Remark on the Triebel-Lizorkin space boundedness of rough singular integrals associated to surfaces

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Abstract. In the present paper, we consider the boundedness of the rough singular integral operator $T_{\Omega,h,\phi}$ along a surface $\Gamma = \{x = \phi(|y|)y/|y|)\}$ on the Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in \mathbb{R}$, $1 < p, q < \infty$ for $\Omega \in H^1(S^{n-1})$ and Ω belonging to some class $W\mathcal{F}_\alpha(S^{n-1})$, which relates to the Grafakos-Stefanov class. We improve recent results about these operators.

1 Introduction.

The purpose of this paper is to improve recent results in [10].

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$. Suppose $\Omega \in L^1(S^{n-1})$ satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0, \tag{1}$$

where $y' = y/|y|$.

For a suitable function ϕ and a measurable function h on $[0,\infty)$, we denote by $T_{\Omega,\phi,h}$ the singular integral operator along the surface

$$\Gamma = \{x = \phi(|y|)y' : y \in \mathbb{R}^n\}$$

defined as follows:

$$T_{\Omega,h,\phi} f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - \phi(|y|)y')\, dy, \tag{2}$$

for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. If $\phi(t) = t$, then $T_{\Omega,h,\phi}$ is the classical singular integral operator $T_{\Omega,h}$, which is defined by

$$T_{\Omega,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - y)\, dy. \tag{3}$$

When $h \equiv 1$, we denote simply $T_{\Omega,h,\phi}$ and $T_{\Omega,h}$ by $T_{\Omega,\phi}$ and T_Ω, respectively.

Let us recall the definitions of some function spaces. First recall the definitions of the *homogeneous Triebel-Lizorkin spaces* $\dot{F}^{\alpha}_{p,q} = \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and the *homogeneous Besov spaces* $\dot{B}^{\alpha}_{p,q} = \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$. For $0 < p, q \leq \infty$ $(p \neq \infty)$ and $\alpha \in \mathbb{R}$, $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) : ||f||_{\dot{F}^{\alpha}_{p,q}} = \left\| \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} |\Psi_k * f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\} \tag{4}$$

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and $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)=\left\{f\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n):||f||_{\dot{B}^{\alpha}_{p,q}}=\left(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|\Psi_k*f\|^q_{L^p}\right)^{1/q}<\infty\right\}, \tag{5}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, and $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$, $\widehat{\Psi_k}(\xi)=\Phi(2^{-k}\xi)$ for $k\in\mathbb{Z}$ and $\Phi\in C_c^\infty(\mathbb{R}^n)$ is a radial function satisfying the following conditions: (i) $0\le\Phi\le1$; (ii) $\operatorname{supp}\Phi\subset\{\xi:1/2\le|\xi|\le2\}$; (iii) $\Phi>c>0$ if $3/5\le|\xi|\le5/3$; (iv) $\sum_{j\in\mathbb{Z}}\Phi(2^{-j}\xi)=1$ $(\xi\ne0)$. Note that the space $\mathcal{S}_\infty(\mathbb{R}^n)$ given by

$$\mathcal{S}_\infty(\mathbb{R}^n):=\bigcap_{\alpha\in(\mathbb{N}\cup\{0\})^n}\left\{f\in\mathcal{S}(\mathbb{R}^n)\;:\;\int_{\mathbb{R}^n}x^\alpha f(x)\,dx=0\right\}$$

is dense in $\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$ and $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ as long as $\alpha\in\mathbb{R}$ and $p,q\in(0,\infty)$ ([9, Theorem 5.1.5]).

The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F^{\alpha}_{p,q}(\mathbb{R}^n)$ and $B^{\alpha}_{p,q}(\mathbb{R}^n)$ respectively, are obtained by adding the term $\|\Phi_0*f\|_p$ to the right-hand side of (4) or (5) with $\sum_{k\in\mathbb{Z}}$ replaced by $\sum_{k=0}^\infty$, where $\Phi_0\in\mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\widehat{\Phi_0}\subset\{\xi:|\xi|\le2\}$, and $\widehat{\Phi_0}(\xi)>c>0$ if $|\xi|\le5/3$.

The following properties of the Triebel-Lizorkin space and Besov space are well known. Let $1<p,q<\infty,\alpha\in\mathbb{R}$, and $1/p+1/p'=1,1/q+1/q'=1$.

$$\begin{aligned}
&\text{(a)}\quad \dot{F}^0_{2,2}=\dot{B}^0_{2,2}=L^2,\dot{F}^0_{p,2}=L^p \text{ and } \dot{F}^\alpha_{p,p}=\dot{B}^\alpha_{p,p} \text{ for } 1<p<\infty, \text{ and } \dot{F}^0_{\infty,2}=\mathrm{BMO};\\
&\text{(b)}\quad F^\alpha_{p,q}\sim\dot{F}^\alpha_{p,q}\cap L^p \text{ and } ||f||_{F^\alpha_{p,q}}\sim||f||_{\dot{F}^\alpha_{p,q}}+||f||_{L^p}\ (\alpha>0);\\
&\text{(c)}\quad B^\alpha_{p,q}\sim\dot{B}^\alpha_{p,q}\cap L^p \text{ and } ||f||_{B^\alpha_{p,q}}\sim||f||_{\dot{B}^\alpha_{p,q}}+||f||_{L^p}\ (\alpha>0);\\
&\text{(d)}\quad (\dot{F}^\alpha_{p,q})^*=\dot{F}^{-\alpha}_{p',q'} \text{ and } (F^\alpha_{p,q})^*=F^{-\alpha}_{p',q'};\\
&\text{(e)}\quad (\dot{B}^\alpha_{p,q})^*=\dot{B}^{-\alpha}_{p',q'} \text{ and } (B^\alpha_{p,q})^*=B^{-\alpha}_{p',q'};\\
&\text{(f)}\quad \left(\dot{F}^{\alpha_1}_{p,q_1},\dot{F}^{\alpha_2}_{p,q_2}\right)_{\theta,q}=\dot{B}^\alpha_{p,q}\ (\alpha_1\ne\alpha_2,0<p<\infty,0<q,q_1,q_2\le\infty,\\
&\qquad\qquad \alpha=(1-\theta)\alpha_1+\theta\alpha_2,0<\theta<1)..
\end{aligned} \tag{6}$$

See [9] for more properties of $\dot{F}^\alpha_{p,q}$ and $\dot{B}^\alpha_{p,q}$.

Next, we give the definition of the *Hardy space* $H^1(S^{n-1})$.

$$H^1(S^{n-1})=\left\{\omega\in L^1(S^{n-1})\middle|\,\|f\|_{H^1(S^{n-1})}=\left\|\sup_{0\le r<1}\left|\int_{S^{n-1}}\omega(y')P_{r(\cdot)}(y')d\sigma(y')\right|\right\|_{L^1(S^{n-1})}<\infty\right\},$$

where $P_{ry'}(x')$ denotes the Poisson kernel on S^{n-1} defined by $P_{ry'}(x')=(1-r^2)/|ry'-x'|^n$, $0\le r<1$ and $x',y'\in S^{n-1}$.

Besides $H^1(S^{n-1})$, there are two important function spaces $L(\log L)(S^{n-1})$ and the block spaces $B_q^{(0,0)}(S^{n-1})$ in the theory of singular integrals. Let $L(\log L)^\alpha(S^{n-1})$ (for $\alpha>0$) denote the class of all measurable functions Ω on S^{n-1} which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(S^{n-1})} = \int_{S^{n-1}} |\Omega(y')| \log^\alpha(2 + |\Omega(y')|)\, d\sigma(y') < \infty.$$

Denote by $L(\log L)(S^{n-1})$ $L(\log L)^1(S^{n-1})$. A well-known fact is $L(\log L)(S^{n-1}) \subset H^1(S^{n-1})$, cf. [8].

We turn to the block space $B_q^{(0,v)}(S^{n-1})$. Let $1 < q \le \infty$ and $v > -1$. A q-block on S^{n-1} is an $L^q(S^{n-1})$ function b which satisfies $\operatorname{supp} b \subset$ and $\|b\|_q \le |I|^{-1/q'}$, where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) \cap S^{n-1}$ is a cap on S^{n-1} for some $x_0' \in S^{n-1}$ and $\theta_0 \in (0, 1]$. The *block space* $B_q^{(0,v)}(S^{n-1})$ is defined by

$$B_q^{(0,v)}(S^{n-1}) = \Big\{\Omega \in L^1(S^{n-1});\, \Omega = \sum_{j=1}^\infty \lambda_j b_j,\, M_q^{(0,v)}(\{\lambda_j\}) < \infty\Big\}, \tag{7}$$

where $\lambda_j \in \mathbb{C}$ and b_j is a q-block supported on a cap I_j on S^{n-1}, and

$$M_q^{(0,v)}(\{\lambda_j\}) = \sum_{j=1}^\infty |\lambda_j| \big\{1 + \log^{(v+1)}\big(|I_j|^{-1}\big)\big\}. \tag{8}$$

For $\Omega \in B_q^{(0,v)}(S^{n-1})$, denote

$$\|\Omega\|_{B_q^{(0,v)}(S^{n-1})} = \inf\Big\{M_q^{(0,v)}(\{\lambda_j\}); \Omega = \sum_{j=1}^\infty \lambda_j b_j, b_j \text{ is a } q\text{-block}\Big\}.$$

Then $\|\cdot\|_{B_q^{(0,v)}(S^{n-1})}$ is a norm on the space $B_q^{(0,v)}(S^{n-1})$, and $\Big(B_q^{(0,v)}(S^{n-1}), \|\cdot\|_{B_q^{(0,v)}(S^{n-1})}\Big)$ is a Banach space. The following inclusion relations are known.

$$\begin{aligned}
&\text{(a)} \quad B_q^{(0,v_1)}(S^{n-1}) \subset B_q^{(0,v_2)}(S^{n-1}) \quad \text{if } v_1 > v_2 > -1;\\
&\text{(b)} \quad B_{q_1}^{(0,v)}(S^{n-1}) \subset B_{q_2}^{(0,v)}(S^{n-1}) \quad \text{if } 1 < q_2 < q_1 \text{ for any } v > -1;\\
&\text{(c)} \quad \bigcup_{p>1} L^p(S^{n-1}) \subset B_q^{(0,v)}(S^{n-1}) \quad \text{for any } q > 1,\, v > -1;\\
&\text{(d)} \quad \bigcup_{q>1} B_q^{(0,v)}(S^{n-1}) \not\subset \bigcup_{q>1} L^q(S^{n-1}) \quad \text{for any } v > -1;\\
&\text{(e)} \quad B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+v}(S^{n-1}) \quad \text{for any } q > 1,\, v > -1;\\
&\text{(f)} \quad \bigcup_{q>1} B_q^{(0,0)}(S^{n-1}) \subset H^1(S^{n-1}).
\end{aligned} \tag{9}$$

Besides them, there is another class of kernels which lead L^p and Triebel-Lizorkin space boundedness of singular integral operators $T_{\Omega,h}$. It is closely related to the class $\mathcal{F}_\alpha$ introduced by Grafakos and Stefanov [4].

For $\beta > 0$ we say $\Omega \in \mathcal{F}_\beta(S^{n-1})$ if

$$\sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \log^\beta \frac{2}{|y' \cdot \xi'|} d\sigma(y') < \infty, \tag{10}$$

and $\Omega \in W\mathcal{F}_\beta(S^{n-1})$ ($\tilde{\mathcal{F}}_\beta(S^{n-1})$ in [6]) if

$$\sup_{\xi' \in S^{n-1}} \left(\int_{S^{n-1}} \int_{S^{n-1}} |\Omega(y')\Omega(z')| \log^\beta \frac{2e}{|(y'-z')\cdot \xi'|} d\sigma(y') d\sigma(z') \right)^{\frac{1}{2}} < \infty. \tag{11}$$

We note that $\cup_{r>1} L^r(S^{n-1}) \subset W\mathcal{F}_{\beta_2}(S^{n-1}) \subset W\mathcal{F}_{\beta_1}(S^{n-1})$ for $0 < \beta_1 < \beta_2 < \infty$.

About the inclusion relation between $\mathcal{F}_{\beta_1}(S^{n-1})$ and $W\mathcal{F}_{\beta_2}(S^{n-1})$, the following is known: when $n = 2$, Lemma 1 in [3] shows $\mathcal{F}_\beta(S^1) \subset W\mathcal{F}_\beta(S^1)$. It is also known that $W\mathcal{F}_{2\alpha}(S^1) \setminus \big(\mathcal{F}_\alpha(S^1) \cup H^1(S^1)\big) \neq \emptyset$. cf. [7].

To state our claims, we need one more function space. For $1 \le \gamma \le \infty$, $\Delta_\gamma(\mathbb{R}_+)$ is the collection of all measurable functions $h : [0, \infty) \to \mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty.$$

Note that

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\alpha(\mathbb{R}_+) \quad \text{for } \alpha < \beta,$$

and all these inclusions are proper.

In this short note, we report that Theorems 1.1, 1.2 and 1.3 in [10] are improved essentially in the following form. In the following theorems, the statement "$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$" means that

$$\|T_{\Omega,h,\phi} f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C \|T_{\Omega,h,\phi} f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)},$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$. In any case, by density we can extend the above inequality and have them for all $f \in \dot{F}^\alpha_{pq}(\mathbb{R}^n)$. We use similar abbreviation to $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$.

Theorem 1. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0, \infty)$ satisfying*

$$\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0, \infty). \tag{12}$$

Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Suppose $\Omega \in H^1(S^{n-1})$, satisfying the cancellation condition (1). Then

(i) *$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$ (hexagon $P_1P_2P_3P_4P_5P_6$ in the case $1 < \gamma \le 2$), where $P_1 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_2 = (\frac{1}{2}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_3 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $P_4 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_5 = (\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_6 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $R_2 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, and $R_4 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$.*

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q satisfying $|\frac{1}{2} - \frac{1}{p}| < \min\{\frac{1}{2}, \frac{1}{\gamma'}\}$ and $1 < q < \infty$.*

See the following Figures 1-1 and 1-2 for the conclusion (i) of Theorem 1.

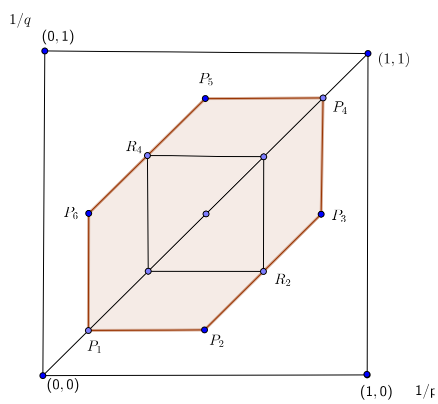

Figure 1 − 1 $(1 < \gamma < 2)$

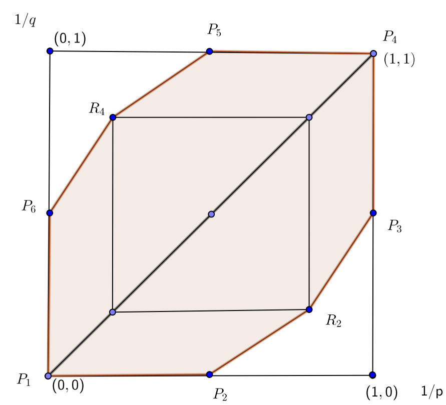

Figure 1 − 2 $(2 \le \gamma \le \infty)$

The following theorem shows that if Ω belongs to $L\log L(S^{n-1})$ or block spaces, then we can get better results than Theorem 1.

Theorem 2. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0,\infty)$ satisfying the same condition as in Theorem 1. Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, and $\Omega \in L^1(S^{n-1})$ satisfy the cancellation condition* (1). *Then*

(i) *if $\Omega \in L(\log L)(S^{n-1})$, $T_{\Omega,h,\phi}$ is bounded on $\dot F^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p,q with $(\frac{1}{p},\frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2Z_2Q_3Q_4Z_4$ when $1 < \gamma < 2$ and $Q_1Q_2S_2Q_3Q_4S_4$ when $2 \le \gamma \le \infty$, where $Q_1 = (0,0)$, $Q_2 = (\frac{1}{\gamma'}, 0)$, $Q_3 = (1,1)$, $Q_4 = (\frac{1}{\gamma}, 1)$, $S_2 = (1, \frac{1}{\gamma})$, $S_4 = (\frac{1}{\gamma}, 0)$, $Z_2 = (1, \frac{1}{2})$, and $Z_4 = (\frac{1}{2}, 0)$.*

(ii) *if $\Omega \in \cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})$, $T_{\Omega,h,\phi}$ is bounded on $\dot F^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p,q with $(\frac{1}{p},\frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$*

(iii) *if $\Omega \in L(\log L)(S^{n-1}) \cup \big(\cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})\big)$, $T_{\Omega,h,\phi}$ is bounded on $\dot B^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and $1 < p,q < \infty$.*

See the following Figures 1-3 and 1-4 for the conclusion of Theorem 2(i).

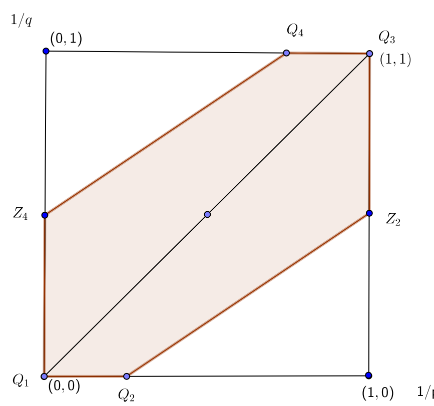

Figure 1 − 3 $(1 < \gamma < 2)$

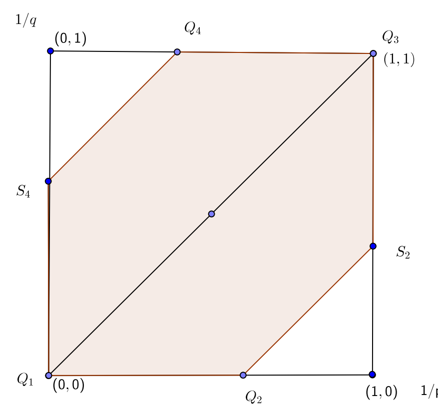

Figure 1 − 4 $(2 \leq \gamma \leq \infty)$

As a corresponding result to the case Ω belongs to $W\mathcal{F}_\alpha$, we have the following:

Theorem 3. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0, \infty)$ satisfying the same condition as in Theorem 1. Let $h \in \Delta_\gamma$ for some $1 < \gamma \leq \infty$. Suppose $\Omega \in W\mathcal{F}_\beta = W\mathcal{F}_\beta(S^{n-1})$ for some $\beta > \max(\gamma', 2)$, and satisfies the cancellation condition* (1). *Then*

(i) *the singular integral operator $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{S}_2\mathcal{Q}_3\mathcal{Q}_4\mathcal{S}_4$, where $\mathcal{Q}_1 = \big(\frac{\max(\gamma',2)}{2\beta}, \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_2 = \big(\frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'}), \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_3 = \big(1 - \frac{\max(\gamma',2)}{2\beta}, 1 - \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_4 = \big(\frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2}), 1 - \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{S}_2 = \big(, 1 - \frac{\max(\gamma',2)}{2\beta}, \frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2})\big)$, and $\mathcal{S}_4 = \big(\frac{\max(\gamma',2)}{2\beta}, \frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'})\big)$.*

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$, $\frac{2\beta}{2\beta - \max(\gamma',2)} < p < \frac{2\beta}{\max(\gamma',2)}$ and $1 < q < \infty$.*

See the Figure 1-5 for the conclusion (i) of Theorem 3.

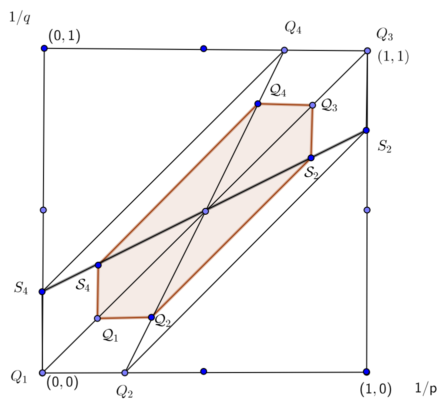

Figure 1 − 5

Remark 1. In [10] we have shown these theorems under the stronger assumption on ϕ, i.e, when ϕ is a positive increasing function on $(0,\infty)$ satisfying the doubling condition $\phi(2t) \le c_1\phi(t)$ $(t>0)$ for some $c_1 > 1$ besides (12). Note also that we improve Theorems 1.2 and 1.3 in [10] even in the case $\phi(t) = t$.

Example 1. As typical examples of ϕ satisfying the condition (12), we list the following four: $t^\alpha e^t$ $(\alpha > 0)$, $t^\alpha \log^\beta(1+t)$ $(\alpha > 0,\ \beta \ge 0)$, $(2t^2 - 2t + 1)t^{1+\alpha}$ $(\alpha \ge 0)$, and $\phi(t) = 2t^2 + t$ $(0 < t < \frac{\pi}{2})$, $= 2t^2 + t \sin t$ $(t \ge \frac{\pi}{2})$. Note that linear combinations with positive coefficients of functions ϕ's satisfying the above two conditions also satisfies them. Note that the first example satisfies (12), but does not satisfy the doubling condition.

2 Proofs of Theorems. One can prove these theorems by a change of variable and the corresponding theorems in case $\phi(t) = t$ in [10], like in [2] or [5].

To prove the theorems, we prepare the following three lemmas: Lemma 1, Lemma 2 and Lemma 4. The first one is Lemma 2.2 in [2], and the second one is Lemma 2.3 in [2].

Lemma 1. *Let ϕ and φ be the same as in Theorem 1. If $b \in \Delta_\gamma$ for some $\gamma \ge 1$, then*

$$\frac{1}{R}\int_0^R |b(|\Phi|^{-1}(t))\varphi(|\Phi|^{-1}(t))|^\gamma \, dt \le C_\gamma(\|\varphi\|_\infty^{\gamma-1} + \|\varphi\|_\infty^\gamma), \quad R > 0, \tag{13}$$

that is, $b(|\Phi|^{-1})\varphi(|\Phi|^{-1}) \in \Delta_\gamma$.

Lemma 2. *Let ϕ and φ be the same as in Theorem 1. Then*

$$T_{\Omega,\phi,h}f(x) = \begin{cases} T_{\Omega,\varphi(\phi^{-1})h(\phi^{-1})}f(x), & \text{if } \phi \text{ is nonnegative and increasing,} \\ -T_{\Omega,\varphi(\phi^{-1})h(\phi^{-1})}f(x), & \text{if } \phi \text{ is nonnegative and decreasing,} \\ T_{\tilde{\Omega},\varphi(\phi^{-1}(-\cdot))h(\phi^{-1}(-\cdot))}f(x), & \text{if } \phi \text{ is nonpositive and decreasing,} \\ -T_{\tilde{\Omega},\varphi(\phi^{-1}(-\cdot))h(\phi^{-1}(-\cdot))}f(x), & \text{if } \phi \text{ is nonpositive and increasing,} \end{cases} \tag{14}$$

where $\tilde{\Omega}(y) = \Omega(-y)$.

To state the third one we prepare some definitions and a lemma. For $\Omega \in L^1(S^{n-1})$, $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, a suitable function ϕ on $\mathbb{R}_+$, and $k \in \mathbb{Z}$, we define the measures $\sigma_{\Omega,h,\phi,k}$ on $\mathbb{R}^n$ and the maximal operator $\sigma^*_{\Omega,h,\phi}f(x)$ by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_{\Omega,h,\phi,k}(x) = \int_{\mathbb{R}^n} f(\phi(|x|x')\frac{\Omega(x')h(|x|)}{|x|^n}\chi_{\{2^{k-1}<|x|\le 2^k\}}(x)\, dx, \tag{15}$$

$$\sigma^*_{\Omega,h,\phi}f(x) = \sup_{k\in\mathbb{Z}}\big||\sigma_{\Omega,h,\phi,k}| * f(x)\big|, \tag{16}$$

where $|\sigma_{\Omega,h,\phi,k}|$ is defined in the same way as $\sigma_{\Omega,h,\phi,k}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

we also define the maximal functions $M_{\Omega,h,\phi}$ by

$$M_{\Omega,h,\phi}f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{2^{kn}} \int_{\{2^{k-1}<|y|\le 2^k\}} |\Omega(y')h(|y|)f(x - \phi(|y|)y')|\, dy. \tag{17}$$

We see easily that $M_{\Omega,h,\phi}$ is equivalent to $\sigma^*_{\Omega,h,\phi}(|f|)$.

In [10], we have shown the following auxiliary lemma.

Lemma 3. *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, and $\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0,\infty)$. Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Then, for $\gamma' < p, q < \infty$ we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |M_{\Omega,h,\phi}f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}. \tag{18}$$

Using this we get our third lemma.

Lemma 4. *Let ϕ be the same as above, and $\ell(j) \in \mathbb{Z}$ for $j \in \mathbb{Z}$. Then, if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$, we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,\ell(j)} * f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}, \tag{19}$$

where $Q_1 = (0, 0)$, $Q_2 = (\frac{1}{\gamma'}, 0)$, $Q_3 = (1, 1)$, $Q_4 = (\frac{1}{\gamma}, 1)$, $S_2 = (1, \frac{1}{\gamma})$, and $S_4 = (\frac{1}{\gamma}, 0)$.

Proof. By Lemma 3, we see that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|\sigma_{\Omega,h,\phi,\ell(j)}*f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|M_{\Omega,h,\phi}f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)},$$

if $\gamma' < p, q < \infty$. By duality, we see that the estimate (19) holds if $1 < p, q < \gamma$. Interpolating these two cases, we see that the estimate (19) holds, if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$. □

Now we can prove our theorems.

Using Lemmas 1 and 2 and applying Theorem 1.1 in [10] for $\phi(t) = t$, we get our Theorem 1.

Next, using Lemma 4 in place of Lemma 2.4(ii) in [10], we modify the proof of the inequality (3.4) in [10], and obtain that estimate if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_3Q_3Q_4S_4$. Thus we get our Theorem 3(i) under the additional assumption $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, in particular when $\phi(t) = t$. Similarly, we get our Theorem 2(ii) under the additional assumption $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$. So, using Lemmas 1 and 2 and applying Theorems 2(ii) and 3(i) for $\phi(t) = t$, we get our Theorems 2(ii) and 3(i), respectively.

Next, we consider Theorem 2(i) i.e. the case $\Omega \in L(\log L)(S^{n-1})$. Similarly to the case Ω belonging to block spaces, we see that $T_{\Omega,h,\phi}$ is bounded on $\dot F^{\alpha}_{p,q}(\mathbb{R}^n)$ if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$.

On the other hand, by Theorem 1.3 in [1] we know that $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n) = \dot F^0_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, if $\Omega \in L(\log L)(S^{n-1})$ and $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. So, using Lemmas 1 and 2, we see that $T_{\Omega,h,\phi}$ is bounded on $L^p(\mathbb{R}^n) = \dot F^0_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$.

Hence, interpolating between this case and the case $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$, we see that $T_{\Omega,h,\phi}$ is bounded on $\dot F^{\alpha}_{p,q}(\mathbb{R}^n)$ if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the quadrilateral $Q_1Q_2Z_2Z_4$ or $Q_3Q_4Z_4Z_2$. Interpolating between the cases $Q_1Q_2Z_2Z_4$ and $Q_3Q_4Z_4Z_2$, we have the desired conclusion of Theorem 2(i).

Theorems 2(iii) and 3(ii) follow by using the property (f) of Triebel-Lizorkin spaces and interpolating the cases $\dot F^{\alpha+1}_{p,p}(\mathbb{R}^n)$ and $\dot F^{\alpha-1}_{p,p}(\mathbb{R}^n)$. This completes the proofs of our theorems. □

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