A NOTE ON A TWO-PERSON ZERO-SUM STOPPING GAME

Cloud Makasu

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Abstract. In this short note, a two-person zero-sum stopping game when the observation processes are exponential Brownian motions is formulated and solved explicitly under certain conditions. The present problem extends the planar Brownian motion case treated by Vinnichenko and Mazalov [6].

1 Introduction For a planar Brownian motion on a closed interval $[0,1]$ and absorbed at the end points, a two-person zero-sum stopping game is considered by Vinnichenko and Mazalov [6]. It is proved therein that the optimal stopping times are the so-called Azéma-Yor stopping times [2] and the value of the game is the smallest upper convex function of the payoff (see Dynkin and Yushkevich [3]), both given explicitly. The proof is essentially a result of decomposing the zero-sum stopping game into two pure optimal stopping problems [5]. In the one-dimensional case, a variant of the zero-sum stopping game in [6] is treated by Yasuda [7]. The main purpose of this note is to extend the result by Vinnichenko and Mazalov [6] to the case of exponential Brownian motions. We note that our method of proof is based on similar arguments used in [6]. However, it must be stressed that our main result is not contained in [6].

For the reader's convenience, we shall use similar notation as in [6]. We consider the following two-person zero sum stopping game. Let $i = 1, 2$ and $x_t^{(i)}$ be a geometric Brownian motion starting at a_i in the closed interval $[0, 1]$ and absorbed at the end points, with drift-diffusion coefficients $(\mu_i x^i, \sigma_i x^i)$ where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. The strategy of player i is the stopping time τ_i with respect to the process $x_t^{(i)}$. Players stop their observation processes at the states $x_{\tau_1}^{(1)}$, $x_{\tau_2}^{(2)}$, respectively. Then if $x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}$ the payoff of player 1 is $+1$; if $x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}$ then the payoff is -1. Otherwise, the payoff is assumed to be zero. Player 1 seeks to maximize the expected payoff

$$H(\tau_1, \tau_2) = \mathbf{E}\left\{ I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}} \right\}$$

with player 2 seeking to minimize it, where I_A denotes the indicator function of the set A.

The motivation of the present problem arises from an application in mathematical finance. Consider two investors observing the evolution of prices of two types of stocks. The problem is that of deciding when to invest depending on the values of the two stocks.

2 Main result Fix $i = 1, 2$. Let $\Delta_i = 1 - \frac{2\mu_i}{\sigma_i^2}$, where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. Now following [6], we let $a_1 \le a_2$, $\Delta_1 \le \Delta_2$ and put

$$a = \min\left\{ \left(\frac{2}{\Delta_2}(\Delta_2 + 1) a_2^2 \right)^{1/(\Delta_2+1)}, (2 - 2a_2)^{1/\Delta_2} \right\}.$$

Define

$$\psi_1(x) = 1 - \frac{a_1}{a_2} + \frac{a_1}{2a_2^2} x^{\Delta_1}, \quad \psi_2(x) = \frac{1}{2a_2} x^{\Delta_2}$$

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and

$$s_i^*(x) = \begin{cases} 0 & \text{if } x < 0 \\ \psi_i(x) & \text{if } 0 \le x < a \\ \psi_i(a) & \text{if } a \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases} \tag{1}$$

To this end, we shall also introduce the following barycenter function

$$g_i(x) = a_i + \frac{1}{1 - s_i^*(x-0)} \int_x^1 (u - a_i) ds_i^*(u) = \begin{cases} a_i & \text{if } x = 0 \\ \min\left\{1, a_2 + \frac{\Delta_2 x^{\Delta_2}}{2a_2 - x^{\Delta_2}} \left(\frac{a_2}{\Delta_2} - \frac{x}{\Delta_2 + 1}\right)\right\} & \text{if } 0 < x \le 1 \end{cases}$$

and the Azéma-Yor stopping times

$$\tau_i^* = \inf\left\{t : g_i(x_t^{(i)}) \le \sup_{0 \le s \le t} x_s^{(i)}\right\}. \tag{2}$$

The main result of this note is stated in the next theorem.

THEOREM 2.1. Let $i = 1, 2$ and $x_t^{(i)}$ be an exponential Brownian motion starting at a_i in the closed interval $[0, 1]$ and absorbed at the end points, with drift-diffusion coefficients $(\mu_i x^i, \sigma_i x^i)$ where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. For $a_1 \le a_2$ and $\Delta_1 \le \Delta_2$, the value of the two-person zero-sum stopping game

$$\sup_{\tau_1} \inf_{\tau_2} \mathbf{E}\left\{I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}}\right\} = \inf_{\tau_2} \sup_{\tau_1} \mathbf{E}\left\{I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}}\right\}$$

is given by $H^* = \frac{a_1^{\Delta_2} - a_2}{a_2}$ and the pair of stopping times (τ_1^*, τ_2^*) given by (2) is the equilibrium point provided that

$$2a_1 a_2 - a_1 a_2^{\Delta_1} - a_2 a_1^{\Delta_2} = 0.$$

Proof. This follows with minor modifications of the proof in [6]. Hence, we shall omit the details.

REMARK 2.1. In the special case when $\Delta_1 = \Delta_2 = 1$, then the above result coincides with the result in [6] for the planar Brownian motion case.

3 Conclusion A two-person zero-sum stopping game when the observation processes are assumed to be exponential Brownian motions is solved explicitly, under certain conditions. The present result extends the one obtained by Vinnichenko and Mazalov [6] for the planar Brownian motion case, and has several applications in mathematical finance.

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Communicated by *Vladimir V. Mazalov*

Cloud Makasu
Dept. of Maths and Applied Maths
University of the Western Cape
Private Bag X17
Bellville 7535
Cape Town, South Africa
E-mail: cmakasu@uwc.ac.za