*Scienticae Mathematicae Japonicae*

# $g(x)$-NIL CLEAN RINGS

HANI A. KHASHAN AND ALI H. HANDAM

Received July 9,2014

ABSTRACT. An element in a ring $R$ with identity is called nil clean if it is the sum of an idempotent and a nilpotent, $R$ is called nil clean if every element of $R$ is nil clean. Let $C(R)$ be the center of a ring $R$ and $g(x)$ be a fixed polynomial in $C(R)[x]$. Then $R$ is called $g(x)$-nil clean if every element in $R$ is a sum of a nilpotent and a root of $g(x)$. In this paper, we investigate many properties and examples of $g(x)$-nil clean rings. Moreover, we characterize nil clean rings as $g(x)$-nil clean rings where $g(x) \in (x-(a+1))(x-b)C(R)[x]$, $a, b \in C(R)$ and $b - a \in N(R)$.

## 1. INTRODUCTION

Throughout this paper $R$ denotes an associative ring with identity and all modules are unitary. The group of units, the set of idempotents and the set of nilpotent elements in $R$ are denoted by $U(R)$, $Id(R)$ and $N(R)$ respectively. Following Han and Nicholson [11], an element $r \in R$ is called clean if $r = e + u$ for some $e \in Id(R)$ and $u \in U(R)$. A ring $R$ is called clean if every element of $R$ is clean. The notion of clean rings was first introduced by Nicholson [14] in 1977 in his study of lifting idempotents and exchange rings. Since then, some stronger concepts have been considered (e.g. uniquely clean, strongly clean and some special clean rings), see [4, 6, 15, 17, 18, 19, 20]. As well as some weaker ones (e.g. almost clean and weakly clean rings), see [1]. Recently, in 2013, Diesl [9] studied a stronger concept than clean rings, namely, nil-clean rings. They are rings in which every element is a sum of an idempotent element and a nilpotent element. In fact, nil clean rings were firstly presented in [12] as a special case of rings in which every element is a sum of nilpotent and potent elements.

Let $C(R)$ denotes the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. Then following Camillo and Simón [5], $R$ is called $g(x)$-clean if for each $r \in R$ , $r = s + u$ where $u \in U(R)$ and $g(s) = 0$. Of course $(x^2 - x)$-clean rings are precisely the clean rings.

Nicholson and Zhou [16] proved that if $g(x) \in (x - a)(x - b)C(R)[x]$ with $a, b \in C(R)$ and $b, b - a \in U(R)$ and ${}_RM$ is a semisimple left $R$-module, then $End({}_RM)$ is $g(x)$-clean. Recently, Fan and Yang [10], studied more properties of $g(x)$-clean rings. Among many conclusions, they prove that if $g(x) \in (x - a)(x - b)C(R)[x]$ where $a, b \in C(R)$ with $b - a \in U(R)$, then $R$ is a clean ring if and only if $R$ is $(x - a)(x - b)$-clean.

In this paper, we define and study $g(x)$-nil clean rings as a special class of $g(x)$-clean rings. For a ring $R$ and $g(x) \in C(R)[x]$, an element $r \in R$ is called $g(x)$-nil clean if $r = s+b$ for some $b \in N(R)$ and $g(s) = 0$. Moreover, $R$ is called $g(x)$-nil clean if every element in $R$ is $g(x)$-nil clean.

2000 *Mathematics Subject Classification.* 16N40, 16U99.

*Key words and phrases.* clean ring, $g(x)$-clean ring, nil clean ring, $g(x)$-nil clean ring.

In section 2, we study many properties of $g(x)$-nil clean rings analogous to those of nil clean and $g(x)$-clean rings. In particular, for a commutative ring $R$, we justify a condition under which the the amalgamated duplication $R \bowtie I$ of a ring $R$ along an ideal $I$ is $g(x)$-nil clean. Also, we consider the idealization $R(M)$ of any $R$-module $M$ and prove that $R(M)$ is $g(x)$-nil clean ring if and only if $R$ is so.

In section 3, we study $(x^2 + cx + d)$-nil clean rings where $c, d \in C(R)$. We give many characterizations for a nil clean ring $R$ in terms of some $g(x)$-nil clean rings. In particular for $n \in \mathbb{N}$, we focus on $(x^2 - (n-1)x)$-nil clean and $(x^n - x)$-nil clean rings.

## 2. $g(x)$- NIL CLEAN RINGS

In this section, we give some properties of $g(x)$-nil clean rings which are similar to those of $g(x)$-clean rings.

**Definition 2.1.** Let $R$ be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element

$r \in R$ is called $g(x)$-nil clean if $r = b + s$ where $g(s) = 0$ and $b$ is a nilpotent of $R$. We say that $R$ is $g(x)$-nil clean if every element in $R$ is $g(x)$-nil clean.

Clearly, nil clean rings are $(x^2 - x)$-nil clean. However, there are $g(x)$-nil clean rings which are not nil clean. For example, it can be easily proved that $\mathbb{Z}_3$ is an $(x^3 + 2x)$-nil clean ring which is not nil clean. For a non commutative $g(x)$-nil clean ring we have the following example.

**Example 2.2.** Consider the ring $R = \left\{ \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}_4 \right\}$.Then one can see that for any $x, y \in R$, $(x - x^2)(y - y^2) = 0$. Hence, $R$ is $(x - x^2)^2$-nil clean.

**Proposition 2.3.** *Every $g(x)$-nil clean ring is $g(x)$-clean ring.*

*Proof.* Suppose $R$ is a $g(x)$-nil clean ring and let $x \in R$. Then $x - 1 = b + s$ where $b$ is nilpotent and $g(s) = 0$. Thus, $x = (b + 1) + s$ where $b + 1 \in U(R)$. Therefore, $R$ is $g(x)$-clean. □

The converse of Proposition 2.3 is not be true in general. For example, one can verify that $\mathbb{Z}_{10}$ is $(x^7 - x)$-clean ring which is not $(x^7 - x)$-nil clean ring.

Let $R$ and $S$ be rings and $\phi : C(R) \to C(S)$ be a ring homomorphism with $\phi(1_R) = 1_S$. For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, we let $g^\star(x) := \sum_{i=0}^{n} \phi(a_i) x^i \in C(S)[x]$. In particular, if $g(x) \in \mathbb{Z}[x]$, then $g^\star(x) = g(x)$.

**Proposition 2.4.** *Let $\theta : R \to S$ be a ring epimorphism. If $R$ is $g(x)$-nil clean, then $S$ is $g^\star(x)$-nil clean.*

*Proof.* Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ and consider $g^\star(x) := \sum_{i=0}^{n} \theta(a_i) x^i \in C(S)[x]$. For every $\alpha \in S$, there exist $r \in R$ such that $\theta(r) = \alpha$. Since $R$ is $g(x)$-nil clean, there exist $s \in R$ and $u \in N(R)$ such that $r = u + s$ and $g(s) = 0$. So $\alpha = \theta(r) = \theta(u + s) = \theta(u) + \theta(s)$ with

$\theta(u) \in N(S)$ and $g^\star(\theta(s)) = \sum_{i=0}^{n}\theta(a_i)(\theta(s))^i = \sum_{i=0}^{n}\theta(a_i)\theta(s^i) = \sum_{i=0}^{n}\theta\left(a_i s^i\right) = \theta\left(\sum_{i=0}^{n} a_i s^i\right) = \theta(g(s)) = \theta(0) = 0$. Therefore, $S$ is $g^\star(x)$-nil clean. □

**Proposition 2.5.** *If $R$ is a $g(x)$-nil clean ring and $I$ is an ideal of $R$, then $\overline{R} = R/I$ is $g^\star(x)$-nil clean. Moreover, The converse is true if $I$ is nil and the roots of $g^\star(x)$ lift modulo $I$.*

*Proof.* For the first statement, we use Proposition 2.4 and the fact that $\theta : R \to R/I$ defined by $\theta(r) = \overline{r} = r + I$ is an epimorphism. Now, suppose $R/I$ is $g^\star(x)$-nil clean and let $r \in R$. Then $\overline{r} = \overline{s} + \overline{b}$ where $\overline{b} \in N(\overline{R})$ and $g^\star(\overline{s}) = \overline{0}$. Since the roots of $g^\star(x)$ lift modulo $I$, we may assume that $s \in R$ with $g(s) = 0$. Now, $r - s$ is nilpotent modulo $I$ and $I$ is nil imply that $r - s$ is nilpotent. Therefore, $R$ is $g(x)$-nil clean. □

**Proposition 2.6.** *Let $R_1, R_2, ..., R_k$ be rings and $g(x) \in \mathbb{Z}[x]$. Then $R = \prod_{i=1}^{k} R_i$ is $g(x)$-nil clean if and only if $R_i$ is $g(x)$-nil clean for all $i \in \{1, 2, ..., n\}$.*

*Proof.* $\Rightarrow)$ : For each $i \in \{1, 2, ..., k\}$, $R_i$ is a homomorphic image of $\prod_{i=1}^{k} R_i$ under the projection homomorphism. Hence, $R_i$ is $g(x)$-nil clean by Proposition 2.4.

$\Leftarrow)$ : Let $(x_1, x_2, ..., x_k) \in \prod_{i=1}^{k} R_i$ . For each $i$, write $x_i = n_i + s_i$ where $n_i \in N(R_i)$, $g(s_i) = 0$. Let $n = (n_1, n_2, ..., n_k)$ and $s = (s_1, s_2, ..., s_k)$. Then it is clear that $n \in N(R)$and $g(s) = 0$. Therefore, $R$ is $g(x)$-nil clean. □

In general, the ring of polynomials $R[t]$ over a ring $R$ is not $g(x)$-clean. This is also true for commutative $g(x)$-nil clean rings.

**Proposition 2.7.** *If $R$ is any commutative ring, then the ring of polynomials $R[t]$ is not nil clean (and hence not $(x^2 - x)$-nil clean).*

*Proof.* Since $R$ is commutative, $N(R[t]) = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k \mid a_0, a_1, \cdots, a_k \in N(R)$ and $k \in \mathbb{N}\}$. If $t$ is nil clean, we may write $t = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + e$ where $e \in Id(R[t]) = Id(R)$ and $a_0, a_1, \cdots, a_k \in N(R)$. Hence, $1 = a_1 \in J(R)$ which is a contradiction. Therefore $R[t]$ is not nil clean. □

Let $\theta : R[[t]] \to R$ be defined by $\theta(f) = f(0)$. As a consequence of Proposition 2.3, if $R[[t]]$ is $g^\star(x)$-nil clean, then $R$ is $g(x)$-nil clean.

Let $R$ be a commutative ring and $M$ an $R$-module. Nagata [13] introduced the idealization $R(M)$ of $R$ and $M$. The idealization of $R$ and $M$ is the ring $R(M) = R \oplus M$ with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. This construction has been extensively studied and has many applications in different contexts, see [2] and [3].

Note that if $(r, m) \in R(M)$, then $(r, m)^k = (r^k, kr^{k-1}m)$ for any $k \in \mathbb{N}$. The proof of the following lemma is immediate.

**Lemma 2.8.** *Let $R$ be a commutative ring and $M$ an $R$-module. Then $(b, m)$ is nilpotent in $R(M)$ if and only if $b$ is nilpotent in $R$.*

We recall that $R$ naturally embeds into $R(M)$ via $r \rightarrow (r,0)$. Thus any polynomial $g(x) = \sum\limits_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum\limits_{i=0}^{n} (a_i,0)x^i \in R(M)[x]$ and conversely.

**Theorem 2.9.** *Let $R$ be a commutative ring and $M$ an $R$-module. Then the idealization $R(M)$ of $R$ and $M$ is $g(x)$-nil clean if and only if $R$ is $g(x)$-nil clean.*

*Proof.* $\Rightarrow$) : Note that $R \simeq R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence $R$ is $g(x)$-nil clean by Proposition 2.4.

$\Leftarrow$) : Let $g(x) = \sum\limits_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Write $r = b + s$ where $b \in N(R)$ and $g(s) = 0$. Then for $m \in M$, $(r,m) = (b,m) + (s,0)$ where $(b,m) \in N(R(M))$ by Lemma 2.8. Moreover, we have

$$\begin{aligned} g(s,0) &= a_0(1,0) + a_1(s,0) + a_2(s,0)^2 + ... + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + a_2(s^2,0) + ... + a_n(s^n,0) \\ &= (a_0 + a_1 s + a_2 s^2 + ... + a_n s^n, 0) = (g(s),0) = (0,0). \end{aligned}$$

Therefore, $R(M)$ is $g(x)$-nil clean. □

Let $R$ be a commutative ring with identity 1 and let $I$ be a proper ideal of $R$. The amalgamated duplication of $R$ along $I$ is defined as $R \bowtie I = \{(a, a+i) : a \in R \text{ and } i \in I\}$. It is easy to check that $R \bowtie I$ is a subring with identity $(1,1)$ of $R \times R$ (with the usual componentwise operations). Moreover, $\varphi : R \rightarrow R \bowtie I$ defined by $\varphi(a) = (a,a)$ is a ring monomorphism and so $R \cong \{(a,a) : a \in R\} \subseteq R \bowtie I$. For more properties of $R \bowtie I$, one can see [7] and [8]. In the following theorem, we investigate the $g(x)$-nil cleanness of $R \bowtie I$.

**Theorem 2.10.** *Let $R$ be a commutative ring, $I$ be a proper ideal of $R$ and $g(x) = \sum\limits_{k=0}^{n} a_k x^k \in R[x]$ . If $R$ is $g(x)$- nil clean ring and $I \subseteq N(R)$, then $R \bowtie I$ is $g(x)$- nil clean ring. Moreover, the converse is true if $R \bowtie I$ is domain-like (every zero divisor of $R \bowtie I$ is nilpotent).*

*Proof.* Assume $R$ is $g(x)$-nil clean. Let $(a, a+i) \in R \bowtie I$ and write $a = b + s$ where $b \in N(R)$ and $g(s) = 0$. Then $(a, a+i) = (b+s, b+s+i) = (b, b+i) + (s,s)$. Since $I \subseteq N(R)$, then $(b, b+i) \in N(R \bowtie I)$. Moreover, we have $g((s,s)) = \sum\limits_{k=0}^{n} (a_k, a_k)(s,s)^k = \sum\limits_{k=0}^{n} (a_k, a_k)(s^k, s^k) = (\sum\limits_{k=0}^{n} a_k s^k, \sum\limits_{k=0}^{n} a_k s^k) = (0,0)$. Therefore, $R \bowtie I$ is $g(x)$-nil clean.

Conversely, suppose that $R \bowtie I$ is domain-like $g(x)$-nil clean. Let $(0) \times I = \{(0,a) : a \in I\}$. Then clearly $(0) \times I$ is an ideal of $R \bowtie I$ with $R \bowtie I/(0) \times I \simeq R$. Thus, $R$ is $g(x)$-nil clean by Proposition (2.3). Let $i$ be a nonzero element in $I$ and consider $(0,i) \in R \bowtie I$ . Then $(0,i)(i,0) = (0,0)$ and so $(0,i)$ is a zero divisor in $R \bowtie I$. By assumption, $(0,i) \in N(R \bowtie I)$ and so $(0,i)^m = (0,0)$ for some $m \geq 1$. Therefore, $i^m = 0$ and $I \subseteq N(R)$. □

The proof of the following Lemma is straightforward.

**Lemma 2.11.** *Let $R$ be a ring. For any $n \in \mathbb{N}$, we have*

$$N(T_n(R)) = \begin{bmatrix} Nil(R) & R & R & \cdots & R & R \\ 0 & Nil(R) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Nil(R) & R \\ 0 & 0 & 0 & \cdots & 0 & Nil(R) \end{bmatrix}$$ *where $T_n(R)$ is the upper triangular matrix ring over $R$.*

**Theorem 2.12.** *Let $R$ be a ring, $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$ and $n \in \mathbb{N}$. Then $R$ is $g(x)$-nil clean if and only if $T_n(R)$ is $g(x)$-nil clean.*

*Proof.* $\Leftarrow)$ : Define $f : T_n(R) \longrightarrow R$ by $f(A) = a_{11}$ where $A = (a_{ij}) \in T_n(R)$. Then clearly $f$ is a ring epimorphism and $R$ is $g(x)$-nil clean.

$\Rightarrow)$ : Suppose that $R$ is $g(x)$-nil clean and let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & a_{nn} \end{bmatrix}$$ $\in T_n(R)$. Since $R$ is $g(x)$-nil clean, then for every $1 \leq i \leq n$, there exist $u_{ii} \in N(R)$ and $s_{ii} \in R$ such that $a_{ii} = u_{ii} + s_{ii}$ with $g(s_{ii}) = 0$. Write $A = B + C$ where $$B = \begin{bmatrix} u_{11} & b_{12} & b_{13} & \ldots & b_{1,n-1} & b_{1n} \\ 0 & u_{22} & b_{23} & \ldots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & u_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & u_{nn} \end{bmatrix}$$ and

$$C = \begin{bmatrix} s_{11} & 0 & \ldots & 0 \\ 0 & s_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & s_{nn} \end{bmatrix}.$$ Then $B$ is nilpotent in $T_n(R)$ and $g(C) = a_0 I_n + a_1 C + ... +$

$$a_m C^m = \begin{bmatrix} g(s_{11}) & 0 & \ldots & 0 \\ 0 & g(s_{22}) & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & g(s_{nn}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$ Therefore, $T_n(R)$ is $g(x)$-nil clean. □

**Theorem 2.13.** *Let $A$ and $B$ be rings and let $M =_B M_A$ be a bimodule. If the formal triangular matrix $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ is $g(x)$-nil clean, then both, $A$ and $B$ are $g(x)$-nil clean.*

*Proof.* Let $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ be $g(x)$-nil clean. For every $a \in A$, $b \in B$ and $m \in M$, write $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = \begin{bmatrix} n_1 & 0 \\ n_2 & n_3 \end{bmatrix} + \begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix}$ where $\begin{bmatrix} n_1 & 0 \\ n_2 & n_3 \end{bmatrix} \in N(T)$ and $g\left(\begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix}\right) = 0$. Then $a = n_1 + s_1$ and $b = n_3 + s_3$. It is easy to see that $n_1 \in N(A)$, $n_2 \in N(B)$ and $g(s_1) = g(s_3) = 0$. Therefore, $A$ and $B$ are $g(x)$-nil clean. □

## 3. $(x^2+cx+d)$-nil Clean Rings

In this section we first consider $g(x)$-nil clean rings where $g(x)=(x-(a+1))(x-b)$, $a,b\in C(R)$. Then we turn to some special types of polynomials such as $x^n-1$, $x^n-x$ and $x^n+x$.

For a ring $R$, a semisimple $R$-module ${}_RM$ and $a,b\in C(R)$, Nicholson and Zhou [16] proved that if $g(x)\in(x-a)(x-b)C(R)[x]$ where $b,b-a\in U(R)$, then $End({}_RM)$ is $g(x)$-clean. More recently, Fan and Yang proved the following.

**Lemma 3.1.** [10]. *Let $R$ be a ring, $a,b\in C(R)$ and $g(x)\in(x-a)(x-b)C(R)[x]$ where $b-a\in U(R)$. Then*

*(1) $R$ is clean if and only if $R$ is $(x-a)(x-b)$-clean*

*(2) If $R$ is clean, then $R$ is $g(x)$-clean.*

Now, we prove the following main result.

**Theorem 3.2.** *Let $R$ be a ring and $a,b\in C(R)$. Then $R$ is nil clean and $b-a\in N(R)$ if and only if $R$ is $(x-(a+1))(x-b)$-nil clean.*

*Proof.* $\Rightarrow$) : Let $r\in R$. Since $R$ is nil clean, then $\frac{r-(a+1)}{(b-a)-1}=e+u$ , where $e^2=e$ and $u\in N(R)$. Hence, $r=e((b-a)-1)+(a+1)+u((b-a)-1)=t+v$ where $t$ is a root of $(x-(a+1))(x-b)$ and $v\in N(R)$. Indeed,

$$\begin{aligned}&[e(b-a)-1)+(a+1)-(a+1)][e(b-a)-1)+(a+1)-b]\\ =\ & e^2((b-a)-1)^2-e((b-a)-1)((b-a)-1)=0\end{aligned}$$

Thus, $R$ is $(x-(a+1))(x-b)$-nil clean.

$\Leftarrow$) : Conversely, suppose $R$ is $(x-(a+1))(x-b)$-nil clean. Then $a=s+u$ where $(s-(a+1))(s-b)=0$ and $u\in N(R)$. Thus, $s-a\in N(R)$ and so $s-a-1\in U(R)$. It follows that $s=b$ and $b-a\in N(R)$. Now, let $r\in R$. Since $R$ is nil $(x-(a+1))(x-b)$-clean, then $r((b-a)-1)+(a+1)=s+u$ where $s$ is a root of $(x-(a+1))(x-b)$ and $u\in N(R)$. Hence, $r=\frac{s-(a+1)}{(b-a)-1}+\frac{u}{(b-a)-1}$ where $\frac{u}{(b-a)-1}\in N(R)$ and

$$\begin{aligned}\left(\frac{s-(a+1)}{(b-a)-1}\right)^2 &= \frac{(s-(a+1))(s-b+b-(a+1))}{((b-a)-1)^2}\\ &= \frac{(s-(a+1))(s-b)+(s-(a+1))(b-(a+1))}{((b-a)-1)^2}=\frac{s-(a+1)}{(b-a)-1}.\end{aligned}$$

Therefore, $R$ is nil clean. □

Next, we give some special cases of Theorem 3.2.

**Corollary 3.3.** *Let $R$ be a ring and $a\in C(R)$. Then $R$ is nil clean if and only if $R$ is $(x^2-(2a+1)x+a(a+1))$-nil clean.*

*Proof.* We just take $a=b$ in Theorem 3.2. □

For example, we conclude that $(x^2-3x+2)$-nil clean rings, $(x^2-5x+6)$-nil clean rings and $(x^2-7x+12)$-nil clean rings are equivalent to nil clean rings.

**Lemma 3.4.** [9]. *If a ring $R$ is nil clean, then 2 is a (central) nilpotent element in $R$.*

As 2 is a central nilpotent in any nil clean ring $R$, then $2n \in N(R)$ for any integer $n$. So the, previous lemma provides us with more characterizations of nil clean rings.

**Corollary 3.5.** *Let $R$ be a ring and $n$ be any integer. For any $b \in C(R)$, the following are equivalent*

*(1) $R$ is nil clean.*

*(2) $R$ is $(x^2 - (2b+1-2n)x + (b^2 + b(1-2n)))$-nil clean.*

*(3) $R$ is $(x^2 - (2b+1+2n)x + (b^2 + b(1+2n)))$-nil clean.*

*Proof.* In Theorem 3.2, we take $a = b - 2n$ to get (1)⇔(2) and $a = b + 2n$ to get (1)⇔(3). □

In particular, a ring $R$ is nil clean if and only if $R$ is $(x^2 - (2n+1)x)$-nil clean ( $(x^2 + (2n-1)x)$-nil clean). For example, $(x^2+x)$-nil clean, $(x^2+3x)$-nil clean, $(x^2-3x)$-nil clean and $(x^2 - 5x)$-nil clean rings are all equivalent to nil clean rings.

*Remark* 3.6. The equivalence of $(x^2 + x)$-nil clean rings and nil clean rings is a global property. That is, it holds for a ring $R$ but it may fail for a single element. For example, $1 \in \mathbb{Z}_{12}$ is nil clean but it is not $(x^2 + x)$-nil clean in $\mathbb{Z}_{12}$.

*Remark* 3.7. In [10], The authors give more characterizations of clean rings in terms of $g(x)$-clean rings under the additional assumption that 2 is a unit. But in a nil clean ring $R$, if we assume that $2n + 1 \in N(R)$ for some integer $n$, then $1 \in N(R)$ by lemma 3.4. Thus, $1 = 0$ and $R = \{0\}$.

**Definition 3.8.** A ring $R$ is called $g(x)$-nil$^*$clean if every $0 \neq r \in R$, $r = s + b$ where $b \in N(R)$ and $g(s) = 0$.

Of course, every $g(x)$-nil clean ring is $g(x)$-nil$^*$clean. On the other hand, the following are examples of $g(x)$-nil$^*$clean rings which are not $g(x)$-nil clean.

**Example 3.9.** Let $p$ be a prime integer. Then the field $Z_p$ is $(x^{p-1} - 1)$-nil$^*$clean which is not $(x^{p-1} - 1)$-nil clean.

*Proof.* Let $0 \neq r \in Z_p$. Then $r = 0 + r$ where $0 \in N(R)$ and $r^{p-1} - 1 = 0$ in $Z_p$ by Fermat Theorem. Hence, $Z_p$ is $(x^{p-1} - 1)$-nil$^*$clean. On the other hand, since $Z_p$ is reduced, then 0 can't be written as a sum of a nilpotent and a root of $x^{p-1} - 1$. Therefore $Z_p$ is not $(x^{p-1} - 1)$-nil clean. □

Next, we give a general example.

**Example 3.10.** Let $R$ be a non zero ring, $n \in \mathbb{N}$ and $g(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \in C(R)[x]$ where $a_0 \in U(R)$. Then $R$ is not $g(x)$-nil clean. In particular, If $R$ is any non zero ring and $n \in \mathbb{N}$, then $R$ is not $(x^n - 1)$-nil clean.

*Proof.* Suppose $R$ is $g(x)$-nil clean and write $0 = s + b$ where $b \in N(R)$ and $g(s) = 0$. Then $s(s^{n-1} + a_{n-1}s^{n-2} + ... + a_1) = -a_0 \in U(R)$ and so $s \in U(R)$. Since also $s = -b \in N(R)$, then $R = \{0\}$, a contradiction. □

*Remark* 3.11. Let $R$ be a ring and $g(x) \in C(R)[x]$, The concepts of $g(x)$-nil clean and $g(x)$-nil*clean coincide if there is a non unit root of $g(x)$ such that $0 = s + b$ for some $b \in N(R)$. In particular, they coincide if all roots of $g(x)$ are non units.

**Proposition 3.12.** *Let $R$ be a ring and $n \in \mathbb{N}$. Then $R$ is $(x^n - 1)$-nil*clean if and only if for every $0 \neq r \in R$, $r = v + b$ where $b \in N(R)$ and $v^n = 1$.*

*Proof.* $\Rightarrow$) Let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $s^n - 1 = 0$. Then $s^n = 1$ and the result follows.

$\Leftarrow$) Conversely, let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $v^n = 1$. Then clearly $v$ is a root of $x^n - 1$ and $R$ is $(x^n - 1)$-nil*clean. □

It is well known that if a ring $R$ is commutative, then the sum of a nilpotent element and a unit in $R$ is again a unit. Thus, we have the following Corollary.

**Corollary 3.13.** *Any commutative $(x^n - 1)$-nil*clean is a field.*

**Proposition 3.14.** *Let $R$ be a ring and $2 \leq n \in \mathbb{N}$. If $R$ is $(x^{n-1} - 1)$-nil*clean, then $R$ is $(x^n - x)$-nil clean.*

*Proof.* If $r = 0$, then clearly $r$ is an $(x^n - x)$-nil clean element. Suppose $0 \neq r \in R$. Then $r = v + b$ where $b \in N(R)$ and $v^{n-1} = 1$ and so $v$ is a root of $x^n - x$. Therefore, $R$ is $(x^n - x)$-nil clean. □

The converse of Proposition 3.14 is true under a certain condition.

**Theorem 3.15.** *Let $R$ be a ring and let $0 \neq a \in R$ such that $(a+1)R$ or $R(a+1)$ contain no non trivial idempotents. Then $a$ is $(x^n - x)$-nil clean if and only if $a$ is $(x^{n-1} - 1)$-nil clean. In particular, if for every $a \in R$, $(a + 1)R$ or $R(a + 1)$ contain no non trivial idempotents, then $R$ is $(x^n - x)$-nil clean if and only if $R$ is $(x^{n-1} - 1)$-nil*clean*

*Proof.* $\Leftarrow$) : We use Proposition 3.14.

$\Rightarrow$) : Suppose $a$ is $(x^n - x)$-nil clean and $(a + 1)R$ contains no non trivial idempotents. Then $a = s + b$ where $b \in N(R)$ and $s^n = s$. Now, $as^{n-1} = s + bs^{n-1}$ and so $a(1 - s^{n-1}) = b(1 - s^{n-1})$. Set $y = 1 + b$. Then $y \in U(R)$ and $(a + 1)(1 - s^{n-1}) = (b + 1)(1 - s^{n-1}) = y(1 - s^{n-1})$. This implies that $y(1 - s^{n-1})y^{-1} = (a + 1)(1 - s^{n-1})y^{-1} \in (a + 1)R$. obviously, $y(1 - s^{n-1})y^{-1}$ is an idempotent. If $1 - s^{n-1} \neq 0$, then $y(1 - s^{n-1})y^{-1} \neq 0$. Thus, $(a + 1)R$ contains a non trivial idempotent, a contradiction. If $R(a + 1)$ contains no non trivial idempotents, then we get a similar contradiction. Therefore, $1 - s^{n-1} = 0$ and $s$ is a root of $x^{n-1} - 1$. Thus, $a$ is $(x^{n-1} - 1)$-nil clean. The other part of the Theorem follows clearly. □

Recall that for a ring $R$ and $n \in \mathbb{N}$, $U_n(R)$ denotes the set of elements in $R$ that can be written as a sum of no more than $n$ units. If $R$ is $(x^n - 1)$-nil*clean and $1 \neq r \in R$, then $r - 1 = v + b$ where $b \in N(R)$ and $v^n = 1$ and so $r = v + (b + 1) \in U_2(R)$. Since also clearly $1 \in U_2(R)$, then $R = U_2(R)$. This result can be generalized as follows.

**Proposition 3.16.** *let $R$ be a ring, $n \in \mathbb{N}$ and $g(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in C(R)[x]$ where $1 \pm a_0 \in N(R)$. If $R$ is $g(x)$-nil*clean, then $R = U_2(R)$. In particular, if $R$ is $(x^{n-2} + x^{n-3} + ... + x + 1)$-nil*clean, then $R = U_2(R)$ is $(x^n - x)$-nil clean.*

*Proof.* Let $1 \neq r \in R$ and write $r-1 = s+b$ where $b \in N(R)$ and $s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = 0$. Then $r = s + (b+1)$ where $b+1 \in U(R)$. Moreover, $s(s^{n-1} + a_{n-1}s^{n-2} + ... + a_1) = -a_0 \in U(R)$ and so $s \in U(R)$. Thus, $r \in U_2(R)$. Since also $1 \in U_2(R)$, then $R = U_2(R)$. In particular, suppose $R$ is $(x^{n-2} + x^{n-3} + ... + x + 1)$-nil* clean, then $R = U_2(R)$ by taking $a_0 = 1 \in U(R)$. Now, if $r = 0$, then $r$ is clearly an $(x^n - x)$-nil clean element. Let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $s^{n-2} + s^{n-3} + ... + s + 1 = 0$. Then $s^n - s = s(s-1)(s^{n-2} + s^{n-3} + ... + s + 1) = 0$ and so $R$ is $(x^n - x)$-nil clean. □

By choosing $n = 4$ in the previous proposition, we conclude that if $R$ is $(x^2 + x + 1)$-nil*clean, then $R = U_2(R)$ is $(x^4 - x)$-nil clean.

In the next Proposition, we determine conditions under which the group ring $RG$ is $(x^n - x)$-nil clean for some integer $n$.

**Proposition 3.17.** *Let $R$ be a Boolean ring and $G$ any cyclic group of order $p$ (prime). Then $RG$ is $(x^{2^{p-1}} - x)$- nil clean ring.*

*Proof.* Let $G = < g >$ be a cyclic group of order $p$ and $x = a_0 + a_1 g + a_2 g^2 + ... + a_{m-1} g^{m-1} \in RG$. Using mathematical induction, it can be shown that $x^{2^k} = \sum_{i=0}^{m-1} a_i g^{2^k * i}, k = 1, 2, \ldots$ . It follows from Fermat theorem that $2^{p-1} = 1 + np$ for some $n \in \mathbb{N}$. So, $x^{2^{p-1}} = \sum_{i=0}^{m-1} a_i g^{2^{p-1} * i} = \sum_{i=0}^{m-1} a_i g^{(1+np)*i} = \sum_{i=0}^{m-1} a_i g^i = x$. Thus, $RG$ is $(x^{2^{p-1}} - x)$-nil clean ring. □

Next we give examples showing that $(x^n - x)$-nil cleanness of a ring $R$ does not imply nil cleanness of $R$ whether $n$ is odd or even.

**Example 3.18.** The field $\mathbb{Z}_3$ is $(x^3 - x)$-nil clean which is not nil clean. Also, by Proposition 3.17 the group ring $\mathbb{Z}_2(C_3)$ is $(x^4 - x)$-nil clean which is not nil clean.

**Proposition 3.19.** *Let $R$ be a ring and $n \in \mathbb{N}$. Then $R$ is $(ax^{2n} - bx)$-nil clean if and only if $R$ is $(ax^{2n} + bx)$-nil clean.*

*Proof.* $\Rightarrow$) : Suppose $R$ is $(ax^{2n} - bx)$-nil clean and let $r \in R$. Then $-r = u + s$ where $u \in N(R)$ and $as^{2n} - bs = 0$. Thus, $r = (-u) + (-s)$ where $-u \in N(R)$ and $a(-s)^{2n} + b(-s) = as^{2n} - bs = 0$. Therefore, $R$ is $(ax^{2n} + bx)$-nil clean.

$\Leftarrow$) : Suppose $R$ is $(ax^{2n} + bx)$-nil clean and let $r \in R$. Then $-r = u + s$ where $u \in N(R)$ and $as^{2n} + bs = 0$. Thus, $r = (-u) + (-s)$ where $-u \in N(R)$ and $a(-s)^{2n} - b(-s) = as^{2n} + bs = 0$. Therefore, $R$ is $(ax^{2n} - bx)$-nil clean. □

By Proposition 3.19, we conclude that $\mathbb{Z}_2(C_3)$ is also $(x^4 + x)$-nil clean. On the other hand, the equivalence in Proposition 3.19 need not be true if we replace the even power $2n$ by an odd power $2n+1$. By a simple calculations, we can see that the field $\mathbb{Z}_3$ is $(x^3 - x)$-nil clean ($(x^5 - x)$-nil clean) but not $(x^3 + x)$-nil clean ($(x^5 + x)$-nil clean). However, we don't know whether $(x^n + x)$-nil cleanness implies the $(x^n - x)$-nil cleanness of $R$ or not.

Recall that a ring $R$ is called unit $n$-regular if for any $a \in R$, $a = a(ua)^n$ for some $u \in U(R)$. In [10], the authors ask about the relation between the following conditions on a ring $R$

(1) $R$ is $(x^n - x)$-clean for all $n \geq 3$.

(2) $R$ is a unit $n$-regular.

In general, condition (1) does not imply condition (2) for odd or even integer $n$. For example, the ring $\mathbb{Z}_4$ is $(x^3 - x)$-clean which is not unit 3-regular and the ring $\mathbb{Z}_8$ is $(x^4 - x)$-clean which is not unit 4-regular. However, we still don't know whether condition (2) implies condition (1) or not. On the other hand if we replace $(x^n - x)$-cleanness by $(x^n - x)$-nil cleanness in condition (1), then non of the two conditions implies the other. For example, $\mathbb{Z}_4$ is also $(x^4 - x)$-nil clean which is not unit 4-regular and $\mathbb{Z}_3$ is unit 4-regular which is not $(x^4 - x)$-nil clean.

## References


[1] M. -S. Ahn and D. D. Anderson, Weakly clean rings and almost clean rings, Rocky Mount. J. Math. 36 (2006), 783–798.

[2] M. M. Ali, Idealization and Theorems of D. D. Anderson, Comm. Algebra, 34 (2006), 4479-4501.

[3] D. D. Anderson and M. Winders, Idealization of a module, J. Comm. Algebra, 1(1) (2009), 3-56.

[4] G. Borooah, A. J. Diesl, T. J. Dorsey, Strongly clean matrix rings over commutative local rings, J. Pure Appl. Algebra 212 (1)(2008), 281–296.

[5] V. P. Camillo, J. J. Simón, The Nicholson-Varadarajan theorem on clean linear transformations, Glasgow Math. J. 44 (2002), 365-369.

[6] H. Chen, On uniquely clean rings, Comm. Algebra, 39 (2011), 189-198.

[7] M. Chhiti, N. Mahdou, Homological dimension of the amalgamated duplication of a ring along a pure ideal, Afr. Diaspora J. Math. (N. S.) 10, no. 1, (2010), 1–6.

[8] M. D'Anna, M. Fontana, An amalgamated duplication of a ring along an ideal, J. Algebra Appl. 6 (3)(2007), 443–459.

[9] A. J. Diesl, Nil clean rings, Journal of Algebra, 383 (2013), 197-211.

[10] L. Fan, X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, Comm. Algebra. 36 (2008), 269-278.

[11] J. Han and W. K. Nicholson, Extensions of clean rings, Communications in Algebra 29(6) (2001), 2589–2595.

[12] Y. Hirano, H. Tominaga and A. Yaqub, On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element, 30 (1988).

[13] A. Lambert, T. G. Lucas, Nagata's principle of idealization in relation to module homomorphisms and conditional expectations, Kyungpook Math. J. 40(2000), 327-337.

[14] W. K. Nicholson, Lifting idempotents and exchange rings, Transactions of the American Mathematical Society 229 (1977), 269–278.

[15] W. K. Nicholson, Strongly clean rings and Fitting's lemma. Comm. Alg. 27 (1999), 3583–3592.

[16] W. K. Nicholson, Y. Zhou, Endomorphisms that are the sum of a unit and a root of a fixed polynomial, Canad. Math. Bull. 49 (2006), 265-269.

[17] Z. Wang and J. Chen, A note on clean rings, Algebra Colloquium, 14 (3)(2007), 537-540.

[18] G. Xiao, W. Tong, $n$-clean rings and weakly unit stable range rings. Comm. Alg. 33 (2003),1501-1517.

[19] G. Xiao, W. Tong, $n$-clean rings, Algebra Colloquium. 13 (4)(2006), 599-606.

[20] Y. Q. Ye, Semiclean rings, Comm. Algebra 31 (2003), 5609-5625.


Communicated by *Klaus Denecke*


Department of Mathematics, Al al-Bayt University, P.O.Box: 130095, Al Mafraq, Jordan

*E-mail address*: hakhashan@aabu.edu.jo

Department of Mathematics, Al al-Bayt University, P.O.Box: 130095, Al Mafraq, Jordan

*E-mail address*: ali.handam@windowslive.com