC}_{\tilde{C}}(E_m)$ for any m. We denote by $\mathbf{ACG}_{\tilde{C}}([a,b])$ the class of all $\tilde{C}$-generalized absolutely continuous interval functions on $[a,b]$.

Theorem 2.2. *For any $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in (\tilde{C})([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (\tilde{C})\int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in (\tilde{C})([a,b])$ satisfies $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$.

On the other hand, in [9, 12] Nakanishi gave the following criteria for the restricted Denjoy integrability. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in (\mathbf{L})(F_n)$ for any n;

(3) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in (\mathbf{L})(F_n)$ for any n;

(4) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in (\mathbf{L})(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1,2,\ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1,\ldots,k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1,2,\ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

Next Nakanishi gave the following theorem for the restricted Denjoy integrability.

Theorem 2.3. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* (A), (B), (C) *and* (D). *Moreover, if the pair of f and F satisfies one of* (A), (B), (C) *and* (D), *then*

$$F(I) = (D^*)\int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

Motivated by the results of Nakanishi, in [8] Kawasaki and Suzuki gave similar criteria and theorem as Theorem 2.3 for the C-integrability.

3 Main results Firstly we consider the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

(A)$_{\tilde{C}}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1,\ldots,k_0,k_0+1,\ldots,k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1,\ldots,k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k,x_k) \mid k = k_0+1,\ldots,k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0+1,\ldots,k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k,x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in N$ implies $x_k \in I_k$.

(B)$_{\tilde{C}}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1,\ldots,k_0,k_0+1,\ldots,k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1,\ldots,k_0\}$ with $I_k \cap M_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k,x_k) \mid k = k_0+1,\ldots,k_1\}$ satisfying

(4.1) $x_k \in M_n$ for any $k = k_0+1,\ldots,k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k,x_k) < \frac{1}{\varepsilon_n}$;

(4.3) $x_k \in N$ implies $x_k \in I_k$.

(C)$_{\tilde{C}}$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a countable subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a,b]$ satisfying

(3.1) $x_k \in F_n$ for any k;

(3.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3.3) $x_k \in N$ implies $x_k \in I_k$;

(3.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

$(\mathrm{D})_{\tilde{C}}$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a countable subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a,b]$ satisfying

(4.1) $x_k \in M_n$ for any k;

(4.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(4.3) $x_k \in N$ implies $x_k \in I_k$;

(4.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(\mathrm{A})_{\tilde{C}}$ implies $(\mathrm{B})_{\tilde{C}}$ and $(\mathrm{C})_{\tilde{C}}$ implies $(\mathrm{D})_{\tilde{C}}$. Now we give the following theorems for the $\tilde{C}$-integrability.

Theorem 3.1. *Let $f \in (\tilde{C})([a,b])$ and let F be the additive interval function on $[a,b]$ defined by*

$$F(I) = (\tilde{C}) \int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies $(\mathrm{A})_{\tilde{C}}$.

Proof. Since $f \in (\tilde{C})([a,b])$, we obtain $f \in (\mathbf{D}^*)([a,b])$. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Since by Theorem 2.3 the pair of f and F satisfies (A), for $\left\{\frac{\varepsilon_n}{2}\right\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1, \dots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$. By Theorem 2.1 there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1} (f(x_k)|I_k| - F(I_k))\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \dots, k_1\}$ in $[a,b]$ satisfying (3.2) and (3.3). Since $f\chi_{F_n} \in (\mathbf{L})([a,b])$, by the Saks-Henstock lemma for the McShane integral, for instance see [6, Lemma 10.6], for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1} \left(f(x_k)\chi_{F_n}|I_k| - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \dots, k_1\}$ in $[a,b]$. Since $f = f\chi_{F_n}$ on F_n, for any n there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1} (F(I_k) - f(x_k)|I_k|)\right| + \left|\sum_{k=k_0+1}^{k_1} \left(f(x_k)\chi_{F_n}|I_k| - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \dots, k_1\}$ in $[a,b]$ satisfying (3.1), (3.2) and (3.3). Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{k_1} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| \\
&\quad \leq \left|\sum_{k=1}^{k_0} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n
\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying (3.1), (3.2) and (3.3), that is, (3) holds. □

Theorem 3.2. *If the pair of a function f from an interval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{A})_{\tilde{C}}$*, then the pair of f and F satisfies* $(\mathrm{C})_{\tilde{C}}$*. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{B})_{\tilde{C}}$*, then the pair of f and F satisfies* $(\mathrm{D})_{\tilde{C}}$*.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_{\tilde{C}}$ hold. We show (3) of $(\mathrm{C})_{\tilde{C}}$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n,\varepsilon)$ such that, if $|E| < \rho(n,\varepsilon)$, then

$$\left|(L)\int_{E\cap F_n} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n,\varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n,\varepsilon) \ge n$, and put $\eta = \rho(m(n,\varepsilon),\varepsilon)$. By (3) of $(\mathrm{A})_{\tilde{C}}$ there exists a subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for $m(n,\varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a,b]$ satisfying (3.1), (3.2), (3.3) and (3.4) of $(\mathrm{C})_{\tilde{C}}$. Then we obtain

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0} |I_k| < \eta = \rho(m(n,\varepsilon),\varepsilon)$, we obtain

$$\left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} F(I_k)\right| &\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| + \left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Next we show (4) of $(\mathrm{C})_{\tilde{C}}$. Let I be a subinterval of $[a,b]$. In the case of $I \cap F_n = \emptyset$ (4) of $(\mathrm{C})_{\tilde{C}}$ is clear. Consider the case of $I \cap F_n \neq \emptyset$. Let $\{J_p \mid p = 1, 2, \ldots\}$ be the sequence of all connected components of $I^i \setminus F_n$. Since $I \cap F_m \neq \emptyset$ holds for any $m \ge n$, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| < \varepsilon_m.$$

Since $\overline{J_p} \cap F_m \neq \emptyset$ holds for any p, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left|\sum_{p=1}^{\infty}\left(F(\overline{J_p}) - (L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \le \varepsilon_m$$

for any $m \geq n$. On the other hand, we obtain

$$(L)\int_{I\cap F_m} f(x)dx = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx$$

for any $m \geq n$. Therefore we obtain

$$\begin{aligned}
&\left|F(I) - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p})\right)\right| \\
&\quad \leq \left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| \\
&\qquad + \left|(L)\int_{I\cap F_m} f(x)dx - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \\
&\qquad + \left|-\sum_{p=1}^{\infty} F(\overline{J_p}) + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right| \\
&\quad < \varepsilon_m + 0 + \varepsilon_m = 2\varepsilon_m
\end{aligned}$$

for any $m \geq n$ and hence

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}).$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_{\tilde{C}}$, then the pair of f and F satisfies $(\mathrm{D})_{\tilde{C}}$. □

Theorem 3.3. *If the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies $(\mathrm{D})_{\tilde{C}}$, then $f \in (\tilde{C})([a, b])$ and*

$$F(I) = (\tilde{C})\int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Proof. By (1) and (4) there exist a countable subset $N \subset [a, b]$ and a increasing sequence $\{M_n\}$ of non-empty measurable sets such that $\bigcup_{n=1}^{\infty} M_n = [a, b]$ and for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a, b]$ satisfying (4.1), (4.2), (4.3) and (4.4). Therefore we obtain

$$\begin{aligned}
\sum_{k=1}^{k_0} |F(I_k)| &= \left|\sum_{F(x_k)>0} F(I_k)\right| + \left|\sum_{F(x_k)<0} F(I_k)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

and hence $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$. By Theorem 2.2 there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $g \in (\tilde{C})([a,b])$ such that

$$F(I) = (\tilde{C}) \int_I g(x)dx$$

for any interval $I \subset [a,b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [14, Theorem (5.1)] $g_n \in (\mathbf{D}^*)(I)$ for any interval $I \subset [a,b]$ and by (3)

$$\begin{aligned} (D^*) \int_I g_n(x)dx &= (D^*) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (D^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (\tilde{C}) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}), \end{aligned}$$

where $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*) \int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a,x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a,x]) = f(x)$ for almost every $x \in [a,b]$. □

By Theorems 3.1, 3.2 and 3.3 we obtain the following new criteria for the $\tilde{C}$-integrability.

Theorem 3.4. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is $\tilde{C}$-integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_{\tilde{C}}$, $(\mathrm{B})_{\tilde{C}}$, $(\mathrm{C})_{\tilde{C}}$ *and* $(\mathrm{D})_{\tilde{C}}$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_{\tilde{C}}$, $(\mathrm{B})_{\tilde{C}}$, $(\mathrm{C})_{\tilde{C}}$ *and* $(\mathrm{D})_{\tilde{C}}$, *then*

$$F(I) = (\tilde{C}) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

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