Criteria for the C-integral

TOSHIHARU KAWASAKI ICHIRO SUZUKI

Received October 31, 2013 ; revised March 7, 2014

ABSTRACT. The C-integral was introduced by Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. Moreover Bongiorno, Piazza and Preiss gave some criteria for the C-integral. On the other hand, Nakanishi gave some criteria for the restricted Denjoy integral. In this paper we will give new criteria for the C-integral in the style of Nakanishi.

1 Introduction and preliminaries Throughout this paper we denote by $\mathbf{(L)}(S)$ and $\mathbf{(D^*)}(S)$ the class of all Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a,b]$ into $(0,\infty)$ and a δ-fine McShane partition is a collection $\{(I_k, x_k) \mid k = 1,\dots,k_0\}$ of non-overlapping intervals $I_k \subset [a,b]$ satisfying $I_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$ and $\sum_{k=1}^{k_0} |I_k| = b-a$. If $\sum_{k=1}^{k_0} |I_k| \leq b-a$, then we say that the collection is a δ-fine partial McShane partition.

In [3] Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. It is given as follows:

Definition 1.1. A function f from an inteval $[a,b]$ into $\mathbb{R}$ is said to be C-integrable if there exists a number A such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1,\dots,k_0\}$ with $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$, where $d(I_k, x_k) = \inf_{x \in I_k} d(x, x_k)$. The constant A is denoted by

$$A = (C)\int_{[a,b]} f(x)dx.$$

We denote by $\mathbf{(C)}([a,b])$ the class of all C-integrable functions from $[a,b]$ into $\mathbb{R}$.

We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b]$. We denote by $\mathbf{(N)}([a,b])$ the class of all Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [3] they also gave a criterion for the C-integral as follows:

Theorem 1.1. *Let f be a function from an inteval $[a,b]$ into $\mathbb{R}$. Then $f \in \mathbf{(C)}([a,b])$ if and only if there exists $h \in \mathbf{(N)}([a,b])$ such that $f - h \in \mathbf{(L)}([a,b])$.*

2010 *Mathematics Subject Classification.* 26A36, 26A39.
Key words and phrases. C-integral, Denjoy integral, Lebesgue integral, Newton integral.

By the theorem above $\mathbf{(C)}([a,b])$ is the minimal class which contains $\mathbf{(L)}([a,b])$ and $\mathbf{(N)}([a,b])$. Moreover it is contained in the class of all restricted Denjoy integrable functions. Now we refer to the following theorems given by Bongiorno [1, 2].

Theorem 1.2. *Let $f \in \mathbf{(C)}([a,b])$. Then for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C)\int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ with $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a,b]$ is an interval function on $[a,b]$. If an interval function F on $[a,b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a,b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive. For an interval function F on $[a,b]$, for a positive number ε, for a gauge δ and $E \subset [a,b]$ let

$$V_\varepsilon(F,\delta,E) = \sup\left\{ \sum_{k=1}^{k_0} |F(I_k)| \;\middle|\; \begin{array}{l} \{(I_k,x_k) \mid k=1,\ldots,k_0\} \text{ is a } \delta\text{-fine partial McShane} \\ \text{partition with } x_k \in E \text{ and } \sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon} \end{array} \right\}.$$

Moreover let

$$V_C F(E) = \sup_{\varepsilon} \inf_{\delta} V_\varepsilon(F,\delta,E).$$

Theorem 1.3. *An interval function F on $[a,b]$ is the primitive of a C-integrable function if and only if $V_C F$ is absolutely continuous with respect to the Lebesgue measure, that is, for any Lebesgue measurable set E, if $|E| = 0$, then $V_C F(E) = 0$.*

Definition 1.2. Let F be an interval function on $[a,b]$. Then F is said to be C-absolutely continuous on $E \subset [a,b]$ if for any positive number ε there exist a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(α_1) $x_k \in E$ for any k;

(α_2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(α_3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_C(E)$ the class of all C-absolutely continuous interval functions on E. Moreover F is said to be C-generalized absolutely continuous on $[a,b]$ if there exists a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a,b]$ and $F \in \mathbf{AC}_C(E_m)$ for any m. We denote by $\mathbf{ACG}_C([a,b])$ the class of all C-generalized absolutely continuous interval functions on $[a,b]$.

Theorem 1.4. *For any $F \in \mathbf{ACG}_C([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in \mathbf{(C)}([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (C)\int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in \mathbf{(C)}([a,b])$ satisfies $F \in \mathbf{ACG}_C([a,b])$.

On the other hand, in [7, 10] Nakanishi gave the following criteria for the restricted Denjoy integral. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that (A) implies (B) and (C) implies (D). Next Nakanishi gave the following theorems for the restricted Denjoy integral.

Theorem 1.5. *Let $f \in \mathbf{(D^*)}([a,b])$ and let F be an additive interval function on $[a,b]$ defined by*

$$F(I) = (D^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies (A)*.*

Theorem 1.6. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (A)*, then the pair of f and F satisfies* (C)*. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (B)*, then the pair of f and F satisfies* (D)*.*

Theorem 1.7. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (D)*, then $f \in \mathbf{(D^*)}([a,b])$ and*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

By Theorems 1.5, 1.6 and 1.7 we obtain the following criteria for the restricted Denjoy integral.

Theorem 1.8. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* (A)*,* (B)*,* (C) *and* (D)*. Moreover, if the pair of f and F satisfies one of* (A)*,* (B)*,* (C) *and* (D)*, then*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

In this paper, motivated by the results above, we will give new criteria for the C-integral similar to Theorems 1.5, 1.6, 1.7 and 1.8.

2 Main results Firstly we consider the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

$(A)_C$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0+1, \ldots, k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$.

$(B)_C$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1,\dots,k_0,k_0+1,\dots,k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1,\dots,k_0\}$ with $I_k \cap M_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k,x_k) \mid k = k_0+1,\dots,k_1\}$ satisfying

(4.1) $x_k \in M_n$ for any $k = k_0+1,\dots,k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k,x_k) < \frac{1}{\varepsilon_n}$.

$(\mathrm{C})_C$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k,x_k) \mid k = 1,\dots,k_0\}$ in $[a,b]$ satisfying

(3.1) $x_k \in F_n$ for any k;

(3.2) $\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$;

(3.3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1,2,\dots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

$(\mathrm{D})_C$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k,x_k) \mid k = 1,\dots,k_0\}$ in $[a,b]$ satisfying

(4.1) $x_k \in M_n$ for any k;
(4.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(4.3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(\mathrm{A})_C$ implies $(\mathrm{B})_C$ and $(\mathrm{C})_C$ implies $(\mathrm{D})_C$. Now we give the following theorems for the C-integral.

Theorem 2.1. *Let $f \in \mathbf{(C)}([a,b])$ and let F be an additive interval function on $[a,b]$ defined by*

$$F(I) = (C)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies $(\mathrm{A})_C$.

Proof. Since f is C-integrable, it is restricted Denjoy integrable. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. By Theorem 1.5 for $\left\{\frac{\varepsilon_n}{2}\right\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$. Since f is C-integrable, $f - f\chi_{F_n}$ is also C-integrable, where χ_{F_n} is the characteristic function of F_n. Since $f - f\chi_{F_n} = 0$ on F_n, by Theorem 1.2 there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1} (C)\int_{I_k} (f - f\chi_{F_n})(x)dx\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1}\left((C)\int_{I_k} (f - f\chi_{F_n})(x)dx - (f - f\chi_{F_n})(x_k)|I_k|\right)\right| \\
&\quad < \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0 + 1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.1) and (3.2). Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad \le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n
\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0 + 1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a, b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0 + 1, \ldots, k_1\}$ satisfying (3.1) and (3.2), that is, (3) holds. □

Theorem 2.2. *If the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies* $(\mathrm{A})_C$*, then the pair of f and F satisfies* $(\mathrm{C})_C$*. Similarly, if the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies* $(\mathrm{B})_C$*, then the pair of f and F satisfies* $(\mathrm{D})_C$*.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_C$ hold. By Theorem 1.6 (4) of $(\mathrm{C})_C$ holds. Next we show (3) of $(\mathrm{C})_C$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n, \varepsilon)$ such that, if $|E| < \rho(n, \varepsilon)$, then

$$\left| (L) \int_{E \cap F_n} f(x)dx \right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n, \varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n, \varepsilon) \ge n$, and put $\eta = \rho(m(n, \varepsilon), \varepsilon)$. By (3) of $(\mathrm{A})_C$ for $m(n, \varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a, b]$ satisfying (3.1), (3.2) and (3.3) of $(\mathrm{C})_C$. Then we obtain

$$\left| \sum_{k=1}^{k_0} \left(F(I_k) - (L) \int_{I_k \cap F_{m(n,\varepsilon)}} f(x)dx \right) \right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0} |I_k| < \eta = \rho(m(n, \varepsilon), \varepsilon)$, we obtain

$$\left| \sum_{k=1}^{k_0} (L) \int_{I_k \cap F_{m(n,\varepsilon)}} f(x)dx \right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} \left| \sum_{k=1}^{k_0} F(I_k) \right| &\le \left| \sum_{k=1}^{k_0} \left(F(I_k) - (L) \int_{I_k \cap F_{m(n,\varepsilon)}} f(x)dx \right) \right| + \left| \sum_{k=1}^{k_0} (L) \int_{I_k \cap F_{m(n,\varepsilon)}} f(x)dx \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_C$, then the pair of f and F satisfies $(\mathrm{D})_C$. □

Theorem 2.3. *If the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies* $(\mathrm{D})_C$*, then $f \in \mathbf{(C)}([a, b])$ and*

$$F(I) = (C) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Proof. By (1) and (4) we obtain $F \in \mathbf{ACG}_C([a,b])$. By Theorem 1.4 there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $g \in \mathbf{(C)}([a,b])$ such that $g(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and

$$F(I) = (C)\int_I g(x)dx$$

for any interval $I \subset [a,b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [12, Theorem (5.1)] $g_n \in \mathbf{(D^*)}(I)$ for any interval $I \subset [a,b]$ and by (3)

$$\begin{aligned} (D^*)\int_I g_n(x)dx &= (D^*)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(D^*)\int_{\overline{J_p}} g(x)dx \\ &= (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(C)\int_{\overline{J_p}} g(x)dx \\ &= (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}), \end{aligned}$$

where $\{J_p \mid p = 1,2,\ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*)\int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a,x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a,x]) = f(x)$ for almost every $x \in [a,b]$. □

By Theorems 2.1, 2.2 and 2.3 we obtain the following new criteria for the C-integral.

Theorem 2.4. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is C-integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_C$, $(\mathrm{B})_C$, $(\mathrm{C})_C$ *and* $(\mathrm{D})_C$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_C$, $(\mathrm{B})_C$, $(\mathrm{C})_C$ *and* $(\mathrm{D})_C$, *then*

$$F(I) = (C)\int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

References

[1] B. Bongiorno, *Un nuovo integrale per il problema della primitive*, Le Matematiche **51** (1996), 299–313.

[2] ———, *On the C-integral*, AMS Special Session on Nonabsolute Integration (University of Toronto, Toronto, September, 2000).

[3] B. Bongiorno, L. Di Piazza, and D. Preiss, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc. **62** (2000), 117–126.

[4] A. M. Bruckner, R. J. Freissner, and J. Foran, *The minimal integral which includes Lebesgue integrable functions and derivatives*, Coll. Math. **50** (1986), 289–293.

[5] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., Providence, 1994.

[6] R. Henstock, *The general theory of integration*, Clarendon Press, Oxford, 1991.

[7] S. Nakanishi (formerly S. Enomoto), *Sur une totalisation dans les espaces de plusieurs dimensions, I*, Osaka Math. J. **7** (1955), 59–102.

[8] ______, *Sur une totalisation dans les espaces de plusieurs dimensions, II*, Osaka Math. J. **7** (1955), 157–178.

[9] S. Nakanishi, *The Denjoy integrals defined as the completion of simple functions*, Math. Japon. **37** (1992), 89–101.

[10] ______, *A new definition of the Denjoy's special integral by the method of successive approximation*, Math. Japon. **41** (1995), 217–230.

[11] W. F. Pfeffer, *The Riemann approach to integration*, Cambridge University Press, Oxford, 1993.

[12] S. Saks, *Theory of the integral*, Warsaw, 1937.

Communicated by *Jun Kawabe*

T. Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail: toshiharu.kawasaki@nifty.ne.jp

I. Suzuki
Nihon Bunka University, Tokyo 192–0986, Japan
E-mail: i-suzuki@pd5.so-net.ne.jp