Simplicity of the C^*-algebras of skew product k-graphs

Ben Maloney and David Pask

Received 21 March 2014; Revised 19 May 2014

Abstract. We consider conditions on a k-graph Λ, a semigroup S and a functor $\eta : \Lambda \to S$ that ensure that the C^*-algebra of the skew-product graph $\Lambda \times_\eta S$ is simple. Our results give some necessary and sufficient conditions for the AF-core of a k-graph C^*-algebra to be simple.

1 Introduction In [24] Robertson and Steger investigated C^*-algebras that they considered to be higher-rank versions of the Cuntz-Krieger algebras. Subsequently in [9] Kumjian and Pask introduced higher-rank graphs, or k-graphs, as a graphical means to provide combinatorial models for the Cuntz-Krieger algebras of Robertson and Steger. They showed how to construct a C^*-algebra that is associated to a k-graph. Since then k-graphs and their C^*-algebras have attracted a lot of attention from many authors (see [1, 3–5, 9, 12–14, 17–19, 21, 23]).

Roughly speaking, a k-graph is a category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying a certain factorisation property. A 1-graph is then the path category of a directed graph. Given a functor $\eta : \Lambda \to S$, where S is a semigroup with identity, we may form the skew product k-graph $\Lambda \times_\eta S$. Skew product graphs play an important part in the development of k-graph C^*-algebras. For example [9, Corollary 5.3] shows that $C^*(\Lambda \times_d \mathbb{Z}^k)$ is isomorphic to $C^*(\Lambda) \times_\gamma \mathbb{T}^k$ where $\gamma : \mathbb{T}^k \to \operatorname{Aut} C^*(\Lambda)$ is the canonical gauge action. Skew product graphs feature in nonabelian duality: In [13] it is shown that if a right-reversible semigroup (Ore semigroup) S acts freely on a k-graph Λ then the crossed product $C^*(\Lambda) \times S$ is stably isomorphic to $C^*(\Lambda/S)$. On the other hand if S is a group G then $C^*(\Lambda \times_\eta G)$ is isomorphic to the crossed product $C^*(\Lambda) \times_{\delta_\eta} G$ where δ_η is the coaction of G on $C^*(\Lambda)$ induced by η.

The main purpose of this paper is to investigate necessary and sufficient conditions for the C^*-algebra of a skew product k-graph to be simple. We will be particularly interested in the specific case when $S = \mathbb{N}^k$ and $\eta = d$. It can be shown that simplicity of $C^*(\Lambda \times_d \mathbb{N}^k)$ is equivalent to simplicity of the fixed point algebra (AF core) $C^*(\Lambda)^\gamma$. This is important as many results in

2010 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. C^*-algebra; Graph algebra; k-graph.

the literature apply particularly when AF core is simple; see [8, Proposition 3.8] for example.

We begin by introducing some basic facts we will need during this paper.

2 Background

2.1 Basic facts about k-graphs All semigroups in this paper will be countable, cancellative and have an identity, hence any semigroup may be considered as a category with a single object. The semigroup $\mathbb{N}^k$ is freely generated by $\{e_1, \ldots, e_k\}$ and comes with the usual order structure: if $n = \sum_{i=1}^k n_i e_i$ and $m = \sum_{i=1}^k m_i e_i$ then $m > n$ (resp. $m \geq n$) if $m_i > n_i$ (resp. $m_i \geq n_i$) for all i. For $m, n \in \mathbb{N}^k$ we define $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$ for $i = 1, \ldots, k$.

A directed graph E is a quadruple (E^0, E^1, r, s) where E^0, E^1 are countable sets of vertices and edges. The direction of an edge $e \in E^1$ is given by the maps $r, s : E^1 \to E^0$. A path λ of length $n \geq 1$ is a sequence $\lambda = \lambda_1 \cdots \lambda_n$ of edges such that $s(\lambda_i) = r(\lambda_{i+1})$ for $i = 1, \ldots, n-1$. The set of paths of length $n \geq 1$ is denoted E^n. We may extend r, s to E^n for $n \geq 1$ by $r(\lambda) = r(\lambda_1)$ and $s(\lambda) = s(\lambda_n)$ and to E^0 by $r(v) = v = s(v)$.

A *higher-rank graph* or *k-graph* is a combinatorial structure, and is a k-dimensional analogue of a directed graph. A k-graph consists of a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$, known as the degree map, with the following factorisation property: for every morphism $\lambda \in \Lambda$ and every decomposition $d(\lambda) = m+n$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

For $n \in \mathbb{N}^k$ we define $\Lambda^n := d^{-1}(n)$ to be those morphisms in Λ of degree n. Then by the factorisation property Λ^0 may be identified with the objects of Λ, and are called vertices. For $u, v \in \Lambda^0$ and $X \subseteq \Lambda$ we set

$$uX = \{\lambda \in X : r(\lambda) = u\} \quad Xv = \{\lambda \in X : s(\lambda) = v\} \quad uXv = uX \cap Xv.$$

A k-graph Λ is visualised by a k-coloured directed graph E_Λ with vertices Λ^0 and edges $\sqcup_{i=1}^k \Lambda^{e_i}$ together with range and source maps inherited from Λ called its 1-skeleton. The 1-skeleton is provided with square relations $\mathcal{C}_\Lambda$ between the edges in E_Λ, called *factorisation rules*, which come from factorisations of morphisms in Λ of degree $e_i + e_j$ where $i \neq j$. By convention the edges of degree e_1 are drawn blue (solid) and the edges of degree e_2 are drawn red (dashed). For more details about the 1-skeleton of a k-graph see [21]. On the other hand, if G is a k-coloured directed graph with a complete, associative collection of square relations $\mathcal{C}$ completely determines a k-graph Λ such that $E_\Lambda = G$ and $\mathcal{C}_\Lambda = \mathcal{C}$ (see [6]).

A k-graph Λ is *row-finite* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, $v\Lambda^n$ is finite. A k-graph has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and nonzero $n \in \mathbb{N}^k$. A k-graph has *no sinks* is $\Lambda^n v \neq \emptyset$ for all $v \in \Lambda^0$ and nonzero $n \in \mathbb{N}^k$.

For $\lambda \in \Lambda$ and $m \leq n \leq d(\lambda)$, we define $\lambda(m,n)$ to be the unique path in Λ^{n-m} obtained from the k-graph factorisation property such that $\lambda = \lambda'(\lambda(m,n))\lambda''$ for some $\lambda' \in \Lambda^m$ and $\lambda'' \in \Lambda^{d(\lambda)-n}$.

Examples 2.1. (a) In [9, Example 1.3] it is shown that the path category $E^* = \cup_{i\geq 0} E^i$ of a directed graph E is a 1-graph, and vice versa. For this reason we shall move seamlessly between 1-graphs and directed graphs.

(b) For $k \geq 1$ let T_k be the category with a single object v and generated by k commuting morphisms $\{f_1, \ldots, f_k\}$. Define $d : T_k \to \mathbb{N}^k$ by $d(f_1^{n_1} \ldots f_k^{n_k}) = (n_1, \ldots, n_k)$ then it is straightforward to check that T_k is a k-graph. We frequently identify T_k with $\mathbb{N}^k$ via the map $f_1^{n_1} \cdots f_k^{n_k} \mapsto (n_1, \ldots, n_k)$.

(c) For $k \geq 1$ define a category Δ_k as follows: Let $\operatorname{Mor} \Delta_k = \{(m,n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \leq n\}$ and $\operatorname{Obj} \Delta_k = \mathbb{Z}^k$; structure maps $r(m,n) = m$, $s(m,n) = n$, and composition $(m,n)(n,p) = (m,p)$. Define $d : \Delta_k \to \mathbb{N}^k$ by $d(m,n) = n - m$, then one checks that (Δ_k, d) is a row-finite k-graph. We identify $\operatorname{Obj} \Delta_k$ with $\{(m,m) : m \in \mathbb{Z}^k\} \subset \operatorname{Mor} \Delta_k$.

(d) For $n \geq 1$ let $\underline{n} = \{1, \ldots, n\}$. For $m, n \geq 1$ let $\theta : \underline{m} \times \underline{n} \to \underline{m} \times \underline{n}$ a bijection. Let $\mathbb{F}^2_\theta$ be the 2-graph which has 1-skeleton which consists of with single vertex v and edges $f_1, \ldots, f_m, g_1, \ldots, g_n$, such that f_i have the same colour (blue) for $i \in \underline{m}$ and g_j have the same colour (red) for $j \in \underline{n}$ together with complete associative square relations $f_i g_j = g_{j'} f_{i'}$ where $\theta(i,j) = (i', j')$ for $(i,j) \in \underline{m} \times \underline{n}$ (for more details see [3, 4, 19]).

2.2 Skew product k-graphs Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor into a semigroup S. We can make the cartesian product $\Lambda \times S$ into a k-graph $\Lambda \times_\eta S$ by taking $(\Lambda \times_\eta S)^0 = \Lambda^0 \times S$, defining $r, s : \Lambda \times_\eta S \to (\Lambda \times_\eta S)^0$ by

$$r(\lambda, t) = (r(\lambda), t) \quad \text{and} \quad s(\lambda, t) = (s(\lambda), t\eta(\lambda)), \tag{1}$$

defining the composition by

$$(\lambda, t)(\mu, u) = (\lambda\mu, t) \quad \text{when } s(\lambda, t) = r(\mu, u) \text{ (so that } u = t\eta(\lambda)\text{)},$$

and defining $d : \Lambda \times_\eta S \to \mathbb{N}^k$ by $d(\lambda, t) = d(\lambda)$. As in [13] it is straightforward to show that this defines a k-graph.

Remark 2.2. If Λ is row-finite with no sources and $\eta : \Lambda \to S$ a functor then $\Lambda \times_\eta S$ is row-finite with no sources.

A k-graph morphism is a degree preserving functor between two k-graphs. If a k-graph morphism is bijective, then it is called an isomorphism.

Examples 2.3. (i) Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then the map $\pi : \Lambda \times_\eta S \to \Lambda$ given by $\pi(\lambda, s) = \lambda$ is a surjective k-graph morphism.

(ii) For $\ell \geq 1$ the map $(\ell, m) \mapsto (m, \ell + m)$ gives an isomorphism from $T_k \times_d \mathbb{Z}^k$ to Δ_k.

Definition 2.4. Let Λ, Γ be row-finite k-graphs. A surjective k-graph morphism $p : \Lambda \to \Gamma$ has *r-path lifting* if for all $v \in \Lambda^0$ and $\lambda \in p(v)\Gamma$ there is $\lambda' \in v\Lambda$ such that $p(\lambda') = \lambda$. If λ' is the unique element with this property then p has *unique r-path lifting*.

Example 2.5. Let Λ be a row-finite k-graph and $\eta : \Lambda \to S$ a functor where S is a semigroup, and $\Lambda \times_\eta S$ the associated skew product graph. The map $\pi : \Lambda \times_\eta S \to \Lambda$ described in Examples 2.3(i) has unique r-path lifting.

2.3 Connectivity A k-graph Λ is *connected* if the equivalence relation on Λ^0 generated by the relation $\{(u, v) : u\Lambda v \neq \emptyset\}$ is $\Lambda^0 \times \Lambda^0$. The k-graph Λ is *strongly connected* if for all $u, v \in \Lambda^0$ there is $N > 0$ such that $u\Lambda^N v \neq \emptyset$. If Λ is strongly connected, then it is connected and has no sinks or sources. The k-graph Λ is *primitive* if there is $N > 0$ such that $u\Lambda^N v \neq \emptyset$ for all $u, v \in \Lambda^0$. If Λ is primitive then it is strongly connected.

Examples 2.6. The graphs T_k and $\mathbb{F}^2_\theta$ in Examples 2.1 are primitive as they have one vertex.

The connectivity of a k-graph may also be described in terms of its component matrices as defined in [9, §6]: Given a k-graph Λ, for $1 \leq i \leq k$ and $u, v \in \Lambda^0$, we define k non-negative $\Lambda^0 \times \Lambda^0$ matrices M_i with entries $M_i(u, v) = |u\Lambda^{e_i} v|$. Using the k-graph factorisation property, we have that $|u\Lambda^{e_i+e_j} v| = |u\Lambda^{e_j+e_i} v|$ for all $u, v \in \Lambda^0$, and so $M_i M_j = M_j M_i$. For $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$ and $u, v \in \Lambda^0$, we have $|u\Lambda^m v| = (M_1^{m_1} \cdots M_k^{m_k})(u, v) = M^m(u, v)$, using multiindex notation. The following lemma follows directly from the above definitions.

Lemma 2.7. *Let Λ be a row-finite k-graph with no sources.*

(a) Then Λ is strongly connected if and only if for all pairs $u, v \in \Lambda^0$ there is $N \in \mathbb{N}^k$ such that $M^N(u, v) > 0$.

(b) Then Λ is primitive if and only if there is $N > 0$ such that $M^N(u, v) > 0$ for all pairs $u, v \in \Lambda^0$.

Remarks 2.8. Following [18, §4], a primitive 1-graph Λ is strongly connected with period 1; that is, the greatest common divisor of all n such that $v\Lambda^n v \neq \emptyset$ for some $v \in \Lambda^0$ is 1.

Lemma 2.9. *Let Λ be a k-graph with no sinks, and Λ^0 finite. Then for all $v \in \Lambda^0$, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$.*

Proof. Let $p = (1, \ldots, 1) \in \mathbb{N}^k$. Since v is not a sink, there exists $\beta_1 \in \Lambda^p v$. Since $r(\beta_1)$ is not a sink, there exists $\beta_2 \in \Lambda^p r(\beta_1)$. Inductively, there exist infinitely many β_i such that $d(\beta_i) = p$ and $r(\beta_i) = s(\beta_{i+1})$. Since Λ^0 is finite, there exists $w \in \Lambda^0$ such that $r(\beta_i) = w$ for infinitely many i. Suppose $r(\beta_n) = w = r(\beta_m)$ with $m > n$. Then $\alpha = \beta_m \ldots \beta_{n+1}$ has the requisite properties, and $w\Lambda v \neq \emptyset$, since $\beta_n \ldots \beta_1 \in w\Lambda v$. □

2.4 The graph C^*-algebra Let Λ be a row-finite k-graph with no sources, then following [9], a *Cuntz-Krieger Λ-family* in a C^*-algebra B consists of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ in B satisfying the *Cuntz-Krieger relations*:

(CK1) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;

(CK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$;

(CK4) $S_v = \sum_{\{\lambda \in v\Lambda^n\}} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The k-graph C^*-algebra $C^*(\Lambda)$ is generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}$. By [9, Proposition 2.11] there exists a Cuntz-Krieger Λ-family such that each vertex projection S_v (and hence by (CK3) each S_λ) is nonzero and so there exists a nonzero universal k-graph C^*-algebra for a Cuntz-Krieger Λ-family. Moreover,

$$C^*(\Lambda) = \overline{\text{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\} \quad \text{(see [9, Lemma 3.1])}.$$

We will use [23, Theorem 3.1] by Robertson and Sims when considering the simplicity of graph C^*-algebras:

Theorem 2.10 (Robertson-Sims)**.** *Suppose Λ is a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ is cofinal and aperiodic.*

We now focus on the two key properties involved in the simplicity criterion of Theorem 2.10, namely aperiodicity and cofinality. Our attention will be directed towards applying these conditions on skew product graphs.

3 Aperiodicity Our definition of aperiodicity is taken from Robertson-Sims, [23, Theorem 3.2].

Definitions 3.1. A row-finite k-graph Λ with no sources has *no local periodicity* at $v \in \Lambda^0$ if for all $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Λ is called *aperiodic* if every $v \in \Lambda^0$ has no local periodicity.

Examples 3.2. (a) The k-graph Δ_k is aperiodic for all $k \geq 1$. First observe that there is no local periodicity at $v = (0, 0)$. Given $m \neq n \in \mathbb{N}^k$, let $N \geq m \vee n$; then $\lambda = (0, N)$ is the only element of $v\Delta_k$. Then $\lambda(m, m) = (m, m) \neq (n, n) = \lambda(n, n)$. A similar argument applies for any other vertex $w = (n, n)$ in Δ_k and so there is no local periodicity at w for all $w \in \Delta_k^0$.

(b) The k-graph T_k is not aperiodic for all $k \geq 1$. For all $n \in \mathbb{N}^k$ one checks that $f_1^{n_1} \cdots f_k^{n_k}$ is the only element of vT_k^n. Hence given $m \neq n \in \mathbb{N}^k$ it follows that for all $\lambda \in v\Lambda^N$ with $N \geq m \vee n$ we have

$$\lambda(m, m + (m \vee n)) = \lambda(n, n + (m \vee n)).$$

Since the map $\pi : \Lambda \times_\eta S \to \Lambda$ has unique r-path lifting, we wish to know if we can deduce the aperiodicity of $\Lambda \times_\eta S$ from that of Λ. A corollary of our main result Theorem 3.3, shows that this is true.

Theorem 3.3. *Let Λ, Γ be row-finite k-graphs with no sources and $p : \Lambda \to \Gamma$ have r-path lifting. If Γ is aperiodic, then Λ is aperiodic.*

Proof. Suppose that Γ is aperiodic. Let $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$. Since Γ is aperiodic, there exists $\lambda \in p(v)\Gamma$ with $d(\lambda) \geq m \vee n$ such that $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. By r-path lifting there is $\lambda' \in v\Lambda$ with $p(\lambda') = \lambda$ such that $d(\lambda') \geq m \vee n$ and

$$\lambda'(m, m + d(\lambda) - (m \vee n)) \neq \lambda'(n, n + d(\lambda) - (m \vee n)),$$

and so Λ is aperiodic. □

The converse of Theorem 3.3 is false:

Example 3.4. The surjective k-graph morphism $p : \Delta_k \to T_k$ given by $p(m, m + e_i) = f_i$ for all $m \in \mathbb{Z}^k$ and $i = 1, \ldots, k$ has r-path lifting. However by Examples 3.2 we see that $\Delta_k \cong T_k \times_d \mathbb{Z}^k$ is aperiodic but T_k is not.

Corollary 3.5. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. If Λ is aperiodic then $\Lambda \times_\eta S$ is aperiodic.*

Proof. Follows from Theorem 3.3 and Example 2.5. □

In some cases the aperiodicity of a skew product graph $\Lambda \times_\eta S$ can be deduced directly from properties of η.

Proposition 3.6. *Suppose S is a semigroup, Λ is a row-finite k-graph with no sources, $\eta : \Lambda \to S$ is a functor, and there exists a map $\phi : S \to \mathbb{Z}^k$ such that $d = \phi \circ \eta$. Then $\Lambda \times_\eta S$ is aperiodic.*

Proof. Fix $(v, s) \in (\Lambda \times_\eta S)^0$ and $m \neq n \in \mathbb{N}^k$. Let $\lambda \in (v, s)(\Lambda \times_\eta S)$ be such that $d(\lambda) \geq m \vee n$. Observe that $\lambda(m, m) = s(\lambda(0, m))$, $\lambda(m, m)$ is of the form $(w, s\eta(\lambda(0, m)))$ for some $w \in \Lambda^0$. Similarly, $\lambda(n, n)$ is of the form $(w', s\eta(\lambda(0, n)))$ for some $w' \in \Lambda^0$.

We claim $\lambda(m, m) \neq \lambda(n, n)$: Suppose, by hypothesis, $\eta(\lambda(0, n)) = \eta(\lambda(0, m))$. Then $n = \phi \circ \eta(\lambda(0, n)) = \phi \circ \eta(\lambda(0, m)) = m$, which provides a contradiction, and $m \neq n$. Then $\eta(\lambda(0, m)) \neq \eta(\lambda(0, n))$, and so $\lambda(m, m) \neq \lambda(n, n)$, and hence $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. □

Corollary 3.7. *Suppose Λ is a row-finite k-graph with no sources. Then $\Lambda \times_d \mathbb{N}^k$ and $\Lambda \times_d \mathbb{Z}^k$ are aperiodic.*

Proof. Apply Proposition 3.6 with $\eta = d$ and $S = \mathbb{N}^k, \mathbb{Z}^k$ respectively. □

4 Cofinality We will use the Lewin-Sims definition of cofinality, [12, Remark A.3]. By [12, Appendix A] this definition is equivalent to the other standard definitions of cofinality:

Definition 4.1. A row-finite, k-graph Λ with no sources is *cofinal* if for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$.

Lemma 4.2. *Let Λ be a row-finite k-graph with no sources.*

(a) If Λ is cofinal then Λ is connected.

(b) Suppose that for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$. Then for $n \geq N$ we have $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^n$.

Proof. Fix $v, w \in \Lambda^0$. If Λ is cofinal it follows that there is $\alpha \in w\Lambda$ such that $w\Lambda s(\alpha)$ and $v\Lambda s(\alpha)$ are non-empty. It then follows that (v, w) belongs to the equivalence relation described in Section 2.3. Since v, w were arbitrary it follows that the equivalence relation is $\Lambda^0 \times \Lambda^0$ and so Λ is connected.

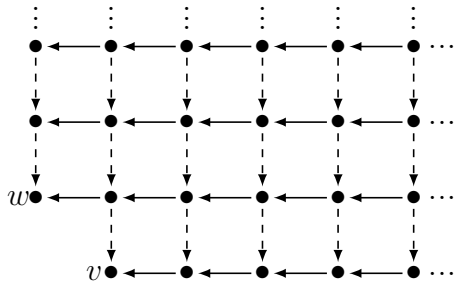
Fix $v, w \in \Lambda^0$, then there is $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$. Let $n \geq N$ and consider $\beta \in w\Lambda^n$ then $\beta' = \beta(0, N) \in w\Lambda^N$ and so by hypothesis there is $\lambda \in v\Lambda s(\beta')$. Then $\lambda\beta(N, n) \in v\Lambda s(\beta)$ and the result follows. □

Lemma 4.3. *Let Λ be a row-finite k-graph with no sources with skeleton E_Λ. If E_Λ is cofinal then Λ is cofinal. Furthermore, Λ is strongly connected if and only if E_Λ strongly connected*

Proof. Fix $v, w \in \Lambda^0 = E^0_\Lambda$. As E_Λ is cofinal there is $n \in \mathbb{N}$ with $vE_\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in wE^n_\Lambda$. Let $N \in \mathbb{N}^k$ be such that $\sum_{i=1}^k N_i = n$. Then for all $\alpha' \in w\Lambda^N$ we have $\alpha' \in E^n_\Lambda$ and so $v\Lambda^N s(\alpha') \neq \emptyset$.

Suppose that Λ is strongly connected and $v, w \in E^0_\Lambda = \Lambda^0$. As Λ is strongly connected there is $\alpha \in v\Lambda w$ with $d(\alpha) > 0$. Let $n = \sum_{i=1}^n d(\alpha)_i$ then $n > 0$ and $vE_\Lambda w \neq \emptyset$, so E_Λ is strongly connected. Suppose that E_Λ is strongly connected, and $v, w \in \Lambda^0 = E^0_\Lambda$. As Λ has no sources, there is $\alpha \in vE^k_\Lambda$ which uses an edge of each of the k-colours. Let $u = s(\alpha)$. Since E_Λ is strongly connected there is $\beta \in uE^n_\Lambda w$. Let λ be the element of Λ which may be represented by $\alpha\beta \in E_\Lambda$. Then $\lambda \in v\Lambda w$ and $d(\lambda) > 0$ and so Λ is strongly connected. □

Remark 4.4. The converse to the first part of Lemma 4.3 is not true: Let Λ be the 2-graph which is completely determined by its 1-skeleton as shown:



Then Λ is cofinal: For example for v, w as shown, $N = (1, 0)$ will suffice. However E_Λ is not cofinal: For example for v, w as shown, for any $n \geq 0$ the vertex which is the source of the vertical path of length n with range w does not connect to v.

The following result establishes a link between cofinality and strongly connectivity for a row-finite k-graph.

Proposition 4.5. *Suppose Λ is a row-finite k-graph with no sources.*

1. *If Λ is strongly connected then Λ is cofinal.*

2. *If Λ is cofinal, has no sinks and Λ^0 finite then Λ is strongly connected.*

Proof. Suppose Λ is strongly connected. Fix $v, w \in \Lambda^0$ then for $N = e_1$ we have $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in w\Lambda^N$ since Λ is strongly connected, and so Λ is cofinal.

Suppose Λ is cofinal. Fix $u, v \in \Lambda^0$. Then by Lemma 2.9, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$. Let $\alpha' \in w\Lambda v$. Given $u, w \in \Lambda^0$, since Λ is cofinal and has no sources, by Lemma 4.2(ii) there exists $N \in \mathbb{N}^k$ such that for all $n \geq N$ and all $\alpha'' \in w\Lambda^n$, there exists $\beta \in u\Lambda s(\alpha'')$. Since $d(\alpha) > 0$ we may choose $t \in \mathbb{N}$ such that $td(\alpha) > N$. Then $\alpha^t \in w\Lambda^n$ where $n > N$, and so by cofinality of Λ exists $\beta \in u\Lambda s(\alpha^t) = u\Lambda w$. Hence $\beta\alpha\alpha' \in u\Lambda v$ with $d(\beta\alpha\alpha') > d(\alpha) > 0$ and so Λ is strongly connected. □

Example 4.6. The condition that Λ^0 is finite in Proposition 4.5(2) is essential: For instance Δ_k is cofinal by Lemma 4.3 since its skeleton is cofinal; however it is not strongly connected by Lemma 4.3 since its skeleton is not strongly connected.

Since the map $\pi : \Lambda \times_\eta S \to \Lambda$ has unique r-path lifting, we wish to know if we can deduce the cofinality of $\Lambda \times_\eta S$ from that of Λ. By Theorem 4.7 the image of a cofinal k-graph under a map with r-path lifting is cofinal, however Example 4.9 shows that the converse is not true. For a cofinal k-graph Λ, we must then seek additional conditions on the functor η which guarantees that $\Lambda \times_\eta S$ is cofinal. In Definition 4.10 we introduce the notion of (Λ, S, η) cofinality to address this problem.

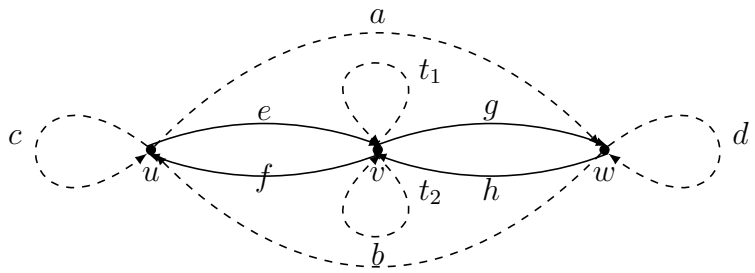
Theorem 4.7. *Suppose Λ, Γ be row-finite k-graphs with no sources and $p : \Lambda \to \Gamma$ have r-path lifting. If Λ is cofinal then Γ is cofinal.*

Proof. Suppose that Λ is cofinal. Fix $v, w \in \Gamma^0$. Let $v', w' \in \Lambda^0$ be such that $p(v') = v$ and $p(w') = w$. As Λ is cofinal there is an N such that for all $\alpha' \in w'\Lambda^N$ there is $\beta' \in v'\Lambda s(\alpha')$. Then for $\alpha \in v\Gamma^N$ there is $\alpha' \in v'\Lambda^N$ with $p(\alpha') = \alpha$. By hypothesis there is $\beta' \in v'\Lambda s(\alpha')$, and so $\beta = p(\beta')$ satisfies $s(\beta) = s(\alpha)$ and $r(\beta) = v$, hence $v\Lambda s(\alpha) \neq \emptyset$ as required. □

Corollary 4.8. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. If $\Lambda \times_\eta S$ is cofinal then Λ is cofinal.*

The converse of Theorem 4.7 is false:

Example 4.9. Consider the following 2-graph Λ with 1-skeleton



and factorisation rules: $ec = t_1e$ and $ha = t_2e$ for paths from u to v; $cf = ft_1$ and $bg = ft_2$ for paths from v to u. Also $hd = t_1h$ and $eb = t_2h$ for paths from w to v; $dg = gt_1$ and $af = gt_2$ for paths from v to w. By Lemma 4.3 Λ is strongly connected as its skeleton is strongly connected. Note there are no paths of degree $e_1 + e_2$ from a vertex to itself.

Since $M_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$, we calculate that $M^{(2j_1,j_2)} = 2^{j_1+j_2-1}M_2$ and $M^{(2j_1+1,j_2)} = 2^{j_1+j_2+1}M_1$. Hence $M^{(2j-1,2j-1)} = \left(\begin{smallmatrix} 0 & 4j & 0 \\ 4j & 0 & 4j \\ 0 & 4j & 0 \end{smallmatrix}\right)$ and $M^{(2j,2j)} = \left(\begin{smallmatrix} 4j & 0 & 4j \\ 0 & 8j & 0 \\ 4j & 0 & 4j \end{smallmatrix}\right)$. In particular by Lemma 2.7 (b) Λ is not primitive, even though it is strongly connected.

We claim that the skew product graph $\Lambda \times_d \mathbb{Z}^2$ is not cofinal. Consider $v_1 = (v, (m, n))$ and $v_2 = (v, (m + 1, n))$ in $(\Lambda \times_d \mathbb{Z}^2)^0$. We claim that for all $N \in \mathbb{N}^2$, for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, we have $v_2(\Lambda \times_d \mathbb{Z}^2)s(\alpha) \neq \emptyset$. Let $N = (N_1, N_2)$. Suppose N_1 is even. Then for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, $s(\alpha) = (v, (m + N_1, n + N_2))$. In order for this vertex to connect to $(v, (m+1, n))$, we have $M^{(N_1-1,N_2)}(v, v) \neq 0$. But $N_1 - 1$ is odd, and this matrix entry is zero. If N_1 is odd, then $s(\alpha) = (u, (m + N_1, n + N_2))$ or $s(\alpha) = (w, (m + N_1, n + N_2))$. In order for either of these vertices to connect to $(v, (m+1, n))$, we must have $M^{(N_1-1,N_2)}(u, v) \neq 0$, or $M^{(N_1-1,N_2)}(w, v) \neq 0$. But $N_1 - 1$ is even, and so both of these matrix entries are zero. Hence $\Lambda \times_d \mathbb{Z}^2$ is not cofinal, even though Λ is cofinal.

To establish a sufficient condition for $\Lambda \times_\eta S$ to be cofinal, we need Λ to be cofinal and an additional condition on η.

Definition 4.10. Let Λ be a row-finite k-graph with no sources and $\eta : \Lambda \to S$ a functor, where S is a semigroup. The system (Λ, S, η) is *cofinal* if for all $v, w \in \Lambda^0$, $a, b \in S$, there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ such that $a\eta(\beta) = b\eta(\alpha)$.

Proposition 4.11. *Let Λ be a row-finite k-graph with no sources and $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then the system (Λ, S, η) is cofinal if and only if $\Lambda \times_\eta S$ is cofinal.*

Proof. Suppose $\Lambda \times_\eta S$ is cofinal. Fix $a, b \in S$ and $v, w \in \Lambda^0$. By hypothesis there is $N \in \mathbb{N}^k$ such that $(v,a)(\Lambda \times_\eta S)s(\alpha, b)$ is non-empty for every $(\alpha, b) \in (w,b)(\Lambda \times_\eta S)^N$. In particular for all $\alpha \in w\Lambda^N$ there exists $\beta \in w\Lambda^N$ such that $a\eta(\beta) = b\eta(\alpha)$, and so (Λ, S, η) is cofinal.

Now suppose (Λ, S, η) is cofinal. Fix $(v,a), (w,b) \in (\Lambda \times_\eta S)^0$. By hypothesis there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ with $a\eta(\beta) = b\eta(\alpha)$. In particular for all $(\alpha, b) \in (w,b)(\Lambda \times_\eta S)^N$ there is $(\beta, a) \in (v,a)\Lambda s(\alpha, b)$, and so $\Lambda \times_\eta S$ is cofinal. □

Theorem 4.12. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if the system (Λ, S, η) is cofinal.*

Proof. If the system (Λ, S, η) is cofinal, then by Proposition 4.11, $\Lambda \times_\eta S$ is cofinal. By Corollary 3.5, $\Lambda \times_\eta S$ is aperiodic and so by [23, Theorem 3.1], $C^*(\Lambda \times_\eta S)$ is simple.

Now suppose that $C^*(\Lambda \times_\eta S)$ is simple. Then by [23, Theorem 3.1], $\Lambda \times_\eta S$ is cofinal. By Proposition 4.11 this implies that (Λ, S, η) is cofinal. □

The condition of (Λ, S, η) cofinality is difficult to check in practice. For 1-graphs it was shown in [18, Proposition 5.13] that $\Lambda \times_d \mathbb{Z}^k$ is cofinal if Λ is primitive[1]. We seek an equivalent condition for k-graphs which guarantees (Λ, S, η) cofinality.

5 Primitivity and left-reversible semigroups A semigroup S is said to be *left-reversible* if for all $s, t \in S$ we have $sS \cap tS \neq \emptyset$. It is more common to work with right-reversible semigroups, which are then called Ore semigroups (see [13]). In analogy with the results of Dubriel it can be shown that a left-reversible semigroup has an enveloping group Γ such that $\Gamma = SS^{-1}$.

In equation (1) we see that functor $\eta : \Lambda \to S$ multiplies on the right in the semigroup coordinate in the definition of the source map in a skew product graph $\Lambda \times_\eta S$. This forces us to consider left-reversible semigroups here. In order to avoid confusion we have decided not to call them Ore.

Examples 5.1. (i) Any abelian semigroup is automatically right- and left-reversible. Moreover, any group is a both a right- and left-reversible semigroup.

(ii) Let $\mathbb{N}$ denote the semigroup of natural numbers under addition and $\mathbb{N}^\times$ denote the semigroup of nonzero natural numbers under multiplication. Let $S = \mathbb{N} \times \mathbb{N}^\times$ be gifted with the associative binary operation $\star$ given

[1] Actually strongly connected with period 1 which is equivalent to primitive

by $(m_1, n_1) \star (m_2, n_2) = (m_1 n_2 + m_2, n_1 n_2)$, then one checks that S is a nonabelian left-reversible semigroup. It is not right-reversible; for example, $S(m, n) \cap S(p, q) = \emptyset$ when $n = q = 0$ and $m \neq p$.

(iii) The free semigroup $\mathbb{F}_n^+$ on $n \geq 2$ generators is not an left-reversible semigroup since for all $s, t \in \mathbb{F}_n^+$ with $s \neq t$ we have $s\mathbb{F}_n^+ \cap t\mathbb{F}_n^+ = \emptyset$ as there is no cancellation, and so not only the left-reversibility but also the right-reversibility conditions cannot be satisfied.

A *preorder* is a reflexive, transitive relation $\leq$ on a set X. A preordered set $(X, \leq)$ is *directed* if the following condition holds: for every $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$. A subset Y of X is *cofinal* if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. We say that sets $X \leq Y$ if $x \leq y$ for all $x \in X$ and for all $y \in Y$. We say that $t \in S$ is *strictly positive* if $\{t^n : n \geq 0\}$ is a cofinal set in S.

The following result appears as [15, Lemma 2.2] for right-reversible semigroups.

Lemma 5.2. *Let S be a left-reversible semigroup with enveloping group Γ, and define $\geq_l$ on Γ by $h \geq_l g$ if and only if $g^{-1}h \in S$. Then $\geq_l$ is a left-invariant preorder that directs Γ, and for any $t \in S$, tS is cofinal in S.*

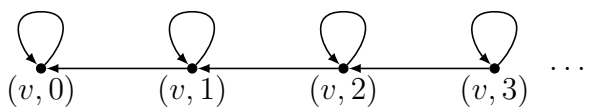
Our first attempt at a condition on η which guarantees cofinality of (Λ, S, η) is one which ensures that η takes arbitrarily large values on paths which terminate a given vertex.

Definition 5.3. Let Λ be a k-graph with no sources and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. We will say that η is *upper dense* if for all $w \in \Lambda^0$ and $a, b \in S$ there exists $N \in \mathbb{N}^k$ such that $b\eta(w\Lambda^N) \geq_l a$.

Lemma 5.4. *Let (Λ, d) be a row-finite k-graph with no sources then d is upper dense for Λ.*

Proof. Since Λ has no sources it is immediate that $w\Lambda^N \neq \emptyset$ for all $w \in \Lambda^0$ and $N \in \mathbb{N}^k$. For any $b, a \in \mathbb{N}^k$ we have $b + d(w\Lambda^N) = b + N \geq a$ provided $N \geq a$. □

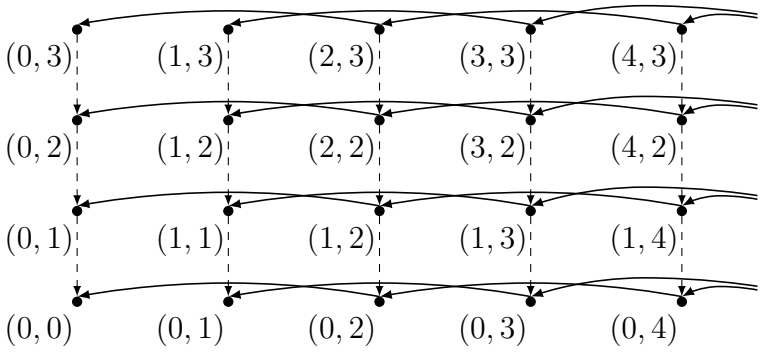
Examples 5.5. (i) Let B_2 be the 1-graph which is the path category of the directed graph with a single vertex v and two edges e, f. Define a functor $\eta : B_2 \to \mathbb{N}$ by $\eta(e) = 1$ and $\eta(f) = 0$. We may form the skew product $B_2 \times_\eta \mathbb{N}$ with 1-skeleton:



Fix $a, b \in \mathbb{N}$, then since $n \in \eta(vB_2^n)$ for all $n \in \mathbb{N}$ it follows that if we choose $N = a$, then $b + \eta(vB_2^N) \geq a$ and so η is upper dense. However $(B_2, \mathbb{N}, \eta)$ is not cofinal: Choose $a = 1$, $b = 0$, then for all $N \geq 0$ there is $f^N \in vB_2^N$ is such that

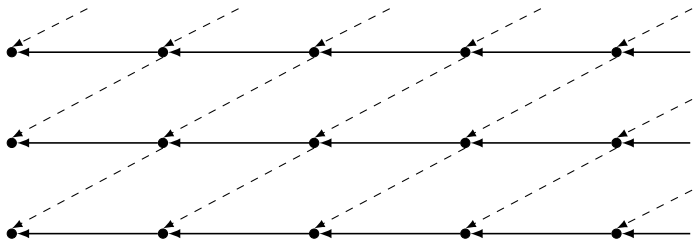
$$b + \eta(f^N) = 0 \neq 1 + \eta(\beta) \text{ for all } \beta \in B_2 v.$$

(ii) Define a functor η from T_2 to $\mathbb{N}^2$ such that $\eta(f_1) = (2, 0)$, and $\eta(f_2) = (0, 1)$. We may form the skew product $T_2 \times_\eta \mathbb{N}^2$ with the following 1-skeleton:



We claim that the functor η is not upper dense: Fix $b = (b_1, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$. Let N_1 be such that $b_1 + 2N_1 \geq a_1$ and N_2 be such that $b_2 + N_2 \geq a_2$ then $b\eta(vT_2^N) \geq_l u$ where $N = (N_1, N_2)$. Moreover $(T_2, \mathbb{N}^2, \eta)$ is not cofinal: Let $b = (0, 0)$ and $a = (1, 0)$ then since $\eta(f_1^{N_1} f_2^{N_2}) = (2N_1, N_2)$ it follows that there cannot be $N = (N_1, N_2) \in \mathbb{N}^2$ such that for $\alpha \in vT_2^N$ there is $\beta \in vT_2 v$ with $b\eta(\alpha) = a\eta(\beta)$.

(iii) Taking T_2 again, we define a functor $\eta : T_2 \to \mathbb{N}^2$ by $\eta(f_1) = (1, 0)$ and $\eta(f_2) = (1, 1)$. The skew product graph has 1-skeleton:



We claim that η is upper dense: Fix $b = (b_2, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$ then there is N_1 such that $b_1 + N_1 \geq a_1$ and N_2 such that $b_2 + N_1 + N_2 \geq a_2$. Then with $N = (N_1, N_2)$ for all $\alpha \in vT_2^N$ we have $b\eta(\alpha) \geq_l a$. In this case $(T_2, \mathbb{N}^2, \eta)$ is cofinal: Fix $b = (b_1, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$. Then there

is N_1 such that $b_1 + N_1 = a_1 + m_1$ for some $m_1 \in \mathbb{N}$ and N_2 such that $b_2 + N_1 + N_2 = a_2 + m_2$ for some $m_2 \in \mathbb{N}$. Hence for all $\alpha \in vT_2^N$ where $N = (N_1, N_2)$ there is $\beta = (f_1^{m_1}, f_2^{m_2}) \in vT_2v$ such that $b\eta(\alpha) = a\eta(\beta)$.

The last two examples show that η being upper dense is not sufficient to guarantee cofinality of (Λ, S, η). The following definition allows for the interaction of the values of η at different vertices of Λ and the following result gives us the required extra condition.

Definition 5.6. Let Λ be a k-graph and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. We say that η is *S-primitive* for Λ if there is a strictly positive $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all $s \in S$ such that $s \geq_l t$.

Remarks 5.7. (i) The condition that t is strictly positive in the above definition guarantees that $\eta(v\Lambda w)$ is cofinal in S for all $v, w \in \Lambda^0$.

(ii) If $\eta : \Lambda \to S$ is S-primitive for Λ where S is a left-reversible semigroup, then if we extend η to $\Gamma = SS^{-1}$ then η is Γ-primitive for Λ.

Examples 5.8. (i) Let Λ be a k-graph. Then the degree functor $d : \Lambda \to \mathbb{N}^k$ is $\mathbb{N}^k$–primitive for Λ if and only if Λ is primitive as defined in Section 2.3. For this reason we will say that Λ is primitive if d is $\mathbb{N}^k$ primitive for Λ.

(ii) As in Examples 5.5 (i) let $\eta : B_2 \to \mathbb{N}$ be defined by $\eta(e) = 1$, $\eta(f) = 0$. Then the functor η is $\mathbb{N}$-primitive since $\eta^{-1}(n)$ is nonempty for all $n \in \mathbb{N}$. Hence $\mathbb{N}$-primitivity does not, by itself, guarantee cofinality.

(iii) As in Examples 5.5 (ii) let η be the functor from T_2 to $\mathbb{N}^2$ such that $\eta(f_1) = (2, 0)$, and $\eta(f_2) = (0, 1)$. Then the functor η is not $\mathbb{N}^2$-primitive for T_2: Take $t = (2m, n) \geq 0$ then if $s = (2m+1, n)$ we have $v\eta^{-1}(s)v = \emptyset$ and $s \geq_l t$. Similarly if $t = (2m + 1) \geq 0$ then if $s = (2m + 2, n)$ we have $v\eta^{-1}(s)v = \emptyset$ and $s \geq_l t$.

(iv) As in Examples 5.5 (iii) let $\eta : T_2 \to \mathbb{N}^2$ be defined by $\eta(f_1) = (1, 0)$ and $\eta(f_2) = (0, 1)$. Then η is not $\mathbb{N}^2$-primitive for T_2 as $v\eta^{-1}(m, n)v = \emptyset$ whenever $n > m$.

The last two examples above illustrate that upper density and primitivity are unrelated conditions on a k-graph. Together they provide a necessary condition for cofinality.

Proposition 5.9. *Let Λ be a k-graph with no sources and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. If (Λ, S, η) is cofinal then η is upper dense. If η is S-primitive for Λ and upper dense then (Λ, S, η) is cofinal.*

Proof. Suppose that (Λ, S, η) is cofinal. Fix $w \in \Lambda^0$ and $a, b \in S$ and let v be any vertex of Λ. By cofinality of (Λ, S, η) there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$ there is $\beta \in v\Lambda s(\alpha)$ such that $a\eta(\beta) = b\eta(\alpha)$. Then any element of $b\eta(w\Lambda^N)$ is of the form

$$b\eta(\alpha) = a\eta(\beta) \geq_l a.$$

Suppose η is S-primitive and upper dense for Λ. Since η is S–primitive for Λ there exists $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all $s \geq_l t$. Fix $v, w \in \Lambda^0$ and $a, b \in S$. Since η is upper dense there exists $N \in \mathbb{N}^k$ such that $b\eta(\alpha) \geq_l at$ for all $\alpha \in w\Lambda^N$. Since S is left-reversible, it is directed, and so by definition $b\eta(\alpha) = atu$ for some $u \in S$. But $tu \geq_l t$ and so since η is S–primitive there exists $\beta \in v\Lambda s(\alpha)$ such that $\eta(\beta) = tu$ and hence $b\eta(\alpha) = a\eta(\beta)$. □

Corollary 5.10. *Let Λ be a row-finite k-graph such that d is $\mathbb{N}^k$ primitive for Λ then $(\Lambda, \mathbb{N}^k, d)$ is cofinal.*

Proof. Since d is $\mathbb{N}^k$ primitive for Λ it follows that Λ has no sources. The result then follows from Lemma 5.4 and Proposition 5.9. □

Example 5.11. Let $\eta : T_2 \to S$ be any functor, then $\eta(S)$ is a subsemigroup of S since T_2 has a single vertex; moreover η is $\eta(S)$–primitive for T_2. Hence if η is upper dense for T_2, it follows that $(T_2, \eta(S), \eta)$ is cofinal. In particular, in Example5.5 (ii) one checks that $(T_2, \eta(\mathbb{N}^2, \eta)$ is cofinal.

Theorem 5.12. *Let Λ be an aperiodic k-graph, $\eta : \Lambda \to S$ be a functor into a left-reversible semigroup, and η be S–primitive for Λ. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if η is upper dense.*

Proof. If η is upper dense then the result follows from Proposition 5.9. On the other hand if $C^*(\Lambda \times_\eta S)$ is simple then the result follows from Theorem 4.12 and Corollary 3.5. □

6 Skew products by a group Let Λ be a row-finite k-graph. A functor $\eta : \Lambda \to G$ defines a coaction δ_η on $C^*(\Lambda)$ determined by $\delta_\eta(s_\lambda) = s_\lambda \otimes \eta(\lambda)$. It is shown in [14, Theorem 7.1] that $C^*(\Lambda \times_\eta G)$ is isomorphic to $C^*(\Lambda) \times_{\delta_\eta} G$. Hence we may relate the simplicity of the C^*-algebra of a skew product graph to the simplicity of the associated crossed product. This can be done by using the results of [20].

Following [14, Lemma 7.9], for $g \in G$ the spectral subspace $C^*(\Lambda)_g$ of the coaction δ_η is given by

$$C^*(\Lambda)_g = \overline{\text{span}}\{s_\lambda s_\mu^* : \eta(\lambda)\eta(\mu)^{-1} = g\}.$$

We define $\mathrm{sp}(\delta_\eta) = \{g \in G : C^*(\Lambda)_g \neq \emptyset\}$, to be the collection of non-empty spectral subspaces. The *fixed point algebra*, $C^*(\Lambda)^{\delta_\eta}$ of the coaction is defined to be $C^*(\Lambda)_{1_G}$. For more details on the coactions of discrete groups on k-graph algebras, see [14, §7] and [20].

We give necessary and sufficient conditions for the skew product graph C^*-algebra to be simple in terms of the fixed-point algebra as our main result in Theorem 6.3. We are particularly interested in the case when η is the degree functor.

Definition 6.1. Let Λ be a row-finite k-graph, G be a discrete group and $\eta : \Lambda \to G$ a functor, then we define

$$\Gamma(\eta) = \{g \in G : g = \eta(\lambda)\eta(\mu)^{-1} \text{ for some } \lambda, \mu \in \Lambda \text{ with } s(\lambda) = s(\mu)\}.$$

Lemma 6.2. *Let Λ be a row-finite graph with no sources and $\eta : \Lambda \to G$ a functor, where G is a discrete group.*

(a) If (Λ, G, η) is cofinal then $\Gamma(\eta) = G$.

(b) $\mathrm{sp}(\delta_\eta) = G$ if and only if $\Gamma(\eta) = G$.

Proof. Fix $g \in G$ and write $g = b^{-1}a$ for some $a, b \in G$. Now fix $v, w \in \Lambda^0$; since (Λ, G, η) is cofinal there exist $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ such that $a\eta(\mu) = b\eta(\lambda)$. Hence $b^{-1}a = \eta(\lambda)\eta(\mu)^{-1}$ and so $g \in \Gamma(\eta)$. Since g was arbitrary the result follows.

The second statement follows by definition. □

Theorem 6.3. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to G$ a functor and δ_η the associated coaction of G on $C^*(\Lambda)$. Then $C^*(\Lambda \times_\eta G)$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\Gamma(\eta) = G$.*

Proof. By [14, Theorem 7.1] it follows that $C^*(\Lambda \times_\eta G)$ is isomorphic to $C^*(\Lambda) \times_{\delta_\eta} G$. Then by [20, Theorem 2.10] $C^*(\Lambda) \times_{\delta_\eta} G$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\mathrm{sp}(\delta_\eta) = G$. The result now follows from Lemma 6.2. □

Example 6.4. Let Λ be a row-finite k-graph with no sources and $d : \Lambda \to \mathbb{N}^k$ be the degree functor. We claim that $\Gamma(d) = \mathbb{Z}^k$. Fix $p \in \mathbb{Z}^k$, and write $p = m - n$ where $m, n \in \mathbb{N}^k$. Since Λ has no sources, for every $v \in \Lambda^0$ there is $\lambda \in \Lambda^m v$ and $\mu \in \Lambda^n v$. Then

$$d(\lambda) - d(\mu) = m - n = p \in \Gamma(d),$$

and so $\Gamma(d) = \mathbb{Z}^k$. Since $\Gamma(d) = \mathbb{Z}^k$, and $(\Lambda, \mathbb{Z}^k, d)$ is aperiodic, we have that $C^*(\Lambda)^{\delta_d}$ is simple if and only $(\Lambda, \mathbb{N}^k, d)$ is cofinal.

We seek conditions on Λ that will guarantee $(\Lambda, \mathbb{N}^k, d)$ is cofinal.

7 The gauge coaction The coaction δ_d of $\mathbb{Z}^k$ on $C^*(\Lambda)$ defined in Section 6 is such that the fixed point algebra $C^*(\Lambda)^{\delta_d}$ is precisely the fixed point algebra $C^*(\Lambda)^\gamma$ for the canonical gauge action of $\mathbb{T}^k$ on $C^*(\Lambda)$ by the Fourier transform (cf. [2, Corollary 4.9].

By [9, Lemma 3.3] the fixed point algebra $C^*(\Lambda)^\gamma$ is AF, and is usually referred to as the AF core. In Theorem 7.2 we use the results of the last two sections to give necessary and sufficient conditions for the AF core $C^*(\Lambda)^\gamma$ to be simple when Λ^0 is finite. When there are infinitely many vertices we show, in Theorem 7.8 that in many cases the AF core is not simple.

The AF core of a k-graph algebra plays a significant role in the development of crossed products by endomorphisms. Results of Takehana and Katayama [8] show that when Λ is a finite 1-graph such that the core $C^*(\Lambda)$ is simple, then every nontrivial automorphism of $C^*(\Lambda)$ is outer (see [17, Proposition 3.4]).

We saw in Example 4.9 that a k-graph being strongly connected is not enough to guarantee that $\Lambda \times_d \mathbb{Z}^k$ is cofinal, and hence by [23, Theorem 3.1] $C^*(\Lambda \times_d \mathbb{Z}^k)$ is not simple and then by Theorem 6.3 the AF core is not simple. Another condition is required to guarantee that $\Lambda \times_d \mathbb{Z}^k$ is cofinal, which is suggested by [18] and was introduced in Section 5:

Theorem 7.1. *Let Λ be a row-finite k-graph with no sinks and sources and Λ^0 finite. If $(\Lambda, d, \mathbb{Z}^k)$ is cofinal then Λ is primitive.*

Proof. We claim that for $v \in \Lambda^0$ there is $N(v) \in \mathbb{N}^k$ such that for all $n \geq N(v)$ we have $v\Lambda^n v \neq \emptyset$. Fix $(v,0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ then for each $w \in \Lambda^0$, when we apply the cofinality condition to $(w,0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ we obtain $N_w \in \mathbb{N}^k$ such that $(v,0)(\Lambda \times_d \mathbb{Z}^k)s(\alpha,0) \neq \emptyset$ for all $(\alpha,0) \in (w,0)(\Lambda \times_d \mathbb{Z}^k)^{N_w}$. Define $N = \max_{w\in\Lambda^0}\{N_w\}$, which is finite since Λ^0 is finite.

By Proposition 4.5 it follows that Λ is strongly connected, hence there exists $\alpha \in v\Lambda v$ with $d(\alpha) = r > 0$. Hence, there exists $t \geq 1$ such that $tr \geq N$. Let $N(v) = tr$.

Let $m = n - tr \geq 0$. Since Λ has no sources, $v\Lambda^m \neq \emptyset$; hence there exists $\gamma \in v\Lambda^m$. Let $w = s(\gamma)$. For $(v,0),(w,0) \in (\Lambda \times_d \mathbb{Z}^k)^0$, we have $(\alpha^t,0) \in (v,0)(\Lambda \times_d \mathbb{Z}^k)^{tr}$ where $tr \geq N \geq N_w$. By cofinality and Lemma 4.2 (b), there exists $(\beta,0) \in (w,0)(\Lambda \times_d \mathbb{Z}^k)(v,tr)$ as $s(\alpha^t,0) = (v,tr)$. As $\beta \in w\Lambda^{tr}v$ it follows that $\gamma\beta \in v\Lambda^n v$, which proves the claim. □

The following result generalises results from [18]:

Theorem 7.2. *Let (Λ, d) be a row-finite k-graph with no sinks or sources, and Λ^0 finite. Then $C^*(\Lambda)^{\delta_d}$ is simple if and only if Λ is primitive.*

Proof. Suppose that Λ is primitive. Then $(\Lambda, \mathbb{Z}^k, d)$ is strongly connected and cofinal by Remarks 2.8. Hence $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple and so $C^*(\Lambda)^{\delta_d}$ is simple by Theorem 6.3.

Suppose that $C^*(\Lambda)^{\delta_d}$ is simple. Recall from Example 6.4 that since Λ has no sources then $\Gamma(d) = \mathbb{Z}^k$. Then by Theorem 6.3, $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple, and hence $(\Lambda, d, \mathbb{Z}^k)$ is cofinal by [23, Theorem 3.1] and Proposition 4.11. By Theorem 7.1 this implies that Λ is primitive. □

Example 7.3. Since it has a single vertex it is easy to see that the 2-graph $\mathbb{F}^2_\theta$ defined in Examples 2.1 (d) is primitive. Hence by Theorem 7.2 we see that $C^*(\mathbb{F}^2_\theta)^\gamma$ is simple for all θ. Indeed in [4, §2.1] it is shown that $C^*(\mathbb{F}^2_\theta)^\gamma \cong \mathrm{UHF}(mn)^\infty$.

We now turn our attention to the case when Λ^0 is infinite. We adapt the technique used in [18] to show that, in many cases the AF core is not simple.

Definition 7.4. Let Λ be a row-finite k-graph with no sources. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ let

$$V(n, v) = \{s(\lambda) : \lambda \in v\Lambda^m, m \leq n\}$$
$$FV(n, v) = V(n, v) \setminus \cup_{i=1}^k V(n - e_i, v).$$

Remarks 7.5. For $v \in \Lambda^0$, $m \leq n \in \mathbb{N}^k$ we have, by definition, that $V(m, v) \subseteq V(n, v)$.

For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ the set $FV(n, v)$ denotes those vertices which connect to v with a path of degree n and there is no path from that vertex to v with degree less than n.

Lemma 7.6. *Let Λ be a row-finite k-graph with no sources. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ then $V(n, v)$ is finite and if $V(n) = V(n - e_i)$ for some $1 \leq i \leq k$ then $V(n + re_i) = V(n - e_i)$ for all $r \geq 0$.*

Proof. Fix, $v \in \Lambda^0$, $n \in \mathbb{N}^k$, since Λ row-finite it follows that $\cup_{m \leq n} v\Lambda^m$ is finite and hence so is $V(n, v)$.

Suppose, without loss of generality that $V(n) = V(n - e_1)$. Let $w \in V(n + e_1)$, then there is $\lambda \in v\Lambda^{n+e_1}w$. Now $\lambda(0, n) \in v\Lambda^n$ and so $s(\lambda(0, n)) \in V(n) = V(n - e_1)$. Hence there is $\mu \in v\Lambda^m s(\lambda(0, n))$ for some $m \leq n - e_1$ and so $\mu\lambda(n, n + e_1) \in v\Lambda^{m+e_1}$. Since $s(\mu\lambda(n, e + e_1)) = s(\lambda) = w$ and $m + e_1 \leq n$ it follows that $w \in V(n)$. As w was an arbitrary element of $V(n + e_1)$ it follows that $V(n + e_1) \subseteq V(n) = V(n - e_1)$. By Remarks 7.5 we have $V(n - e_1) \subseteq V(n + e_1)$ and so $V(n + e_1) = V(n - e_1)$. It then follows that $V(n + re_1) = V(n - e_1)$ for $r \geq 0$ by an elementary induction argument. □

We adopt the following notation, used in [11]: Let Λ be a k-graph for $1 \leq i \leq k$ we set $\Lambda^{\mathbb{N}e_i} = \cup_{r \geq 0} \Lambda^{re_i}$.

Proposition 7.7. *Let Λ be a row-finite k-graph with no sources such that for all $w \in \Lambda^0$ and for $1 \leq i \leq k$, the set $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite. Then for all $n \in \mathbb{N}^k$, $v \in \Lambda^0$ we have $FV(n, v) \neq \emptyset$.*

Proof. Suppose, for contradiction, that $FV(n,v) = \emptyset$ for some $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. Then, without loss of generality we may assume that $V(n) = V(e-e_1)$.

Let $\lambda \in v\Lambda^n$, then $s(\lambda) \in V(n) = V(n-e_1)$. Fix $r \geq 0$, then since Λ has no sources there is $\mu \in s(\lambda)\Lambda^{re_1}$. Then $\lambda\mu \in v\Lambda^{n+re_1}$ and so $s(\lambda\mu) = s(\mu) \in V(n+re_1, v)$. By Lemma 7.6 it follows that $V(n+re_1) = V(n-e_1)$ and so for any $\mu \in s(\lambda)\Lambda^{\mathbb{N}e_1}$ we have $s(\mu) \in V(n-e_1)$. By Remarks 7.5 $V(n-e_1)$ is finite and so we have contradicted the hypothesis that $s^{-1}\left(w\Lambda^{\mathbb{N}e_1}\right)$ is infinite. □

Note that k-graphs satisfying the hypothesis of Proposition 7.7 must have infinitely many vertices. The following result generalises results from [18]:

Theorem 7.8. *Let Λ be a row-finite k-graph with no sources such that for all $w \in \Lambda^0$ and for $1 \leq i \leq k$, the set $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite. Then $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.*

Proof. Suppose, for contradiction, that $\Lambda \times_d \mathbb{Z}^k$ is cofinal.

Fix $v \in \Lambda^0$ then since Λ is row-finite and has no sources $W = s^{-1}\left(v\Lambda^{e_1}\right)$ is finite and nonempty. Without loss of generality let $W = \{w_1, \ldots, w_n\}$.

Since $\Lambda \times_d \mathbb{Z}^k$ is cofinal, for $1 \leq i \leq n$ if we consider $(w_i, 0)$ and $(v, 0) \in \Lambda^0 \times \mathbb{Z}^k$ then there is $N_i \in \mathbb{N}^k$ such that for all $(\alpha, 0) \in (w_i, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)^{N_i}$ we have $(v, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)(s(\alpha), N_i) \neq \emptyset$. Let $N = \max\{N_1, \ldots, N_n\}$. By Proposition 7.7 $FV(N + e_1, v) \neq \emptyset$, hence there is $\lambda \in v\Lambda^{N+e_1}$ such that there is no path of degree less than $N + e_1$ from $s(\lambda)$ to v. Without loss of generality $s(\lambda(0, e_1)) = w_1$, and so $(\lambda(e_1, N + e_1), 0) \in (w_1, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)^N$. Since $N \geq N_1$ and Λ has no sources, by Lemma 4.2(ii) there is $(\alpha, 0) \in (v, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)(s(\lambda), N)$ which implies that $\alpha \in v\Lambda^N s(\lambda)$, contradicting the defining property of $\lambda \in v\Lambda^{N+e_1}$. □

Examples 7.9.

1. Let Λ be a strongly connected k-graph with Λ^0 infinite, then Λ has no sources and for all $w \in \Lambda^0$ we have $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite for $1 \leq i \leq k$. Hence by Theorem 7.8 it follows that $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.

2. Let Λ be a k-graph with Λ^0 infinite, no sources and no paths with the same source and range. Then for all $w \in \Lambda^0$ we have $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite for $1 \leq i \leq k$. Hence by Theorem 7.8 it follows that $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.

References

[1] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on C^*-algebras associated to finite graphs*, J. Math. Anal. Appl., **405** (2013), 388–399.

[2] T. Crisp. *Corners of graph algebras*, J. Operator Theory, **60** (2008), 253–271.

[3] K. R. Davidson, S. C. Power and D. Yang, *Atomic representations of rank 2 graph algebras*, J. Funct. Anal., **255** (2008), 819–853.

[4] K. R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math., **61** (2009), 1239–1261.

[5] D. G. Evans, *On the K-theory of higher rank graph C*-algebras*, New York J. Math., **14** (2008), 1–31.

[6] R. Hazlewood, I. Raeburn, A. Sims and S. Webster, *Remarks on some fundamental results about higher-rank graphs and their C^*-algebras*, Proc. Edinburgh Math. Soc., **56** (2013), 575–597.

[7] S. Kang and D. Pask. *Aperiodicity and the primitive ideal space of a row-finite k-graph C^*-algebra*, to appear Internat. J. Math.

[8] Y. Katayama and H. Takehana, *On automorphisms of generalized Cuntz algebras*, Internat. J. Math., **9** (1998), 493–512.

[9] A. Kumjian and D. Pask, *Higher Rank Graph C^*-algebras*, New York J. Math., **6** (2000), 1–20.

[10] A. Kumjian and D. Pask, *Actions of $\mathbf{Z}^k$ associated to higher rank graphs*, Ergod. Th. & Dynam. Sys., **23** (2003), 1153–1172.

[11] A. Kumjian, D. Pask and A. Sims, *C^*-algebras associated to covering of k-graphs*, Doc. Math., **13** (2008), 161–205.

[12] P. Lewin and A. Sims, *Aperiodicity and Cofinality for Finitely Aligned Higher-Rank Graphs*, Math. Proc. Cambridge Philosophical Soc., **149** (2010), 333–350.

[13] B. Maloney, D. Pask and I. Raeburn, *Skew products of higher-rank graphs and crossed products by semigroups*, Semigroup Forum **88** (2014), 162–176.

[14] D. Pask, J. Quigg and I. Raeburn, *Coverings of k-graphs*, J. Alg., **289** (2005), 161–191.

[15] D. Pask, I. Raeburn and T. Yeend, *Actions of semigroups on directed graphs and their C^*-algebras*, J. Pure Appl. Algebra, **159** (2001), 297–313.

[16] D. Pask and I. Raeburn, *On the K-Theory of Cuntz-Krieger algebras*, Publ. RIMS Kyoto, **32** (1996), 415–443.

[17] D. Pask, I. Raeburn and N. Weaver, *Periodic 2-graph arising from subshifts*, Bull. Aust. Math. Soc., **82** (2010), 120–138.

[18] D. Pask and S-J. Rho. *Some intrinsic properties of simple graph C^*-algebras*, Operator Algebras and Mathematical Physics, Constanza, Romania, 2001, Theta Foundation, **2003**, 325–340.

[19] S. Power, *Classifying higher rank analytic Toeplitz algebras*, New York J. Math., **13** (2007), 271–298.

[20] J. Quigg *Discrete C^*–coactions and C^*–algebraic bundles.* J. Austral. Math. Soc., **60** (1996), 204–221.

[21] I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C^*-algebras*, Proc. Edinburgh Math. Soc., **46** (2003), 99–115.

[22] I. Raeburn, A. Sims and T. Yeend, *The C^*-algebras of finitely aligned higher-rank graphs* J. Funct. Anal., **213** (2004), 206–240.

[23] D. I. Robertson and A. Sims, *Simplicity of C^*-algebras associated to higher rank graphs*, Bull. London Math. Soc., **39** (2007), 337–344.

[24] G. Robertson, T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. reine angew. Math., **513** (1999), 115–144.

Communicated by *Moto O'uchi*

David Pask
School of Mathematics and Applied Statistics
The University of Wollongong NSW 2522
Australia
dpask@uow.edu.au