Classification of idempotents and square roots in the upper triangular matrix Banach algebras and their inductive limit algebras

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Abstract. We study idempotents and square roots in the upper triangular matrix Banach algebras over real or complex numbers. We compute explicitly and determine algebraically the idempotents and the square roots in the cases of size: two by two, three by three, and four by four. We also consider their equivalence classes by homotopy and classify topologically the upper triangular matrix algebras in those cases and in general by the groups generated by the homotopy classes. Moreover, we consider some infinite dimensional, Banach algebras obtained as inductive limits of the upper triangular matrix algebras and obtain several topological classification results for the inductive limits.

1 Introduction We begin to study idempotents and square roots in the upper triangular matrix Banach algebras over real or complex numbers. The upper triangular matrix algebras are typical examples of finite dimensional non self-adjoint Banach algebras over real or complex numbers. We compute explicitly and determine algebraically idempotents and square roots of the upper triangular matrix algebras in the cases of size: two by two, three by three, and four by four. The statements as lists as examples obtained should be useful and convenient for the readers. We also consider the equivalence classes of the idempotents and the square roots by homotopy and classify topologically the upper triangular matrix algebras in the cases and in the general case by the groups generated by the homotopy classes. Moreover, we consider some infinite dimensional, Banach algebras obtained as inductive limits of the upper triangular matrix algebras, and obtain several (topological) classification results for the inductive limits by our V-theory groups mentioned below and also by the scales for the units

As a contrast, C^*-algebras are self-adjoint Banach algebras over complex numbers with the C^*-norm condition. The full matrix algebras over complex numbers are typical examples of finite dimensional C^*-algebras. Projections of C^*-algebras, that are self-adjoint idempotents, and unitaries of C^*-algebras, with adjoints as inverses, play main roles in the K-theory for C^*-algebras, and their associated K-theory classes generate K-theory groups of C^*-algebras ([1], [4] and [5]). By lack of self-adjointness for non self-adjoint Banach algebras, as candidates as substitute, we consider idempotents and square roots and their homotopy classes, that generate our named V-theory groups, first introduced in this paper.

As for inductive limit algebras, AF (approximately finite dimensional) C^*-algebras, that are inductive limits of finite direct sums of full matrix algebras, are classified by K-theory groups (but K_0 only since K_1 trivial) as ordered groups with the scales (see the corresponding results in [1], [4], or [5], due to [2]). Our V-theory groups (V_0 of V_0 and V_1) just correspond to the scales in the C^*-algebra K-theory, in which the symbol V is used for indicating the sets of equivalence classes of projections of matrix algebras over a C^*-algebra

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and the symbol Σ is used for indicating the scales for a C^*-algebra (see [1]). Note that, by lack of self-adjointness, there are no non self-adjoint unitaries, and no unitary equivalence and no stably unitary equivalence as for idempotents in non self-adjoint Banach algebras, but can be used homotopy in the algebras.

However, the scaled ordered, C^*-algebra K-theory groups (K_0) for the inductive limits of non self-adjoint Banach subalgebras obtained in inductive limits of C^*-algebras, such as AF-algebras and UHF-algebras, have been already used to classify those non self-adjoint inductive limit algebras, containing the case we consider here (see [3]). Therefore, our classification results in application to inductive limit non self-adjont algebras are not new, but our formulation in terms of non self-adjoint algebras only, V_0 as well as V_1 (non-trivial while K_1 trivial in that case) seems to be new in this sense, and anyhow to be an equivalent replacement as another method or attempt.

2 The two by two case We denote by $T_2(\mathbb{R})$ the algebra of all upper triangular 2×2 matrices over the real field $\mathbb{R}$ and by $T_2(\mathbb{C})$ the same algebra over the complex field $\mathbb{C}$. We give the topology on the algebras by the Euclidean norm, for convenience, via $T_2(\mathbb{R}) \cong \mathbb{R}^3$ and $T_2(\mathbb{C}) \cong \mathbb{C}^3$ as a space. Let F be either $\mathbb{R}$ or $\mathbb{C}$.

Recall that a matrix element A of $T_2(F)$ is said to be an idempotent if $A^2 = A$.

Proposition 2.1. *All idempotents of $T_2(F)$ are listed up as*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for any $b \in F$.

Proof. Let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(F)$$

with $A^2 = A$, so that $a^2 = a$, $c^2 = c$, and $b(a + c) = b$. Hence $a = 0$ or 1, and $c = 0$ or 1. □

Denote by $P_2(F)$ the set of all idempotents of $T_2(F)$. Define the equivalence relation for elements of $P_2(F)$ by that two elements of $P_2(F)$ are equivalent if there is a continuous path within $P_2(F)$ between the two elements. Write by $E_0(T_2(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_2(F)$.

Corollary 2.2. *All classes of $E_0(T_2(F))$ are listed up as*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As, possibly, a new notion to simplify the situation, we may introduce, as an attempt,

Definition 2.3. We now define the anti-diagonal transpose A^{at} of $A \in T_2(F)$ by

$$A^{at} = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

We denote by $T_2(F)/\sim_{at}$ the set of matrices of $T_2(F)$ identified under the anti-transpose.

Note that the anti-diagonal transpose corresponds to a permutation on $T_2(F) \cong F^3$. Also, one has

$$\{J_2AJ_2\}^t \equiv \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}^t = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} = A^{at}$$

with $\{\cdots\}^t$ the usual transpose, but in $M_2(F)$ the 2×2 matrix algebra over F.

Corollary 2.4. *All idempotents of $T_2(F)/\sim_{at}$ are listed up as*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for any $b \in F$.

Corollary 2.5. *All classes of $E_0(T_2(F)/\sim_{at})$ are listed up as*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Recall that a matrix element A of $T_2(F)$ is said to be a square root if $A^2 = I_2$ the 2×2 identity matrix.

Proposition 2.6. *All square roots of $T_2(F)$ are listed up as*

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{(compound 4 cases)}, \quad \begin{pmatrix} \pm \mathbf{1} & b \\ 0 & \mp \mathbf{1} \end{pmatrix} \quad \text{(not compound 2 cases)}$$

for any $b \in F$ non-zero.

Remark. In what follows, we make the difference of the compound (or composite) in order case or not by denoting ± 1 usual or $\pm \mathbf{1}$ bold as in the statement above.

Proof. Let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(F)$$

with $A^2 = I_2$, so that $a^2 = 1$, $c^2 = 1$, and $b(a + c) = 0$. Hence $a = 1$ or -1, and $c = 1$ or -1, and if b is non-zero, then $a = -c$. □

Denote by $R_2(F)$ the set of all square roots of $T_2(F)$. Define the equivalence relation for elements of $R_2(F)$ by that two elements of $R_2(F)$ are equivalent if there is a continuous path within $R_2(F)$ between the two elements. Write by $E_1(T_2(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_2(F)$.

Corollary 2.7. *All classes of $E_1(T_2(F))$ are listed up as*

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \quad \text{(compound in order 4 cases)}.$$

3 The three by three case We denote by $T_3(F)$ the algebra of all upper triangular 3×3 matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_3(F) \cong F^6$ as a space.

Proposition 3.1. *All idempotents of $T_3(F)$ are listed up as*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & -xz \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $x, y, z \in F$.

Proof. Let

$$A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix} \in T_3(F)$$

with $A^2 = A$, so that $a^2 = a$, $b^2 = b$, $c^2 = c$, and $(a+b)x = x$, $(b+c)z = z$, $(a+c)y + xz = y$. Hence $a = 0$ or 1, and $b = 0$ or 1, and $c = 0$ or 1.

If $a = b = c = 0$, then $x = y = z = 0$.
If $a = 1$ and $b = c = 0$, then $z = 0$.
If $b = 1$ and $a = c = 0$, then $y = xz$.
If $c = 1$ and $a = b = 0$, then $x = 0$.
Moreover, if $a = b = 1$ and $c = 0$, then $x = 0$.
If $a = c = 1$ and $b = 0$, then $y = -xz$.
If $a = 0$ and $b = c = 1$, then $z = 0$.
If $a = b = c = 1$, then $x = y = z = 0$.
These cases correspond to the matrices in the statement in this order. □

Denote by $P_3(F)$ the set of all idempotents of $T_3(F)$. Define the equivalence relation for elements of $P_3(F)$ by that two elements of $P_3(F)$ are equivalent if there is a continuous path within $P_3(F)$ between the two elements. Write by $E_0(T_3(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_3(F)$.

Corollary 3.2. *All classes of $E_0(T_3(F))$ are listed up as*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 3.3. We now define the anti-diagonal transpose A^{at} of $A \in T_3(F)$ by

$$A^{at} = \begin{pmatrix} c & z & y \\ 0 & b & x \\ 0 & 0 & a \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix}.$$

We denote by $T_3(F)/\sim_{at}$ the set of matrices of $T_3(F)$ identified under the anti-transpose.

Note that A^{at} is just the transpose of J_3AJ_3:

$$A^{at} = \{J_3AJ_3\}^t = J_3^tA^tJ_3^t = J_3A^tJ_3,$$

with J_3 the 3×3 matrix of $(1,3),(2,2),(3,1)$ components as 1 and other components as 0.

Corollary 3.4. *All idempotents of $T_3(F)/\sim_{at}$ are listed up as*

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&x&y\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&x&xz\\0&1&z\\0&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&y\\0&1&z\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&x&-xz\\0&0&z\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

for any $x,y,z\in F$.

Corollary 3.5. *All classes of $E_0(T_3(F)/\sim_{at})$ are listed up as*

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Proposition 3.6. *All square roots of $T_3(F)$ are listed up as*

$$\begin{pmatrix}\pm1&0&0\\0&\pm1&0\\0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&0\\0&\mp\mathbf{1}&0\\0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0\\0&\pm\mathbf{1}&z\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&0&y\\0&\pm1&0\\0&0&\mp\mathbf{1}\end{pmatrix},$$
$$\begin{pmatrix}\pm\mathbf{1}&x&0\\0&\mp\mathbf{1}&z\\0&0&\pm\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&y\\0&\mp\mathbf{1}&0\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&0&y\\0&\pm\mathbf{1}&z\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&y\\0&\mp\mathbf{1}&z\\0&0&\pm\mathbf{1}\end{pmatrix}$$

for any non-zero $x,y,z\in F$, where x,y,z satisfy the equation $2(\pm\mathbf{1})y+xz=0$ in the last case, so that $y=2^{-1}(\mp\mathbf{1})xz$. There are compound or not $8+4+4+4+2+2+2+2=28$ cases.

Proof. Let

$$A=\begin{pmatrix}a&x&y\\0&b&z\\0&0&c\end{pmatrix}\in T_3(F)$$

with $A^2=I_3$ the 3×3 identity matrix, so that $a^2=1$, $b^2=1$, $c^2=1$, and $(a+b)x=0$, $(b+c)z=0$, $(a+c)y+xz=0$. Hence $a=\pm1$, and $b=\pm1$, and $c=\pm1$.

If $y=z=0$ and x is non-zero, $a=-b$.

If $x=y=0$ and z is non-zero, $b=-c$.

If $x=z=0$ and y is non-zero, $a=-c$.

If $y=0$ and $xz\neq 0$, then $a=-b$ and $b=-c$.

If $y\neq 0$ and $a=-c$, then $x=0$ or $z=0$.

The rest case is that $xyz\neq 0$ with $2ay+xz=0$.

These cases correspond to the matrices in the statement in this order. ∎

Denote by $R_3(F)$ the set of all square roots of $T_3(F)$. Define the equivalence relation for elements of $R_3(F)$ by that two elements of $R_3(F)$ are equivalent if there is a continuous path within $R_3(F)$ between the two elements. Write by $E_1(T_3(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_3(F)$.

Corollary 3.7. *All classes of $E_1(T_3(F))$ are listed up as*

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \quad \text{(compound in order 8 cases).}$$

4 The four by four case We denote by $T_4(F)$ the algebra of all upper triangular 4×4 matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_4(F) \cong F^{1+2+3+4} = F^{10}$ as a space.

Proposition 4.1. *All idempotents of $T_4(F)$ are listed up as the zero matrix and*

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{12}x_{24} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x_{13} & x_{13}x_{34} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & x_{14} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & x_{13} & -x_{12}x_{24} - x_{13}x_{34} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & x_{12} & x_{13} & x_{12}x_{24} + x_{13}x_{34} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{14} \\ 0 & 1 & x_{23} & -x_{23}x_{34} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & -x_{12}x_{24} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & x_{13} & -x_{13}x_{34} \\ 0 & 1 & x_{23} & -x_{23}x_{34} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for any $x_{i,j} \in F$ $(i < j)$, and the identity matrix.

Proof. Let

$$A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix} \in T_4(F)$$

with $A^2 = A$, so that $a_j^2 = a_j$ $(1 \le j \le 4)$, and $(a_1 + a_2)x_{12} = x_{12}$, $(a_2 + a_3)x_{23} = x_{23}$, $(a_3 + a_4)x_{34} = x_{34}$, $(a_1 + a_3)x_{13} + x_{12}x_{23} = x_{13}$, $(a_2 + a_4)x_{24} + x_{23}x_{34} = x_{24}$, $(a_1 + a_4)x_{14} + x_{12}x_{24} + x_{13}x_{34} = x_{14}$. Hence $a_j = 0$ or 1 $(1 \le j \le 4)$.

If $a_j = 0$ $(1 \le j \le 4)$, then $x_{ij} = 0$ $(1 \le i < j \le 4)$.

If $a_1 = 1$ and $a_j = 0$ $(2 \leq j \leq 4)$, then $x_{23} = x_{34} = 0$, $x_{24} = 0$
If $a_2 = 1$ and $a_j = 0$ $(j \neq 2)$, then $x_{34} = 0$ and $x_{12}x_{23} = x_{13}$, $x_{12}x_{24} = x_{14}$.
If $a_3 = 1$ and $a_j = 0$ $(j \neq 3)$, then $x_{12} = 0$ and $x_{23}x_{34} = x_{24}$, $x_{13}x_{34} = x_{14}$.
If $a_4 = 1$ and $a_j = 0$ $(j \neq 4)$, then $x_{12} = x_{23} = x_{13} = 0$.
Moreover, if $a_1 = a_2 = 1$ and $a_3 = a_4 = 0$, then $x_{12} = x_{34} = 0$.
If $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, then $x_{13} = -x_{12}x_{23}$ and $x_{24} = x_{23}x_{34}$.
If $a_1 = a_4 = 1$ and $a_2 = a_3 = 0$, then $x_{23} = 0$ and $x_{14} = -x_{12}x_{24} - x_{13}x_{34}$.
If $a_2 = a_3 = 1$ and $a_1 = a_4 = 0$, then $x_{23} = 0$ and $x_{14} = x_{12}x_{24} + x_{13}x_{34}$.
If $a_2 = a_4 = 1$ and $a_1 = a_3 = 0$, then $x_{13} = x_{12}x_{23}$ and $x_{24} = -x_{23}x_{34}$.
If $a_3 = a_4 = 1$ and $a_1 = a_2 = 0$, then $x_{12} = x_{34} = 0$.
Furthermore, if $a_1 = a_2 = a_3 = 1$ and $a_4 = 0$, then $x_{12} = x_{23} = x_{13} = 0$.
If $a_1 = a_3 = a_4 = 1$ and $a_2 = 0$, then $x_{34} = 0$, $x_{13} = -x_{12}x_{23}$, $x_{14} = -x_{12}x_{24}$.
If $a_1 = a_2 = a_4 = 1$ and $a_3 = 0$, then $x_{12} = 0$, $x_{24} = -x_{23}x_{34}$, $x_{14} = -x_{13}x_{34}$.
If $a_1 = 0$ and $a_2 = a_3 = a_4 = 0$, then $x_{23} = x_{34} = x_{24} = 0$.
Finally, if $a_j = 1$ $(1 \leq j \leq 4)$, then $x_{ij} = 0$ $(1 \leq i < j \leq 4)$.
These cases correspond to the matrices in the statement in this order. □

Denote by $P_4(F)$ the set of all idempotents of $T_4(F)$. Define the equivalence relation for elements of $P_4(F)$ by that two elements of $P_4(F)$ are equivalent if there is a continuous path within $P_4(F)$ between the two elements. Write by $E_0(T_4(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_4(F)$.

Corollary 4.2. *All classes of $E_0(T_4(F))$ are listed up as the zero class and*

$$\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix},$$

and the identity class.

Definition 4.3. We now define the anti-diagonal transpose A^{at} of $A \in T_4(F)$ by

$$A^{at} = \begin{pmatrix} a_4 & x_{34} & x_{24} & x_{14} \\ 0 & a_3 & x_{23} & x_{13} \\ 0 & 0 & a_2 & x_{12} \\ 0 & 0 & 0 & a_1 \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

We denote by $T_4(F)/\sim_{at}$ the set of matrices of $T_4(F)$ identified under the anti-transpose.

Note that A^{at} is just the transpose of J_4AJ_4:

$$A^{at} = \{J_4AJ_4\}^t = J_4A^tJ_4,$$

with J_4 the 4×4 matrix of $(1,4),(2,3),(3,2),(4,1)$ components as 1 and other components as 0.

Corollary 4.4. *All idempotents of $T_4(F)/\sim_{at}$ are listed up as the zero matrix and*

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{12}x_{24} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & x_{14} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & x_{12} & x_{13} & -x_{12}x_{24} - x_{13}x_{34} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{12}x_{24} + x_{13}x_{34} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & -x_{12}x_{24} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for any $x_{i,j} \in F$ $(i < j)$, and the identity matrix.

Corollary 4.5. *All classes of $E_0(T_4(F)/\sim_{at})$ are listed up as the zero class and*

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

and the identity class.

Proposition 4.6. *All square roots of $T_4(F)$ are listed up as, for any nonzero $x_{ij} \in F$,*

$$\begin{pmatrix}\pm1&0&0&0\\0&\pm1&0&0\\0&0&\pm1&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm\boldsymbol{1}&x_{12}&0&0\\0&\mp\boldsymbol{1}&0&0\\0&0&\pm1&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0&0\\0&\pm\boldsymbol{1}&x_{23}&0\\0&0&\mp\boldsymbol{1}&0\\0&0&0&\pm1\end{pmatrix},$$
$$\begin{pmatrix}\pm1&0&0&0\\0&\pm1&0&0\\0&0&\pm\boldsymbol{1}&x_{34}\\0&0&0&\mp\boldsymbol{1}\end{pmatrix},\quad \begin{pmatrix}\pm\boldsymbol{1}&x_{12}&x_{13}&0\\0&\pm\boldsymbol{1}&x_{23}&0\\0&0&\pm\boldsymbol{1}&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0&0\\0&\pm\boldsymbol{1}&x_{23}&x_{24}\\0&0&\mp\boldsymbol{1}&x_{34}\\0&0&0&\pm\boldsymbol{1}\end{pmatrix},$$
$$\begin{pmatrix}\pm\boldsymbol{1}&x_{12}&0&0\\0&\mp\boldsymbol{1}&0&0\\0&0&\pm\mathit{1}&x_{34}\\0&0&0&\mp\mathit{1}\end{pmatrix}\begin{pmatrix}\text{not compound in}\\\text{each bold and italic,}\\\text{but compound}\\\text{between both}\end{pmatrix},\quad \begin{pmatrix}\pm\boldsymbol{1}&x_{12}&x_{13}&0\\0&\mp\boldsymbol{1}&x_{23}&x_{24}\\0&0&\pm\boldsymbol{1}&x_{34}\\0&0&0&\mp\boldsymbol{1}\end{pmatrix}$$

with $x_{12}x_{24}+x_{13}x_{34} = 0$, where possible cases in the following are written as matrix forms as above, and there are impossible case as tuples as below, with non-zero components (x_{12}, x_{23}) but $x_{13} = 0$ or with non-zero components (x_{23}, x_{34}) but $x_{24} = 0$ or with more other non-zero components:

$$\begin{aligned}&(x_{12}, x_{23}; x_{14}),\quad (x_{12}, x_{23}; x_{24}),\quad (x_{12}, x_{23}; x_{34}),\\ &(x_{12}, x_{23}; x_{14}, x_{24}),\quad (x_{12}, x_{23}; x_{14}, x_{34}),\quad (x_{12}, x_{23}; x_{24}, x_{34}),\\ &\mathit{or}\quad (x_{12}, x_{23}; x_{14}, x_{24}, x_{34}),\end{aligned}$$

and

$$\begin{aligned}&(x_{23}, x_{34}; x_{13}),\quad (x_{23}, x_{34}; x_{14}),\quad (x_{23}, x_{34}; x_{13}, x_{14}),\\ &(x_{23}, x_{34}; x_{12}, x_{13}),\quad \mathit{or}\quad (x_{23}, x_{34}; x_{12}, x_{13}, x_{14});\end{aligned}$$

and moreover,

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

and

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and there is an impossible case with non-zero components $(x_{13}, x_{23}, x_{24}, x_{34})$ *but* $x_{14} = 0$; *and*

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and there is an impossible case with non-zero compoents $(x_{12}, x_{13}, x_{23}, x_{24})$ *but* $x_{14} = 0$; *and furthermore,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathit{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathit{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix} \quad \begin{pmatrix} \text{not compound in} \\ \text{each bold and italic,} \\ \text{but compound} \\ \text{between both} \end{pmatrix},$$

and there are the cases which do not exist, with non-zero components:

$$(x_{12}, x_{13}, x_{14}, x_{24}, x_{34}) \quad \mathit{or} \quad (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$$

(*the full case*); *and moreover,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathit{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathit{1} \end{pmatrix} \begin{pmatrix} \text{not compound in} \\ \text{each bold and italic,} \\ \text{but compound} \\ \text{between both} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and the two impossible cases with non-zero components $(x_{12}, x_{13}, x_{23}, x_{24})$ *and* $(x_{13}, x_{23}, x_{24}, x_{34})$; *and furthermore,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

and there are five impossible cases with non-zero components:

$$\begin{aligned} &(x_{12}, x_{13}, x_{14}, x_{24}), \quad (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}), \quad (x_{12}, x_{13}, x_{14}, x_{24}, x_{34}), \\ &(x_{13}, x_{14}, x_{24}, x_{34}), \quad \mathit{or} \quad (x_{13}, x_{14}, x_{23}, x_{24}, x_{34}); \end{aligned}$$

and

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and finally,

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \pm\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \mp\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}.$$

In total, with respect to the off diagonal part, there are 41 *distinct possible cases and* 23 *distinct impossible cases in* $64 = 2^6$ *all the cases. In more details, together with the diagonal part of components compound or not, there are possible* $\{2^4+3(2^3)+2(2^2)+2\cdot2+2\}+\{2(2^3)+3(2^2)+2\}+\{3(2^2)+2\}+\{2^3+4(2^2)+2\cdot2\}+\{2\cdot2+4(2)\}+\{2(2)\}+\{4(2)+2(2^2)+4(2)\} = 166$ *cases.*

Proof. Let

$$A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix} \in T_4(F)$$

with $A^2 = I_4$ the 4×4 identity matrix, so that $a_j^2 = 1$ $(1 \leq j \leq 4)$, and $(a_1 + a_2)x_{12} = 0$, $(a_2 + a_3)x_{23} = 0$, $(a_3 + a_4)x_{34} = 0$, $(a_1 + a_3)x_{13} + x_{12}x_{23} = 0$, $(a_2 + a_4)x_{24} + x_{23}x_{34} = 0$, $(a_1 + a_4)x_{14} + x_{12}x_{24} + x_{13}x_{34} = 0$. Hence $a_j = \pm 1$ $(1 \leq j \leq 4)$.

If $x_{12} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_2 = 0$.

If $x_{23} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_2 + a_3 = 0$.

If $x_{34} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_3 + a_4 = 0$.

If $x_{12} \neq 0$, $x_{23} \neq 0$, then $(a_1 + a_3)x_{13} \neq 0$ and $a_1 + a_2 = 0$ and $a_2 + a_3 = 0$.

If $x_{23} \neq 0$, $x_{34} \neq 0$, then $(a_2 + a_4)x_{24} \neq 0$ and $a_2 + a_3 = 0$ and $a_3 + a_4 = 0$.

If $x_{12} \neq 0$, $x_{34} \neq 0$, and $x_{ij} = 0$ otherwise, then $a_1 + a_2 = 0$ and $a_3 + a_4 = 0$.

There is another case with $x_{12}x_{23}x_{34} \neq 0$, so that $(a_1+a_3)x_{13} \neq 0$ and $(a_2+a_4)x_{24} \neq 0$.

Note that $x_{12}x_{23} \neq 0$ implies $x_{13} \neq 0$ and also that $x_{23}x_{34} \neq 0$ implies $x_{24} \neq 0$, so that several impossible cases with $x_{12}x_{23} \neq 0$ but $x_{13} = 0$ and with $x_{23}x_{34} \neq 0$ but $x_{24} = 0$ are obtained.

Moreover, if $x_{13} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_3 = 0$. Also, if $x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_2 + a_4 = 0$.

And if $x_{13} \neq 0$, $x_{12}x_{23} = 0$, and $x_{ij} = 0$ otherwise, then $a_1+a_3 = 0$ and either a_1+a_2 or $a_2 + a_3 = 0$. In addition, there are two possible cases with $x_{34} \neq 0$ and another impossible case with $x_{34} \neq 0$.

And if $x_{24} \neq 0$, $x_{23}x_{34} = 0$, and $x_{ij} = 0$ otherwise, then $a_2 + a_4 = 0$ and either $a_2 + a_3 = 0$ or $a_3 + a_4 = 0$. In addition, there are two possible cases with $x_{12} \neq 0$ and another impossible case with $x_{12} \neq 0$.

Furthermore, if $x_{14} \neq 0$ and $x_{ij} = 0$ otherwise, so that $x_{12}x_{24} + x_{13}x_{34} = 0$, then $a_1 + a_4 = 0$. In addition, there are some other cases with $x_{12} \neq 0$ or $x_{24} \neq 0$; $x_{13} \neq 0$ or $x_{34} \neq 0$; $x_{23} \neq 0$ and more in what follows. But if $(a_1 + a_4)x_{14} \neq 0$, then $x_{12}x_{24} \neq 0$ if and only if $x_{13}x_{34} \neq 0$, which implies a contradiction in sigh on the diagonal, so that impossible are the case with $(x_{12}, x_{13}, x_{14}, x_{24}, x_{34})$ and the full case.

Moreover, if $x_{13}x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_3 = 0$ and $a_2 + a_4$. In addition, there are other four possible cases with several other non-zero components and two impossible cases.

Furthermore, if $x_{13}x_{14}x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_4 = 0$, $a_1 + a_3 = 0$ and $a_2 + a_4 = 0$. In addition, there are one more possible case with $x_{23} \neq 0$ and five impossible cases by the contradiction of signs on the diagonal.

And there are the possible cases with $x_{12}x_{13}x_{14} \neq 0$ or $x_{14}x_{24}x_{34} \neq 0$ and with $x_{13}x_{14}x_{23} \neq 0$ or $x_{14}x_{23}x_{24} \neq 0$, and the possible cases with $x_{13}x_{14}x_{34} \neq 0$ or $x_{12}x_{14}x_{24} \neq 0$, so that $a_1 + a_4 \neq 0$.

Finally, there are four cases that complement the list above in all the cases, with $a_1+a_3 \neq 0$, $a_1 + a_4 \neq 0$, $a_2 + a_4 \neq 0$, and $a_1 + a_4 \neq 0$, respectively.

These possible and impossible cases correspond respectively to the matrices and the tuples in the statement in this order. □

Denote by $R_4(F)$ the set of all square roots of $T_4(F)$. Define the equivalence relation for elements of $R_4(F)$ by that two elements of $R_4(F)$ are equivalent if there is a continuous path (or a homotopy) within $R_4(F)$ between the two elements. Write by $E_1(T_4(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_4(F)$.

Corollary 4.7. *All classes of $E_1(T_4(F))$ are listed up as*

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} \quad \text{(compound in order 16 cases).}$$

Proof. Note that any square root in the list of Proposition 4.6 has a homotopy class within $R_4(F)$, equal to one of the $2^4 = 16$ homotopy classes in the statement, by deforming off-diagonal components to zero. □

5 The general case by homotopy We denote by $T_n(F)$ the algebra of all upper triangular $n \times n$ matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_n(F) \cong F^{1+2+3+4+\cdots+n} = F^{2^{-1}n(n+1)}$ as a space.

Denote by $P_n(F)$ the set of all idempotents of $T_n(F)$. Define the equivalence relation for elements of $P_n(F)$ by that two elements of $P_n(F)$ are equivalent if there is a continuous path (or a homotopy) within $P_n(F)$ between the two elements. Write by $E_0(T_n(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_n(F)$.

Let $\{e_{ij}\}_{i,j=1, i \leq j}^n$ be the matrix unit for $T_n(F)$.

Theorem 5.1. *All classes of $E_0(T_n(F))$ are listed up as the zero class and the classes $[e_{ii}]$ for $1 \leq i \leq n$, and $[e_{ii} + e_{jj}]$ for $1 \leq i < j \leq n$, and $[e_{ii} + e_{jj} + e_{kk}]$ for $1 \leq i < j < k$, and moreover, in general, $[e_{i_1 i_1} + e_{i_2 i_2} + \cdots + e_{i_s, i_s}]$ for $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ with $3 \leq s \leq n-1$, and the class of the $n \times n$ identity matrix, and there are 2^n homotopy classes in all.*

Proof. One can prove the claim by induction. Indeed, let $P \in P_n(F)$. Then there are two cases of the block decomposition for P:

$$P = \begin{pmatrix} 1 & c \\ 0_{n-1} & Q \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} 0 & c \\ 0_{n-1} & Q \end{pmatrix}$$

with $Q \in P_{n-1}(F)$, c a $1 \times (n-1)$ row vector and 0_{n-1} the $(n-1) \times 1$ column zero vector, such that $Q^2 = Q$ and $cQ = 0_{n-1}^t$ the transpose of 0_{n-1}. By induction, the class $[Q]$ for Q is one of the classes listed as in the statement in the case of $n-1$. And then in both of two cases, the class $[P]$ can be one of the classes listed as in the statement just in the case of n, by deforming c to the $1 \times (n-1)$ row zero vector within $P_n(F)$ by a continuous path (i.e. a homotopy). □

We define the semigroup $\langle E_0(T_n(F)) \rangle$ generated by $E_0(T_n(F))$ with the addition given by $[p]+[q] = [p+q]$ for $p, q \in P_n(F)$ if p is orthogonal to q, i.e. if $pq = 0$ and by $[p]+[p] = 2[p]$ and by $[p]+[q] = [p - p \wedge q] + 2[p \wedge q] + [q - p \wedge q]$ if $pq \neq 0$, where $p \wedge q$ means the projection corresponding to the intersection of their ranges. It follows that the semigroup $\langle E_0(T_n(F)) \rangle$ becomes an additive semigroup with the zero class as the identity element by this operation.

We define an abelian group $V_0(T_n(F))$ to be the Grothendieck group of the semigroup $\langle E_0(T_n(F)) \rangle$. We say that $V_0(T_n(F))$ is the V_0-group of $T_n(F)$.

Corollary 5.2. *We obtain*

$$V_0(T_n(F)) \cong \mathbb{Z}^n.$$

Proof. Indeed, the group $V_0(T_n(F))$ is generated by the classes $[e_{11}]$, $[e_{22}], \cdots$, and $[e_{nn}]$, and the isomorphism is given by the correspondence:

$$\sum_{j=1}^{n} a_j[e_{jj}] \leftrightarrow (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n.$$

□

Corollary 5.3. *The class of Banach algebras of all upper triangular matrices over real or complex numbers is classified by their V-groups in the sense that $T_n(F) \cong T_m(F)$ as a Banach algebra if and only if $V_0(T_n(F)) \cong V_0(T_m(F))$ as a group.*

Denote by $R_n(F)$ the set of all square roots of $T_n(F)$. Define the equivalence relation for elements of $R_n(F)$ by that two elements of $R_n(F)$ are equivalent if there is a continuous path (or a homotopy) within $R_n(F)$ between the two elements. Write by $E_1(T_n(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_n(F)$.

Theorem 5.4. *All classes of $E_1(T_n(F))$ are listed up as*

$$[(\pm e_{11}) + (\pm e_{22}) + \cdots + (\pm e_{nn})],$$

2^n *classes in all.*

Proof. One can prove the claim by induction. Indeed, let $R \in R_n(F)$. Then there are two cases of the block decomposition for R:

$$R = \begin{pmatrix} \pm 1 & c \\ 0_{n-1} & S \end{pmatrix}$$

with $S \in R_{n-1}(F)$, c a $1 \times (n-1)$ row vector and 0_{n-1} the $(n-1) \times 1$ column zero vector, such that $S^2 = I_{n-1}$ the $(n-1) \times (n-1)$ identity matrix and $\pm c + cS = 0_{n-1}^t$ the transpose of 0_{n-1}. By induction, the class $[S]$ for S is one of the classes listed as in the statement in the case of $n-1$. And then in both of two cases, the class $[R]$ can be one of the classes listed as in the statement just in the case of n, by deforming c to the $1 \times (n-1)$ row zero vector within $R_n(F)$ by a continuous path (i.e. a homotopy). □

We define the group $V_1(T_n(F))$ generated by $E_1(T_n(F))$ with the multiplication given by $[r] \cdot [s] = [rs]$ for $r, s \in R_n(F)$. It follows that the group $V_1(T_n(F))$ is an abelian group with the class of the $n \times n$ identity matrix I_n as the unit. We say that $V_1(T_n(F))$ is the V_1-group of $T_n(F)$.

Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2. Denote by $\oplus$ the diagonal sum.

Corollary 5.5. *We obtain*

$$V_1(T_n(F)) \cong (\mathbb{Z}_2)^n \equiv \Pi^n \mathbb{Z}_2.$$

Proof. Indeed, the group $V_1(T_n(F))$ is generated by the classes $[-1 \oplus I_{n-1}]$, $[1 \oplus -1 \oplus I_{n-2}], \cdots$, and $[I_{n-1} \oplus -1]$, and the isomorphism is given by the correspondence:

$$\Pi_{j=1}^{n}[I_{j-1} \oplus a_j \oplus I_{n-j}] = [a_1 \oplus a_2 \oplus \cdots \oplus a_n]$$
$$\leftrightarrow (a_1, a_2, \cdots, a_n) \in (\mathbb{Z}_2)^n,$$

where each a_j is 1 or -1 and I_0 is removed to be empty. □

Corollary 5.6. *The class of Banach algebras of all upper triangular matrices over real or complex numbers is classified by their V-groups in the sense that $T_n(F) \cong T_m(F)$ as a Banach algebra if and only if $V_1(T_n(F)) \cong V_1(T_m(F))$ as a group.*

6 Inductive limits by V-theory There is a canonical unital inclusion map $i_{n,m}$ from $T_n(F)$ to $T_m(F)$ if n divides m as $m = kn$ for some k a positive integer, defined as $i_{n,m}(p) = \oplus^k p$ the k-fold diagonal sum. Note that there are some other embeddings in the diagonal.

Let $T_\infty(F) = \varinjlim T_{n_j}(F)$ be the inductive limit of $\{T_{n_j}(F)\}_{j=1}^\infty$ with unital connecting maps $i_{n_j,n_{j+1}}$ for an increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that n_j divides n_{j+1} with $n_{j+1} = k_j n_j$ for some k_j a positive integer. Then $T_\infty(F)$ becomes a unital Banach algebra as a Banach algebra completion of the infinite union $\cup_{j=1}^\infty T_{n_j}(F)$ of $T_{n_j}(F)$. Note that $T_\infty(F)$ does depend on the choice of a family of connecting maps in general, as a non-trivial known fact (see [3, Exercises 6.3]).

Proposition 6.1. *Let $T_\infty(F) = \varinjlim T_{n_j}(F)$. We obtain*

$$V_0(T_\infty(F)) \cong \varinjlim \mathbb{Z}^{n_j} \cong \varinjlim \{\mathbb{Z}[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}[\frac{n_j}{n_j}]\}$$

as inductive limits of scaled ordered groups, with $[1] = \lim_{j\to\infty}\{[\frac{1}{n_j}] + \cdots + [\frac{n_j}{n_j}]\}$ as scale. Also,

$$V_1(T_\infty(F)) \cong \varinjlim \Pi^{n_j}\mathbb{Z}_2 \cong \varinjlim \{\mathbb{Z}_2[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_j}{n_j}]\}$$

with $[1] = \lim_{j\to\infty}\{[\frac{1}{n_j}] + \cdots + [\frac{n_j}{n_j}]\}$.

Proof. The inclusion map $i_{n_j,n_{j+1}}$ induces the injective group homomorphism:

$$(i_{n_j,n_{j+1}})_* : V_0(T_{n_j}(F)) \to V_0(T_{n_{j+1}}(F)),$$

so that $(i_{n_j,n_{j+1}})_*$ maps $\mathbb{Z}^{n_j} \cong \mathbb{Z}[\frac{1}{n_j}]\oplus\cdots\oplus\mathbb{Z}[\frac{n_j}{n_j}]$ injectively to $\mathbb{Z}^{n_{j+1}} \cong \mathbb{Z}[\frac{1}{n_{j+1}}]\oplus\cdots\oplus\mathbb{Z}[\frac{n_{j+1}}{n_{j+1}}]$ by Corollary 5.2, where we have $(i_{n_j,n_{j+1}})_*([p]) = [\oplus^{k_j}p]$ for $[p] \in V_0(T_{n_j}(F))$, so that the class $[p]$ is identified with $[\oplus^{k_j}p]$ and with their limit class in $V_0(T_\infty(F))$, and each k-th coordinate base for $\mathbb{Z}^{n_j}$ is identified with $\frac{k}{n_j}$ for $1 \le k \le n_j$. Therefore,

$$V_0(T_\infty(F)) \cong \varinjlim \mathbb{Z}^{n_j} \cong \varinjlim \{\mathbb{Z}[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}[\frac{n_j}{n_j}]\}.$$

Also, induced is the injective group homomorphism:

$$(i_{n_j,n_{j+1}})_* : V_1(T_{n_j}(F)) \to V_1(T_{n_{j+1}}(F)),$$

so that $(i_{n_j,n_{j+1}})_*$ maps $\Pi^{n_j}\mathbb{Z}_2 \cong \mathbb{Z}_2[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_j}{n_j}]$ injectively to $\Pi^{n_{j+1}}\mathbb{Z}_2 \cong \mathbb{Z}_2[\frac{1}{n_{j+1}}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_{j+1}}{n_{j+1}}]$ by Corollary 5.5 and by the same reason as above. □

Next, let $\varinjlim \oplus_{j=1}^k T_{n_{j,k}}(F)$ be a unital inductive limit of finite direct sums $\oplus_{j=1}^k T_{n_{j,k}}(F)$ with unital connecting maps $i_{k,k+1}$ such that each $n_{j,k+1}$ is a weighted sum of $n_{j,k}$, so that $n_{j,k+1} = \sum_{s=1}^k m_{s,k}n_{s,k}$ for some integers $m_{s,k} \ge 0$ and $i_{k,k+1}(x_l) = \oplus_{s=1}^k[\oplus_{j=1}^{m_{s,k}} x_l]$ for $x_l \in T_{n_{l,k}}(F)$. The diagram for such connecting maps is known as the Bratteli diagram (cf. [3] and [4]).

Proposition 6.2. *We obtain*

$$V_0(\varinjlim \oplus_{j=1}^k T_{n_{j,k}}(F)) \cong \varinjlim \oplus_{j=1}^k \mathbb{Z}^{n_{j,k}} \cong \varinjlim\{\oplus_{j=1}^k(\oplus_{s=1}^{n_{j,k}}\mathbb{Z}[\frac{s}{n_{j,k}}])\}$$

as inductive limits of scaled ordered groups, with $[1] = \lim_{k\to\infty}\{\oplus_{j=1}^{k}(\oplus_{s=1}^{n_{j,k}}[\frac{s}{n_{j,k}}])\}$ *as scale. Also,*

$$V_1(\varinjlim \oplus_{j=1}^{k} T_{n_{j,k}}(F)) \cong \varinjlim \oplus_{j=1}^{k}(\mathbb{Z}_2)^{n_{j,k}} \cong \varinjlim\{\oplus_{j=1}^{k}(\oplus_{s=1}^{n_{j,k}}\mathbb{Z}_2[\frac{s}{n_{j,k}}])\}$$

with $[1] = \lim_{k\to\infty}\{\oplus_{j=1}^{k}(\oplus_{s=1}^{n_{j,k}}[\frac{s}{n_{j,k}}])\}$.

Moreover, the V-theory groups V_0 *or* V_1 *with the scaled unit classes are complete invariants for unital inductive limits of finite direct sums of upper triangular matrix Banach algebras.*

Proof. The last consequence follows from the classification theorem for unital AF C^*-algebras which contain canonically those non self-adjoint inductive limits as only subalgebras, by the same way as in [3]. □

On the other hand, let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers. We now denote by $K_\infty(F)$ the inductive limit of $T_{n_j}(F)$ by the non-unital inclusion maps given by $x \mapsto x \oplus O_{n_{j+1}-n_j}$ for $x \in T_{n_j}(F)$, where $O_{n_{j+1}-n_j}$ is the zero square matrix of size $n_{j+1} - n_j$. Then $K_\infty(F)$ becomes a Banach algebra as a Banach algebra completion of the infinite union of $T_{n_j}(F)$. Note that $K_\infty(F)$ does not depend on the choice of a family of connecting maps. Also, $K_\infty(F)$ is a non-unital algebra, so that it has no square roots.

For a non-unital Banach algebra B, one may define its V_1-group to be that of the unitization B^+ by F, so that $V_1(B) = V_1(B^+)$, as one way.

But, on the other way, for a non-unital Banach algebra which can be written as an inductive limit of unital Banach algebras, which may or not depend on a choice of a family of connecting maps, we this time define its V-theory group V_1 to be inductive limits of their V-theory groups V_1, so that the continuity in inductive limits do hold even in the non-unital case, depending on the choice.

Proposition 6.3. *We obtain*

$$V_0(K_\infty(F)) \cong \lim_{j\to\infty} \oplus^{n_j}\mathbb{Z},$$

and $V_1(K_\infty(F)) = \varinjlim V_1(T_{n_j}(F)) = \varinjlim(\mathbb{Z}_2)^{n_j}$.

Proof. Note that

$$V_0(K_\infty(F)) \cong V_0(\varinjlim T_{n_j}(F)) \cong \varinjlim \oplus^{n_j}\mathbb{Z},$$

where $n_j \to \infty$ as $j \to \infty$. □

In general,

Proposition 6.4. *Our* V_0*-theory group is always continuous, with respect to inductive limit Banach algebras, and the* V_1*-theory group is continuous only for unital inductive limit Banach algebras with unital connecting maps.*

Proof. It should follows from continuity of K-theory groups for inductive limits of C^*-algebras (see [5]), by the similar way. But omitted. □

Remark. A more general theory for V-theory groups may be continued to be investigated somewhere else in the future.

Let $\{N_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers. Denote by $\varinjlim \oplus_{j=1}^{k} T_{n_j}(F)$ a canonical inductive limit of the block diagonal sums $\oplus_{j=1}^{k} T_{n_j}(F)$ of $T_{n_j}(F)$ $(1 \le j \le k)$ in $T_{N_k}(F)$ with $\sum_{j=1}^{k} n_j = N_k$ and $n_k = N_k - N_{k-1}$ and $n_1 = N_1$, where the non-unital connecting maps are given by $x \mapsto x \oplus 0_{n_{k+1}}$ for x in the k-fold diagonal sum. Then the inductive limit is non-unital and is an infinite direct sum of block diagonal components, so that $\varinjlim \oplus_{j=1}^k T_{n_j}(F) \cong \oplus_{j=1}^\infty T_{n_j}(F)$. As well, let $\varinjlim T_{N_k}(F)$ be a non-unital inductive limit of $T_{N_k}(F)$ by the same way as above.

Proposition 6.5. *We obatin*

$$V_0(\varinjlim \oplus_{j=1}^k T_{n_j}(F)) \cong \varinjlim \oplus_{j=1}^k \mathbb{Z}^{n_j} \cong \varinjlim \mathbb{Z}^{N_k} \cong V_0(\varinjlim T_{N_k}(F)),$$

as an inductive limit of groups, but not as an inductive limit of scaled ordered groups, with

$$[1] = \lim_{k\to\infty} \{[1_{n_1}] + \cdots + [1_{n_k}]\} \quad \textit{and} \quad [1] = \lim_{k\to\infty} \{[1_{N_k}]\}$$

as the scales of the respective (extended) unit classes. Also,

$$V_1(\varinjlim \oplus_{j=1}^k T_{n_j}(F)) = \varinjlim \oplus_{j=1}^k (\mathbb{Z}_2)^{n_j} \cong \varinjlim (\mathbb{Z}_2)^{N_k} \cong V_1(\varinjlim T_{N_k}(F))$$

as an inductive limit of groups, but not as an inductive limit of scaled oredered groups.

Consequently,

$$\varinjlim \oplus_{j=1}^k T_{n_j}(F) \not\cong \varinjlim T_{N_k}(F).$$

Proof. The last consequence follows from the classification theory for non self-adjoint Banach algebras viewed as sub-Banach algebras of AF C^*-algebras and UHF-algebras (see [3] in details). □

To distinguish non-unital inductive limits of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^\infty$ for any sequence $\{n_j\}_{j=1}^\infty$ of positive integers, we introduce a notion as follows. We may say that the sequence $\{n_j\}_{j=1}^\infty$ of positive integers is a sequence of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^\infty$. We define that two sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ of positive integers is equivalent up to inductive permutation if for any m a positive interger, there are positive intergers k and k' such that $k, k' \geq m$ and the finite sequence $\{a_1, \cdots, a_k\}$ is the same sequence as $\{b_1, \cdots, b_{k'}\}$ by subtracting finitely many l and l' elements so that $k - l = k' - l' = m$ and by a permutation of m elements left, so that the respective unions of left elements are the respective sequences.

Proposition 6.6. *Two non-unital inductive limits of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^\infty$ and $\{T_{m_j}(F)\}_{j=1}^\infty$ for two sequences $\{n_j\}_{j=1}^\infty$ and $\{m_j\}_{j=1}^\infty$ of positive integers, respectively, are isomorphic as Banach algebras if and only if these sequences are equivalent up to inductive permutation.*

Proof. The equivalence between those sequences $\{n_j\}_{j=1}^\infty$ and $\{m_j\}_{j=1}^\infty$ implies that there exist isomorphisms of corresponding finite block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^\infty$ and $\{T_{m_j}(F)\}_{j=1}^\infty$ by permutation of their direct summands, inductively. Therefore, there exists an isomorphism between those inductive limits by the density of unions of isomorphic finite block diagonal sums in the inductive limits.

Conversely, the isomorphism denoted by Φ between the inductive limits denoted by $\mathfrak{I}$ and $\mathfrak{K}$ implies that each finite block diagonal sum of $\mathfrak{I}$ is mapped into a finite block diagonal sum of $\mathfrak{K}$ by Φ. Therefore, it follows that there is the equivalence between the sequences. □

Let $\mathfrak{I}$ be a non-unital inductive limit of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^\infty$ for a sequence $\{n_j\}_{j=1}^\infty$ of positive integers. Then the (inductive or extended) unit I associated to $\mathfrak{I}$ (but not in $\mathfrak{I}$) is equal to

$$\varinjlim \oplus_{j=1}^k I_{n_j}, \quad I_{n_j} \in T_{n_j}(F) \text{ the units.}$$

We say that the limit is the inductive partition of the (extended) unit I. Or we may call it the scale of the inductive limit $\mathfrak{I}$, and write $\Sigma_{\mathfrak{I}}$. Similarly as in the case of sequences above, we define that two inductive partitions $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ of the respective units associated to two inductive limits $\mathfrak{L}$ and $\mathfrak{K}$ of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$, respectively, are equivalent up to inductive permutation if for any m a positive interger, there are positive intergers k and k' such that $k, k' \geq m$ and the element $\oplus_{j=1}^{k} I_{n_j}$ is identified with the element $\oplus_{j=1}^{k'} I_{m_j}$ by subtracting finitely many l and l' diagonal sum components so that $k - l = k' - l' = m$ and by a permutation of m diagonal sum components left, so that the respective left components add up to the respective units. In this case, we write $\Sigma_{\mathfrak{I}} \sim \Sigma_{\mathfrak{K}}$.

Corollary 6.7. *Non-unital inductive limits $\mathfrak{I}$ and $\mathfrak{K}$ of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$ for two sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ of positive integers, respectively, are isomorphic as Banach algebras if and only if the respective inductive partitions of units $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ are equivalent up to inductive permutation, i.e., $\Sigma_{\mathfrak{I}} \sim \Sigma_{\mathfrak{K}}$.*

Proof. The respective inductive partitions of units $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ are by definition, equivalent up to inductive permutation if and only if the sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ are equivalent up to inductive permutation. □

Remark. Those isomorphisms between the inductive limits are given by permutations, that are essentially equivalent to taking unitary equivalences, that are not allowed in the inductive limits. Namely, the isomorphisms exist in the self-adjoint world. If not allowed, i.e., in the non self-adjoint world, the inductive limits can not be isomorphic except the trivial cases. Note also that block diagonal sums are essentially equivalent to direct sums.

We may call the unital or non-unital, inductive limits of finite direct sums of the upper triangular matrix algebras as ATM algebras, in the sense of being approximately triangular matrix algebras. As a summary,

Corollary 6.8. *Two non-unital ATM algebras are isomorphic if and only if their scales are equivalent in our sense, where we suppose that permutations are allowed in isomorphisms.*

As well,

Corollary 6.9. *Two unital or non-unital ATM algebras are isomorphic if and only if their scaled V-theory groups are isomorphic.*

Proof. Note that the unital case can be proved within the same context as in the non-unital case above, without using the classification result in C^*-algebras. □

Remark. This is a sort of classification result in non self-adjoint Banach algebras corresponding to that of AF C^*-algebras. However, our method for the classification is similar to that of the C^*-algebra case, and the results should be the same essentially as contents.

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