A Search Game with Incomplete Information about Target's Energy

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Abstract. This paper deals with a two-person zero-sum search game called a *search allocation game*, in which a searcher distributes search resource to detect a target and the target moves to evade the searcher. When the searcher starts his search operation for the target, the target happens to stay at some position and have some energy for movement. The target knows the initial state consisting of its initial position and initial energy but the searcher does not. The problem is the game with private information about the target's initial state including initial energy. The payoff of the game is the detection probability of the target. We use a convex programming formulation and a linear programming one for the derivation of an equilibrium, which consists of the value of the game, an optimal distribution of searching resource by the searcher and an optimal movement strategy of the target. By some numerical examples, we analyze players' optimal strategies and evaluate the value of information about the target initial state.

1 Introduction Search theory originates in military operations. Koopman [32], who is a founder of search theory, summarized theoretical results of anti-submarine warfare conducted by US Navy in WW2. He [33] mainly researched one-sided problems, in which only the searcher designs a search plan by estimating the target movement. De Guenin [12] studied an optimal distribution of search efforts by adopting general function as a detection probability of target. Kadane [28] and Onaga [36] considered the criterion of searching cost and Iida et al. [26] researched the search problem of a stationary target based on risk criterion. There are other research focused on stationary targets, such as Gittins [11], who considered the optimal strategies of a stationary hider and a searcher in two regions, and Kress et al. [34], who took account of false detection occurrence in the search. Pollock [37], Schweitzer [40] and Dobbie [7] studied moving target problems in two cells and Saretsalo [39] extended their studies to the problem in a multi-dimensional Euclidean space. Iida [24], Brown [4] and Washburn [43] also studied the moving target problems and devised computational algorithms to derive a searcher's optimal strategy in a general way.

Subsequently, research of search theory progressed to search game including not only a searcher but also a target as a decision maker. Game theory is usually categorized into cooperative game and non-cooperative game. Non-cooperative search game has two kinds of models: search-evasion game (SEG) and search allocation game (SAG). In both models, the target uses the moving strategy in the search space but the searcher's strategies are different. In the SEG, the searcher has the moving strategy as well as the target, but in the SAG [17], he distributes searching resource in the search space to detect the target.

We list up the past research on the SEG as follows. Danskin [6] formulated the search game between an anti-submarine-helicopter and a submarine as a datum search game and

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derived an optimal dipping position of the helicopter's sonar. Washburn [42] discussed a multi-stage game, in which the searcher makes a decision of the next position to go after knowing the location of a target at every time. Kan [29] took the searcher's search cost as a payoff in a differential game. Nakai [35] studied an optimal target motion on a line with a safety zone. Kikuta [31] investigated a SEG with the criterion of the traveling cost of the searcher. There are other SEG models such as Eagle [8], Eagle and Washburn [9], Hohzaki and Iida [19], Isler et al. [27], Zora et al. [44], Bhattachary [3] and Stipanovic et al. [41].

For the research on the SAG, Iida et al. [25] handled a two-person-zero-sum search game, in which a mobile target chooses a path and a searcher distributes a limited amount of search efforts. Hohzaki et al. [20] and Hohzaki [13] clarified the relationship between two SAGs defined in a continuous search space and in a discrete space. Hohzaki and Washburn [23] applied the SAG to a datum search in a continuous time. Ruckle [38] and Baston and Kikuta [2] dealt with a kind of the SAG called the ambush game, where player I chooses a crossing point on the border of a lattice space and player II puts obstacles to intercept player I's crossing. Dambreville and Le Carde [5] and Hohzaki [16] considered the SAG taking account of some attributes of searching resource. Hohzaki [14] and Kekka and Hohzaki [30] considered the search game with false contacts by the searcher. Hohzaki and Ikeda [21] extended the target strategy to the movement with energy supply policy in their SAG.

There are other types of search games, such as Baston and Garnaev [1], Gal and Howard [10] and Hohzaki [15, 18]. Baston and Garnaev [1] discussed a non-zero-sum game with a protector, who protects the target not to be detected by the searcher. Gal and Howard [10] discussed a zero-sum game under the situation that the searcher does not know whether the target wants to be searched or evade. Hohzaki [15] and Hohzaki [18] modeled the SAG with many cooperative searchers and the SAG with two competitive searchers against the target into a cooperative game and a nonzero-sum game, respectively.

Almost all SAGs mentioned above assume that the searcher knows the target initial position and energy. Unlike the past models of the SAG, Hohzaki and Joo [22] first studied a search game with target private information of its initial position. As well as the initial position, target's movement energy would be considered to become a private information of the target in realistic search operations. The importance of the target's energy can be seen in various situations of military and non-military operations. When the artillery fires at a retreating enemy in the long-distance area, a precise inference about residual fuel of retreating vehicles makes the firing effective or successful. In maritime operations and air-defense operations, a precise estimation on the mobility of suspicious boats or aircrafts would bring a good result to the search operation for them following the report of their invasion. In search and rescue operations, a rescue team is required to have a good estimation on the mobility and the possession of foods and fuel of missing persons or vehicles in addition to their missing point and missing time. As mentioned above, it is extremely important to consider the energy or the mobility of moving targets in search games. Of course, the target knows his initial position and energy at the beginning of the search but the searcher would not. By those reasons, this paper aims to develop a searching game model taking account of the uncertainty of the target's initial position and initial energy on the searcher's side and to derive players' rational decision making.

In the next section, we model a search game with two players, a searcher and a target, in which the target initial state consisting of its initial position and initial energy is the target's private information but is unknown to the searcher. We formulate the problem into a two-person game with incomplete information of the target initial state. In Section 3, we derive an equilibrium point, which consists of the value of the game, an optimal distribution strategy of searching resource and an optimal movement strategy of the target, by enumerating all target paths. We can easily imagine the combinatorial explosion for generating all target paths. To cope with the problem, we propose a methodology for another type of equilibrium point by a Markov movement of the target in Section 4. In Section 5, we do some sensitivity analyses on equilibria with respect to some system parameters involved in the model and then we evaluate the value of the information about the target initial state, which indicates to what extent the searcher increases the detection probability of the target by acquiring the information.

2 A One-Shot Game with Uncertainty of Target's Initial State In a competitive search game between a searcher and a target, the searcher starts a search operation if he senses the existence of the target to some extent in many cases. In the cases, the search happens to start for the target and therefore, in the beginning of the search, an initial state of the target, such as his initial position and initial possession of moving energy, has some randomness such that it seems to be given by nature. The target knows his initial state, of course, and the searcher anticipates the state in a probabilistic way based on information from his sensors. Under these situations, we consider a two-person zero-sum (TPZS) search game between the target and the searcher with detection probability of the target as payoff.

(A1) A search space consists of a finite discrete geographic space and a finite discrete time. The geographic space is represented by a set of cells $\boldsymbol{K} = \{1, \cdots, K\}$ and the time space by a set of time points $\boldsymbol{T} = \{1, \cdots, T\}$.

(A2) In the beginning of the search, nature determines an initial state of the target according to a probability law. An initial position $s \in S_0 \subset \boldsymbol{K}$ and an initial energy e_0 of the target have probability distribution $\{f(s), s \in S_0\}$ ($\sum_{s\in S_0} f(s) = 1$) and $\{g(e_0), e_0 \in E_0\}$ ($\sum_{e_0\in E_0} g(e_0) = 1$), respectively, which are known to both players. S_0 and E_0 are finite countable sets and the random variables given by $f(s)$ and $g(e_0)$ are independent of each other.

(A3) The target moves from its initial position s as time goes by. Its movement has the following constraints. He is allowed to move from cell i at time t to a limited area of cells $N(i, t)$. He consumes energy $\mu(i, j)$ to move from cell i to j. If he exhausts his initial energy e_0, he cannot move anywhere expect for staying at his current cell.

Let us denote all target paths starting from the initial position s until using up energy e_0 by P_{se_0}. The target chooses a path among them and goes along it. If he takes a path $\omega \in P_{se_0}$, he is in cell $\omega(t) \in \boldsymbol{K}$ at time $t \in \boldsymbol{T}$.

(A4) The searcher anticipates the initial cell s and energy e_0 of the target by the probability distribution $\{f(s), s \in S_0\}$ and $\{g(e_0), e_0 \in E_0\}$, respectively, and starts a search operation.

After he is allowed to start the search at time τ, he distributes his searching resource in the space $\boldsymbol{K}$ to detect the target. We denote a time period of search after τ by $\widehat{\boldsymbol{T}} = \{\tau, \tau + 1, \cdots, T\}$. $\Phi(t)$ resources are available and infinitely divisible. Let us denote a distribution plan of resource by $\varphi = \{\varphi(i, t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, where $\varphi(i, t)$ is the amount of the resource distributed in cell i at time t.

(A5) If the searcher scatters x resources in cell i and the target is there, the searcher detects the target with probability

$$1 - \exp(-\alpha_i x). \quad (1)$$

Parameter α_i indicates the efficiency of detection per unit resource in the cell i.

If the searcher detects the target, the game ends and the searcher obtains reward 1 but the target incurs the same amount of loss.

¿From Assumption (A5), the search game is a TPZS game with detection probability as payoff.

Let us begin with enumerating conditions of a feasible path $\omega \in P_{se_0}$ for the target with its initial position s and energy e_0. We call the initial state (s, e_0) the type of the target. Using notation $e(t)$ which indicates residual energy of the target at time t, we express the feasibility conditions of $\omega \in P_{se_0}$, as follows:

(i) Condition of initial position: $\omega(1) = s$

(ii) Condition of reachable cells: $\omega(t+1) \in N(\omega(t), t),\ t = 1, \cdots, T-1$

(iii) Condition of initial energy: $e(1) = e_0$

(iv) Condition of energy conservation: $e(t+1) = e(t) - \mu(\omega(t), \omega(t+1)),\ t = 1, \cdots, T-1$

(v) Condition of movement energy: $\mu(\omega(t), \omega(t+1)) \leq e(t),\ t = 1, \cdots, T-1$

We generate P_{se_0} for the target of type (s, e_0) by enumerating path ω satisfying the conditions above. Reversely, we can calculate $e(t)$ by $e(t) = e_0 - \sum_{\xi=1}^{t-1} \mu(\omega(\xi), \omega(\xi+1))$ for a specific path ω.

We have a feasible region Ψ for a searcher's strategy φ from Assumption (A4), as follows.

$$\Psi \equiv \left\{ \varphi \;\middle|\; \sum_{i \in \boldsymbol{K}} \varphi(i,t) \leq \Phi(t),\ \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}. \tag{2}$$

Now we are going to formulate the payoff function of the game by using the players' pure strategies φ and $\omega \in P_{se_0}$ of the (s, e_0)-type target. At time t, the target is at cell $\omega(t)$ and $\varphi(\omega(t), t)$ searching resources are distributed there. Therefore, from the expression (1), we have the following function as the payoff.

$$R_{se_0}(\varphi, \omega) = 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right).$$

We denote a mixed strategy of the (s, e_0)-type target by $\pi_{se_0} \equiv \{\pi_{se_0}(\omega),\ \omega \in P_{se_0}\}$, where $\pi_{se_0}(\omega)$ is the probability of taking path ω. The feasible region for the mixed strategy π_{se_0} is

$$\Pi_{se_0} \equiv \left\{ \pi_{se_0} \;\middle|\; \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) = 1,\ \pi_{se_0}(\omega) \geq 0, \omega \in P_{se_0} \right\}. \tag{3}$$

The expected payoff, i.e. the detection probability of target, by a pure strategy φ and a mixed strategy π_{se_0} is given by

$$\begin{aligned} R_{se_0}(\varphi, \pi_{se_0}) &= \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) R_{se_0}(\varphi, \omega) \\ &= \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\} \end{aligned} \tag{4}$$

$$= 1 - \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right).$$

The (s, e_0)-type target aims to minimize the payoff. The searcher does not know the type of the target with certainty and he has to evaluate his payoff taking account of all strategies of all types of target, $\pi \equiv \{\pi_{se_0}, s \in S_0, e_0 \in E_0\}$, based on the probability distribution $\{f(s), s \in S_0\}$ and $\{g(e_0), e_0 \in E_0\}$, as follows.

$$(5) \qquad R(\varphi, \pi) = \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) R_{se_0}(\varphi, \pi_{se_0})$$

$$= \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\}$$

$$= 1 - \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right)$$

The searcher wants to maximize the payoff. In the next section, let us discuss the game with the payoff, which looks different at first glance from each side of the searcher and the target, and derive its equilibrium point.

3 Derivation of Equilibrium Point As seen from Eqs. (4) and (5), all types of targets, each of which aims for the minimization of its own payoff $R_{se_0}(\varphi, \pi_{se_0})$, also minimize the comprehensive payoff $R(\varphi, \pi)$ in the aggregate. Therefore, an optimal strategy of the searcher is given by the maximin optimization of $R(\varphi, \pi)$. Let us begin the maximin optimization. We can carry out the minimization of $R(\varphi, \pi)$ with respect to π as follows:

$$(6) \qquad \min_{\pi} R(\varphi, \pi) = \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \min_{\omega \in P_{se_0}} R_{se_0}(\varphi, \omega)$$

$$= \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \min_{\omega \in P_{se_0}} \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\}$$

by making $\pi_{se_0}(\omega) = 0$ for any path $\omega \notin \Omega^{+se_0} \equiv \{\omega \in P_{se_0} | R_{se_0}(\varphi, \omega) = \min_{p \in P_{se_0}} R_{se_0}(\varphi, p)\}$. Furthermore, the maximization of the above minimized value with respect φ gives us a formulation

$$(P_S^0) \qquad \max_{\varphi, \{\nu_{se_0}\}} \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \nu_{se_0}$$

$$s.t. \qquad 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \geq \nu_{se_0}, \ \omega \in P_{se_0}, \ s \in S_0, e_0 \in E_0,$$

$$\varphi \in \Psi.$$

By the replacement of ν_{se_0} with $\eta_{se_0} \equiv -\log(1 - \nu_{se_0})$, i.e., $\nu_{se_0} \equiv 1 - \exp(-\eta_{se_0})$, and noting $\sum_s f(s) = 1$ and $\sum_{e_0} g(e_0) = 1$, we can equivalently transform the above formulation to

$$(P_S) \qquad \max_{\varphi, \eta} \left\{ 1 - \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \exp(-\eta_{se_0}) \right\}$$

$$\text{s.t.} \quad \sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \geq \eta_{se_0},\ \omega \in P_{se_0}, s\in S_0, e_0 \in E_0, \tag{7}$$

$$\sum_{i\in\boldsymbol{K}} \varphi(i,t) = \Phi(t),\ t\in\widehat{\boldsymbol{T}}, \tag{8}$$

$$\varphi(i,t)\geq 0,\ i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}.$$

Because the feasible region of the above problem is a convex set and the objective function is concave, the problem is a convex programming problem. It is easily solved by any general commercial software package of numerical optimization.

Next let us derive an optimal strategy of the (s,e_0)-type target. The target estimates φ^* by solving problem (P_S) first and is going to take an optimal strategy π_{se_0} to minimize his payoff $R_{se_0}(\varphi^*,\pi_{se_0})$ as follows:

$$\begin{aligned}
\min_{\pi_{se_0}} R_{se_0}(\varphi^*,\pi_{se_0}) &= \min_{\omega\in P_{se_0}} R_{se_0}(\varphi^*,\omega) \\
&= \min_{\omega\in P_{se_0}} \left\{ 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t)\right)\right\} \\
&= 1-\exp\left(-\min_{\omega\in P_{se_0}} \sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t)\right) = 1-\exp(-v^*_{se_0}),
\end{aligned}$$

where $v^*_{se_0}$ is given by

$$v^*_{se_0} = \min_{\omega\in P_{se_0}} \sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t).$$

Comparing the above equation with Eq. (7), we can see that $v^*_{se_0}$ equals an optimal value $\eta^*_{se_0}$ and $1-\exp(-\eta^*_{se_0})$ is the minimum detection probability of the (s,e_0)-type target. Using $\eta^*_{se_0}$, we redefine $\Omega^{+se_0} \equiv \{\omega\in P_{se_0} \,|\, \sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t) = \eta^*_{se_0}\}$.

Hereafter, we want to carry out the minimax optimization of $R(\varphi,\pi)$ to derive an optimal strategy of the target. However a direct approach to the optimization would be difficult. Instead, we consider the conditions of the target strategy π to which the optimal searcher's strategy φ^* given by (P_S) becomes an optimal response. On the other hand, an optimal response π to φ^* is given by minimizing a linear function $R(\varphi^*,\pi)$ of variable π or equivalently by setting $\pi_{se_0}(\omega)=0$ for any $\omega\notin\Omega^{+se_0}$, as seen in the transformation (6).

We derive the necessary and sufficient conditions of the optimal response φ^* to π by the so-called Karush-Kuhn-Tucker (KKT) conditions of $\max_\varphi R(\varphi,\pi)$ with respect to $\varphi\in\Psi$. After defining a Lagrange function

$$L(\varphi;\lambda,\eta) \equiv R(\varphi,\pi) + \sum_{t\in\widehat{\boldsymbol{T}}} \lambda(t)\left(\Phi(t)-\sum_{i\in K}\varphi(i,t)\right) + \sum_{(i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}}} \eta(i,t)\varphi(i,t)$$

with Lagrangian multipliers $\{\lambda(t),\ t\in\widehat{\boldsymbol{T}}\}$ and $\{\eta(i,t)\geq 0,\ (i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}}\}$, we have the KKT conditions as follows:

$$\frac{\partial L}{\partial\varphi(i,t)} = \frac{\partial R(\varphi,\pi)}{\partial\varphi(i,t)} - \lambda(t) + \eta(i,t) = \alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s) \sum_{\omega\in\Omega^{se_0}_{it}} \pi_{se_0}(\omega) \tag{9}$$

$$\times \exp\left(-\sum_{t'\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t')}\varphi(\omega(t'),t')\right) - \lambda(t) + \eta(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$

$$\varphi(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$

$$\sum_{i\in K} \varphi(i,t) = \Phi(t),\ t \in \widehat{\boldsymbol{T}},$$

(10) $$\eta(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$

(11) $$\eta(i,t)\varphi(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}.$$

In the conditions, we use notation $\Omega_{it}^{se_0} \equiv \{\omega \in P_{se_0} | \omega(t) = i\}$. From the previous discussion about optimal target strategy and $v^*_{se_0}$, optimal π_{se_0} should be $\pi_{se_0}(\omega) = 0$ for $\omega \notin \Omega^{+se_0}$ and the condition $\sum_{t'} \alpha_{\omega(t')}\varphi^*(\omega(t'),t') = \eta^*_{se_0}$ holds for any path $\omega \in \Omega^{+se_0}$ with positive selection probability $\pi_{se_0}(\omega) > 0$. Using these expressions, we can rewrite Eq. (9) into

(12) $$\alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s) \exp(-\eta^*_{se_0}) \sum_{\omega\in\Omega_{it}^{+se_0}} \pi_{se_0}(\omega) - \lambda(t) + \eta(i,t) = 0,$$

$$(i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$

where

$$\Omega_{it}^{+se_0} \equiv \{\omega \in P_{se_0} |\ \omega(t) = i,\ \sum_{t'\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t')}\varphi^*(\omega(t'),t') = \eta^*_{se_0}\}.$$

Let us simplify the conditions (9), (10) and (11), as follows. If $\varphi^*(i,t) > 0$, we have $\eta(i,t) = 0$ from Eq. (11) and then

(13) $$\alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s) \exp(-\eta^*_{se_0}) \sum_{\omega\in\Omega_{it}^{+se_0}} \pi_{se_0}(\omega) = \lambda(t)$$

from Eq. (12). If $\varphi^*(i,t) = 0$, from Eq. (10), we have

(14) $$\alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s) \exp(-\eta^*_{se_0}) \sum_{\omega\in\Omega_{it}^{+se_0}} \pi_{se_0}(\omega) \leq \lambda(t).$$

Reversely, if the above two conditions hold, we can easily generate optimal multipliers $\{\lambda^*(t)\}$ and $\{\eta^*(i,t)\}$ so as to satisfy the KKT conditions. Anyway, the original feasibility conditions of π_{se_0} are given by Π_{se_0} of Eq. (3).

We have discussed the conditions of an optimal target strategy $\pi = \{\pi_{se_0}, s \in S_0, e_0 \in E_0\}$ so far. If π satisfies all the conditions derived so far, the optimal searcher's strategy φ^* given by problem (P_S) is optimal to π and, at the same time, π is optimal to φ^*. The two-sided optimality validates that π is in an equilibrium of the game. Summing up the discussion so far, we have a linear programming problem to derive an optimal target strategy π, as follows.

$$(P_T) \quad \min_{\pi,\lambda} \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s) \sum_{\omega\in P_{se_0}} \pi_{se_0}(\omega) \left\{1 - \exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t)\right)\right\}$$

s.t.

$$\alpha_i \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \exp(-\eta^*_{se_0}) \sum_{\omega \in \Omega^{+se_0}_{it}} \pi_{se_0}(\omega) = \lambda(t)$$
$$\textit{for } (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}} \textit{ of } \varphi^*(i,t) > 0,$$
$$\alpha_i \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \exp(-\eta^*_{se_0}) \sum_{\omega \in \Omega^{+se_0}_{it}} \pi_{se_0}(\omega) \le \lambda(t)$$
$$\textit{for } (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}} \textit{ of } \varphi^*(i,t) = 0,$$
$$\sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) = 1,\ s \in S_0,\ e_0 \in E_0,$$
$$\pi_{se_0}(\omega) \ge 0,\ \omega \in P_{se_0},\ s \in S_0,\ e_0 \in E_0.$$

4 Markov Movement Strategy of Target In Section 3, we enumerate all target paths taking account of the target movement constraints in Assumption (A3) in Section 2 and use the path set $\{P_{se_0}, s \in S_0, e_0 \in E_0\}$ to derive an equilibrium point of the game. The proposed formulation is inconvenient for the game with a lot of time points because the number of target paths would increase at an exponential rate of the number of time points. To cope with the exponential expansion of the number of paths, we define a strategy of the target by Markov movement strategy, which was first introduced into search games by Eagle and Washburn [9] and sophisticated by Hohzaki et al. [20]. We represent a state of target by (s, e_0, i, t, e), where (s, e_0) is a target type, and i, t and e are the current cell, the present time and the residual energy, respectively, or a state of the (s, e_0)-type target by (i, t, e). By the Markov strategy, the target specifies the movement from the state (s, e_0, i, t, e) to a cell at the next time $t + 1$ in a probabilistic manner. The Markov strategy at time t depends on not the past tracks of path before t but just a state at the present time t. We will show the equivalence between the path selection strategy discussed in the previous section and the Markov movement strategy later in the process of deriving an equilibrium point for the Markov strategy. Anyway, we have to discriminate between the beginning point of time t and the ending point of the time. Because the search operation is conducted between the two points, the target lies at different levels at two time points from the survival point of view even if the target is in the same state. We refer to the former time point as the BP of time t and the latter one as the EP of t.

Let us denote all energy states of the (s, e_0)-type target by $\boldsymbol{F}_{e_0} \equiv \{0, 1, \cdots, e_0\}$. To make use of dynamic programming, we define an optimized value $z_{se_0}(i, t, e)$ as the maximized non-detection probability of the (s, e_0)-type target who is in a state (i, t, e) at the BP of time t and takes his optimal movement strategy since then until the end of the game. In this section, we adopt the non-detection probability as the payoff for the sake of formulation. Variable $v_{se_0}(i, j, t, e)$ represents a Markov strategy of the (s, e_0)-type target and indicates the probability that the target in the state (i, t, e) moves to cell j at the next time $t + 1$. Let us denote a set of cells to which the target can move at time $t + 1$ from (i, t, e) by $N(i, t, e) \equiv \{j \in \boldsymbol{K} | j \in N(i, t),\ \mu(i, j) \le e\}$ and a set of cells at the previous time $t - 1$, from which the target can transition to the state (i, t, e) at time t, by $N^*_{e_0}(i, t, e) \equiv \{j \in \boldsymbol{K} | i \in N(j, t - 1),\ e + \mu(j, i) \le e_0\}$.

The feasibility conditions of the Markov strategy v_{se_0} are given by

(15) $$V_{se_0} \equiv \{\{v_{se_0}(i, j, t, e), i, j \in \boldsymbol{K}, t \in \boldsymbol{T} \backslash \{T\}, e \in \boldsymbol{F}_{e_0}\} \mid \sum_{j \in N(i,t,e)} v_{se_0}(i, j, t, e) = 1,\ v_{se_0}(i, j, t, e) = 0 (j \notin N(i, t, e)),\ v_{se_0}(i, j, t, e) \ge 0\}.$$

Before the main discussion of deriving an equilibrium, we prove the equivalency between the path selection strategy $\{\pi_{se_0}(\omega)\}$ and the Markov strategy $\{v_{se_0}(i, j, t, e)\}$ of the (s, e_0)-type target. We accomplish the proof by showing that one expression is transformable from the other one as follows, using notation $e(\omega, n) \equiv e_0 - \sum_{k=1}^{n-1} \mu(\omega(k), \omega(k+1))$:

$$\pi_{se_0}(\omega) = \prod_{t=1}^{T-1} v_{se_0}(\omega(t), \omega(t+1), t, e(\omega, t)) \ for \ \omega \in P_{se_0},$$

$$v_{se_0}(i, j, t, e) = \frac{\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e, \omega(t+1)=j\}} \pi_{se_0}(\omega)}{\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e\}} \pi_{se_0}(\omega)}.$$

If $\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e\}} \pi_{se_0}(\omega)$ becomes zero in the denominator, the state (s, e_0, i, t, e) is not reachable and any Markov strategy $v_{se_0}(i, j, t, e)$ is allowable.

We denote a strategy of the searcher by a distribution of searching resource $\{\varphi(i, t)\}$, as same as in Section 3. Considering the transition that the target in state (i, t, e) remains surviving from the search operation at time t and goes to cell j at the next time $t+1$, the optimized value $z_{se_0}(i, t, e)$ has the following recursive equation at any time $t \in \widehat{\boldsymbol{T}}$:

$$z_{se_0}(i, t, e) = \max_{\{v_{se_0}(i,j,t,e), j \in N(i,t,e)\}} e^{-\alpha_i \varphi(i,t)} \sum_{j \in N(i,t,e)} v_{se_0}(i, j, t, e) z_{se_0}(j, t+1, e - \mu(i, j)).$$

Taking account of the feasibility condition V_{se_0} of Eq. (15), we further transform the above expression to

$$\begin{aligned} (16) \qquad z_{se_0}(i, t, e) &= \max_{j \in N(i,t,e)} e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)) \\ &\geq e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)). \end{aligned}$$

In a similar manner, we have the following equation during a time period $\boldsymbol{T} \backslash \widehat{\boldsymbol{T}}$ with no search operation:

$$\begin{aligned} (17)\ z_{se_0}(i, t, e) &= \max_{\{v_{se_0}(i,j,t,e), j \in N(i,t,e)\}} \sum_{j \in N(i,t,e)} v_{se_0}(i, j, t, e) z_{se_0}(j, t+1, e - \mu(i, j)) \\ &= \max_{j \in N(i,t,e)} z_{se_0}(j, t+1, e - \mu(i, j)) \geq z_{se_0}(j, t+1, e - \mu(i, j)). \end{aligned}$$

An equation $z_{se_0}(i, T, e) = \exp(-\alpha_i \varphi(i, T))$ holds at the last time T. Because the maximized non-detection probability of the (s, e_0)-type target over an entire time points is given by $z_{se_0}(s, 1, e_0)$ from its definition, the searcher wants to minimize its expectation $\sum_{e_0} \sum_s g(e_0) f(s) z_{se_0}(s, 1, e_0)$ to obtain a minimax value (a maximin value for the original payoff of the detection probability of target). From the discussion so far, we formulate the minimax optimization into the following problem.

$$\begin{aligned} (P_S^{M0}) \quad & \min_{\varphi, z} \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) z_{se_0}(s, 1, e_0) \\ s.t. \quad & z_{se_0}(i, t, e) \geq z_{se_0}(j, t+1, e - \mu(i, j)), \\ & \qquad j \in N(i, t, e), i \in \boldsymbol{K}, t \in \boldsymbol{T} \backslash \widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & z_{se_0}(i, t, e) \geq e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)), \\ & \qquad j \in N(i, t, e), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}} \backslash \{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & z_{se_0}(i, T, e) = e^{-\alpha_i \varphi(i,T)}, \ i \in \boldsymbol{K}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & \sum_{i \in K} \varphi(i, t) = \Phi(t), \ t \in \widehat{\boldsymbol{T}}, \\ & \varphi(i, t) \geq 0, \ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}. \end{aligned}$$

We substitute $w_{se_0}(i,t,e) \equiv -\log z_{se_0}(i,t,e)$ for $z_{se_0}(i,t,e)$ to have a formulation

$$
\begin{aligned}
(P_S^M) \quad & \min_{\varphi,w} \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s)\exp(-w_{se_0}(s,1,e_0)) \\
s.t. \quad & w_{se_0}(i,t,e) \le w_{se_0}(j,t+1,e-\mu(i,j)), \\
& \qquad j\in N(i,t,e), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in \boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& w_{se_0}(i,t,e) \le \alpha_i\varphi(i,t) + w_{se_0}(j,t+1,e-\mu(i,j)), \\
& \qquad j\in N(i,t,e), i\in \boldsymbol{K}, t\in \widehat{\boldsymbol{T}}\backslash\{T\}, e\in \boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& w_{se_0}(i,T,e) = \alpha_i\varphi(i,T),\ i\in\boldsymbol{K}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& \sum_{i\in K}\varphi(i,t) = \Phi(t),\ t\in\widehat{\boldsymbol{T}}, \\
& \varphi(i,t)\ge 0,\ i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}.
\end{aligned}
$$

Because $z_{se_0}(i,t,e)$ lies in $0 < z_{se_0}(i,t,e) \le 1$ from its definition, $w_{se_0}(i,t,e)$ is nonnegative. The formulation (P_S^M) is a convex minimization problem. In the formulation, there are some variables with no effect on the optimal value, such as $\{w_{se_0}(i,1,e), i\ne s, e\ne e_0\}$. It might be good to set these variables zeros. The setting corresponds to making variables z_{se_0} 1s in the formulation (P_S^{M0}). The variable setting also does not generate any constraint in the problem and therefore they do not have any effect on the optimal value of (P_S^{M0}) at all.

Hereafter, we are going to derive an optimal Markov strategy of the target by using optimal solutions φ^* and $w^*_{se_0}$ already obtained from problem (P_S^M) and $z^*_{se_0}$ from problem (P_S^{M0}). From the definition of $z^*_{se_0}(i,t,e)$, $\widehat{z}^*_{se_0}(i,t,e) \equiv z^*_{se_0}(i,t,e)\exp(\alpha_i\varphi^*(i,t))$ is the maximum non-detection probability after time t given by an optimal movement of the target conditioned that the (s,e_0)-type target is surviving in state (i,t,e) at the EP of the time t. As the Markov movement strategy, we temporarily adopt variables $\{\widehat{v}_{se_0}(i,j,t,e),\ i,j\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{F}_{e_0}\}$ other than variable v_{se_0} for the expressional sake. $\widehat{v}_{se_0}(i,j,t,e)$ indicates the probability that the (s,e_0)-type target has not been detected since the beginning, is in state (i,t,e) at the EP of time t and moves to cell j at the next time $t+1$. The movement strategy indirectly affects the following probabilities. $q_{se_0}(i,t,e)$ is the probability that the (s,e_0)-type target reaches state (i,t,e) at the BP of t with no detection. $q'_{se_0}(i,t,e)$ is the probability that the (s,e_0)-type target reaches state (i,t,e) at the EP of t with no detection.

Considering the state transition of the (s,e_0)-type target, we have the following equations.

$$
\begin{aligned}
& q_{se_0}(s,1,e_0) = 1, s\in S_0, e_0\in E_0, \\
& \sum_{i\in K}\sum_{e\in F_{e_0}} q_{se_0}(i,1,e) = 1, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = q_{se_0}(i,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = q_{se_0}(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q_{se_0}(i,t,e) = \sum_{j\in N^*_{e_0}(i,t,e)} \widehat{v}_{se_0}(j,i,t-1,e+\mu(j,i)), \\
& \qquad\qquad i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{1\}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = \sum_{j\in N(i,t,e)} \widehat{v}_{se_0}(i,j,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0.
\end{aligned}
$$

Focusing a distribution of searching effort, $\{\varphi(i,t), i\in\boldsymbol{K}\}$, at time t, we have an expression for the non-detection probability.

$$h_t(\varphi) \equiv \sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{i\in K}\sum_{e\in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\exp(-\alpha_i\varphi(i,t))\widehat{z}^*_{se_0}(i,t,e)$$

The optimal distribution $\{\varphi^*(i,t), i\in \boldsymbol{K}\}$ at time t must be an optimal solution of the minimization problem of the above objective under constraints of $\sum_i \varphi(i,t) = \Phi(t)$ and $\varphi(i,t)\ge 0$ $(i\in\boldsymbol{K})$. After defining a Lagrange function by

$$\begin{aligned}L(\varphi;\lambda,\mu) \quad\equiv\quad & \sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{i\in K}\sum_{e\in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\exp(-\alpha_i\varphi(i,t))\widehat{z}^*_{se_0}(i,t,e)\\ & +\lambda(t)\left(\sum_{i\in K}\varphi(i,t)-\Phi(t)\right)-\sum_{i\in K}\mu(i,t)\varphi(i,t),\end{aligned}$$

we find KKT conditions as follows.

$$\begin{aligned}(18)\qquad & \frac{\partial L}{\partial\varphi(i,t)} = -\alpha_i\exp(-\alpha_i\varphi(i,t))\sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{e\in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e)\\ & \qquad\qquad +\lambda(t)-\mu(i,t)=0,\ i\in\boldsymbol{K},\\ (19)\qquad & \mu(i,t)\ge 0,\ i\in\boldsymbol{K},\\ (20)\qquad & \mu(i,t)\varphi(i,t)=0,\ i\in\boldsymbol{K},\\ (21)\qquad & \sum_{i\in K}\varphi(i,t)=\Phi(t),\\ (22)\qquad & \varphi(i,t)\ge 0,\ i\in\boldsymbol{K}.\end{aligned}$$

We reconstruct conditions (18)∼(20) into equivalent conditions:

(i) If $\varphi(i,t)>0$,

$$(23)\qquad \alpha_i\exp(-\alpha_i\varphi(i,t))\sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{e\in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e)=\lambda(t).$$

(ii) If $\varphi(i,t)=0$,

$$(24)\qquad \alpha_i\sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{e\in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e)\le\lambda(t).$$

The total non-detection probability is expressed by $\sum_{e_0,s,i,e} g(e_0)f(s)q'_{se_0}(i,T,e)$ as well as $h_t(\varphi)$. Now let us confirm the followings. First, an optimal Markov movement strategy $\widehat{v}^*$ maximizes the total non-dection probability. Secondly, if the conditions (23) and (24) are valid for arbitrary $i\in\boldsymbol{K}$ and $t\in\widehat{\boldsymbol{T}}$, φ becomes an optimal response to the Markov strategy $\widehat{v}$. The discussion so far helps us formulate a linear programming problem to derive an optimal Markov strategy $\widehat{v}^*$ by using already-obtained φ^* and $\widehat{z}^*$.

$$\begin{aligned}(P_T^M)\qquad & \max_{\widehat{v},q,q',\lambda}\sum_{e_0\in E_0}\sum_{s\in S_0}\sum_{i\in\boldsymbol{K}}\sum_{e\in F_{e_0}} g(e_0)f(s)q'_{se_0}(i,T,e),\\ s.t.\qquad & q_{se_0}(s,1,e_0)=1, s\in S_0, e_0\in E_0,\\ & \sum_{i\in K}\sum_{e\in F_{e_0}} q_{se_0}(i,1,e)=1, s\in S_0, e_0\in E_0,\end{aligned}$$

$$q'_{se_0}(i,t,e) = q_{se_0}(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q'_{se_0}(i,t,e) = q_{se_0}(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q_{se_0}(i,t,e) = \sum_{j \in N^*_{e_0}(i,t,e)} \widehat{v}_{se_0}(j,i,t-1,e+\mu(j,i)),$$
$$i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{1\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q'_{se_0}(i,t,e) = \sum_{j \in N(i,t,e)} \widehat{v}_{se_0}(i,j,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$\alpha_i \exp(-\alpha_i\varphi^*(i,t)) \sum_{e_0 \in E_0}\sum_{s \in S_0}\sum_{e \in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e) = \lambda(t)$$
$$for\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t) > 0,$$
$$\alpha_i \sum_{e_0 \in E_0}\sum_{s \in S_0}\sum_{e \in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e) \le \lambda(t)$$
$$for\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t) = 0,$$
$$\widehat{v}_{se_0}(i,j,t,e) \ge 0,\ i,j \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$\widehat{v}_{se_0}(i,j,t,e) = 0,\ j \notin N(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0.$$

Using the optimal solution $\widehat{v}^*_{se_0}(i,j,t,e)$ of the problem (P^M_T), we can reconstruct an optimal form of the original Markov strategy $v^*_{se_0}(i,j,t,e)$, as follows:

$$v^*_{se_0}(i,j,t,e) = \frac{\widehat{v}^*_{se_0}(i,j,t,e)}{\sum_{j \in N(i,t,e)} \widehat{v}^*_{se_0}(i,j,t,e)}. \tag{25}$$

$\widehat{v}^*$ includes the reachability of the target with no detection and there could be some cases that the denominator of the formula is zero for some state (s,e_0,i,t,e). The cases indicate the impossibility of the state (s,e_0,i,t,e) for the target in an optimal movement. For the state (s,e_0,i,t,e), we do not have to specify $v_{se_0}(i,j,t,e)$ or any Markov strategy $v_{se_0}(i,j,t,e)$ is allowed.

5 Numerical Example In this section, we apply our methodology proposed in previous sections to some numerical examples to analyze the features of optimal player's strategy.

We consider a discrete cell space $\boldsymbol{K} = \{1,\ldots,19\}$, shown by Fig. 1.

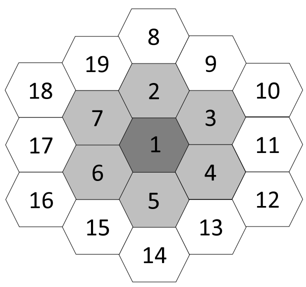

Figure 1: A search space

We set a discrete time space by $\boldsymbol{T} = \{1,2,3\}$ and a searching period by $\hat{\boldsymbol{T}} = \{2,3\}$. A searcher uses available searching resource $\Phi(2) = \Phi(3) = 1$ at each time point of $\hat{\boldsymbol{T}}$. The efficiency of detection of cell i, α_i, is set as follows : $\alpha_1 = 0.5,\ \alpha_2 = \cdots = \alpha_7 = 0.6,\ \alpha_8 =$

$\cdots = \alpha_{19} = 0.7$. Cell 1 in the center has the smallest efficiency and α_i becomes larger as the cell i is located farther from the center. The efficiency is also represented by the gradation of black color in the figure. Darker cell has the smaller efficiency. The target's initial position is assumed to be cell 1 or 2, $S_0 = \{1, 2\}$, and its initial energy 1 or 4, $E_0 = \{1, 4\}$. The searcher infers the target's initial position based on probabilities $f(1)$ and $f(2)$ $\big(f(1)+f(2) = 1\big)$ and the initial energy by $g(1)$ and $g(4)$ $\big(g(1)+g(4) = 1\big)$. The target can move from a present cell i to its neighboring cell j by consuming energy $\mu(i, j) = 1$ and move to its 2nd-neighboring cell j by consuming $\mu(i, j) = 4$ while staying at the same cell needs no energy, $\mu(i, i) = 0$. From now on, we are going to analyze the player's optimal strategy. We compute an optimal searcher's strategy φ^* and an optimal target strategy π^* by solving problem (P_S) and (P_T) in Section 3, respectively.

5.1 Features of optimal strategies Fig. 2 shows the value of game in the case of $f(1) = f(2) = 0.5$ while changing $g(1)$ from 0 through 1 at intervals of 0.1. ¿From now on, we analyze the curve of Fig. 2 by looking at searcher's and target's optimal strategies in detail.

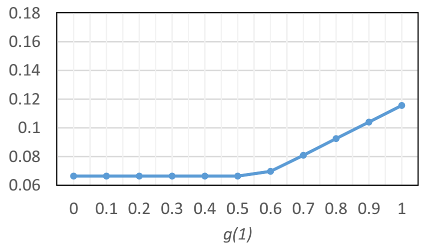

Figure 2: Value of the game in the case of $f(1) = f(2) = 0.5$

(1) Features of optimal searcher's strategy

Changing $g(1)$ from 0 through 1 at intervals of 0.1, we derive the searcher's optimal distribution of his searching resource, φ^*, at $t = 2, 3$ and illustrate it in Fig. 3 just for $g(1) = 0, 0.1, \ldots, 0.5$. For other cases of $g(1) = 0.6, 0.7, \ldots, 1.0$, we show the results in Table 1 without using space-consuming figures. A blank with no number indicates zero.

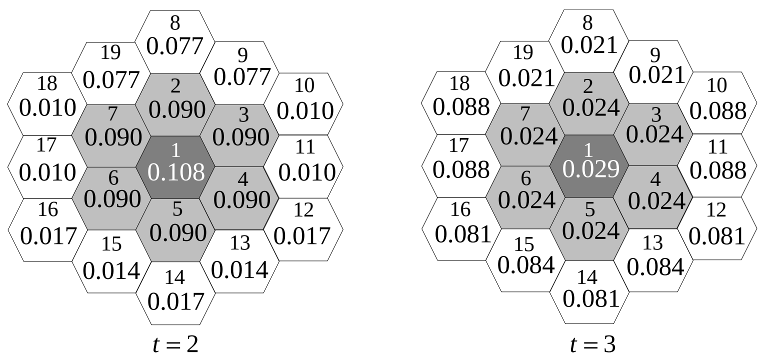

Figure 3: Optimal resource distribution in the case of $g(1) = 0 \sim 0.5$

In Fig. 3, the optimal distribution of resource keeps same even though $g(1)$ changes. Because in the case of low $g(1)$ the target has high energy 4 with high probability and it can move quickly to peripheral cells $8 \sim 19$ at the early time $t = 2$, the searcher has to widely spread its searching resource to all cells. The unchangingness of the optimal resource distribution corresponds to an optimal distribution of target's existence which also spreads to all cells at early time for $g(1) = 0 \sim 0.5$. We check it later, though. As seen in Fig. 3, the searcher distributes larger amount of searching resource in smaller numbered cells at $t = 2$ but in larger numbered cells at $t = 3$. This result explains that the searcher gradually widens its focal area of search considering the target movement. Since cell $s = 2$ is an initial target position as well as cell 1, larger amount of searching resource are distributed in its neighboring cells $8, 9$ and 19 at $t = 2$ compared to the other peripheral cells $10, \cdots, 18$.

Table 1: Optimal resource distribution in the case of $g(1) = 0.6 \sim 1.0$

Cell \ $g(1)$	$g(1) = 0.6$		$g(1) = 0.7$		$g(1) = 0.8$		$g(1) = 0.9$		$g(1) = 1.0$	
	t=2	t=3	t=2	t=3	t=2	t=3	t=2	t=3	t=2	t=3
1	.177	.031	.219	.027	.221	.025	.221	.025	.225	.021
2	.148	.026	.182	.022	.184	.021	.184	.021	.187	.017
3	.105	.069	.110	.094	.111	.094	.111	.094	.060	.144
4	.063	.110	.083	.122	.081	.124	.081	.124	.086	.119
5	.072	.102	.086	.119	.084	.120	.085	.120	.086	.119
6	.063	.110	.083	.122	.081	.124	.081	.124	.086	.119
7	.105	.069	.110	.094	.111	.094	.111	.094	.060	.144
8	.071	.077	.033	.143	.033	.143	.033	.142	.070	.106
9	.075	.074	.048	.128	.047	.128	.047	.128	.070	.106
10	.011	.065								
11	.012	.064								
12										
13										
14										
15										
16										
17	.012	.064								
18	.011	.065								
19	.075	.074	.048	.128	.047	.128	.047	.128	.070	.106
Total	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

As seen from Table 1, as $g(1)$ becomes larger and the target more likely has smaller energy, the searcher distributes more resource in smaller areas in the vicinity of target's initial positions. There is no searching resource distributed in cells $12, \cdots, 16$ in the case of $g(1) = 0.6$ and in cells $10, \cdots, 18$ in the case of $g(1) \geq 0.7$. As $g(1)$ becomes larger, the possible area of target is getting smaller and the searcher carries out more effective search by concentrating his resource on the smaller area.

(2) Features of optimal target's strategy

Here we analyze optimal target strategy. Fig. 4 shows the probability distribution of target's existence weighted over all types of (s, e_0) at time 2 and 3. The probability of target's existence in cell i at time t is calculated by $\sum_{s\in S_0} \sum_{e_0\in E_0} f(s)g(e_0) \sum_{\omega\in\Omega_{it}^{+se_0}} \pi^*_{se_0}(\omega)$. ¿From Fig. 4, we can pick up main features of the optimal target strategy: "diffusiveness", "uniformity" and "preference to ineffective cell of search".

(i) **Diffusiveness**: The target possible area with positive probability quickly spreads to all cells even at the early time $t = 2$ and the target distribution is kept stable although there is a small bias based on efficiency parameter α_i of each cell i. The quick diffusion and spread of the target distribution over wider area is preferable for the target because it intervenes

the searcher's effective search of concentrating searching resource in small area.
(ii) **Uniformity**: From the Fig. 4, the target seems to move intentionally keeping the target distribution uniform all through the searching period and all over the cells. The target tries to deteriorate the detection probability by inducing the dispersion of searching resource by the uniform target distribution.
(iii) **Preference to ineffective cell of search**: The target changes his probability distribution such that more probabilities are allocated in the cells with smaller efficiency α_i. It would be natural that the target keeps off more detectable cells with larger α_i.

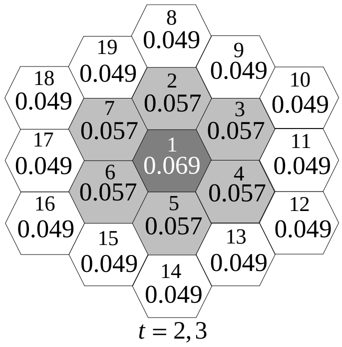

Figure 4: Target's Existence Probability in the case of $0 \leq g(1) \leq 0.5$

By the detail analysis on the optimal player's strategies mentioned above, we can explain the change of the detection probability for varying $g(1)$ in Fig. 2, as follows. In the case of $0 \leq g(1) \leq 0.5$, in which the possibility of high-mobile target is high, the detection probability remains low because the target distribution diffuses widely and uniformly all over the whole space and prevents the searcher from doing an effective search by the concentration of searching resource. On the other hand, in case of $0.6 \leq g(1) \leq 1.0$, the target's possible area is restricted by his poor mobility and the searcher distributes his resource intensively in the restricted area to increase the detection probability.

5.2 Sensitivity analysis on the value of the game and valuation of information

Here we analyze the expected detection probability of the target and the value of information. We change $f(1)$ and $g(1)$ from 0 through 1.0 at intervals of 0.1 and illustrate the value of the game or the detection probability in Fig. 5. Taking $g(1)$ on the x-axis and $f(1)$ on the y-axis, we depict the value of game on the z-axis in Fig. 5. Larger $g(1)$ indicates higher probability of target's having initial energy 1 and larger $f(1)$ higher probability of target's being in initial cell 1. Three curves of Fig. 6 are two-dimensional versions of Fig. 5, given by fixing $g(1)$ to $0, 0.5$, and 1.0, respectively. Figure 7 shows the detection probability with respect to $g(1)$ for each of fixed $f(1) = 0, 0.5$, and 1.0, respectively.

All curves of Fig. 6 have bathtub curves because an ambiguous position of target around $f(1) = 0.5$ gives the searcher some disadvantage that the searcher has to take account of a variety of target paths starting from two initial positions. On the other hand, knowing certainly the initial position near $f(1) = 0$ or 1 helps the searcher concentrate searching resource on the paths starting from the identified position for an effective search. When the target has smaller initial energy, he has less options of paths. That is why the curve of the detection probability takes higher position as $g(1)$ increases, as seen by the comparison among three cases of $g(1) = 0$, 0.5 and 1.0.

We have already analyzed the case of $f(1) = 0.5$ of Fig. 7 in section 5.1. Our analysis still works for other curves of Fig. 7. The detection probability monotonously increases as

$g(1)$ gets larger and it increases sharply over around the center of $g(1) = 0.5$. From Fig. 6, we have found the bathtub form in the change of the detection probability with respect to $f(1)$. By the 'bathtub' effect, the curve of $f(1) = 0.5$ takes a lower position than in the case $f(1) = 0$ but the curve of $f(1) = 1.0$ is located in a little higher position than one of $f(1) = 0.5$.

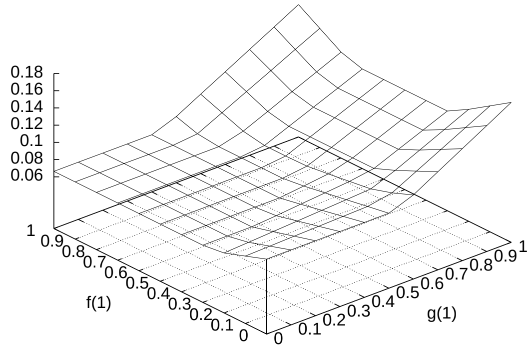

Figure 5: Value of the game for varying $f(1)$ and $g(1)$

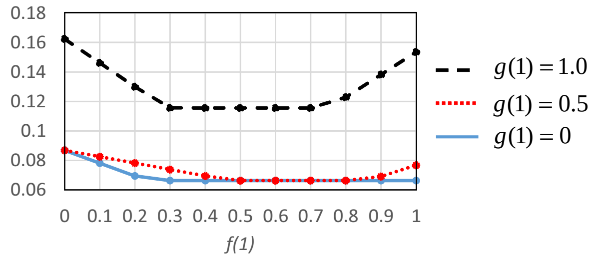

Figure 6: Value of the game for varying $f(1)$

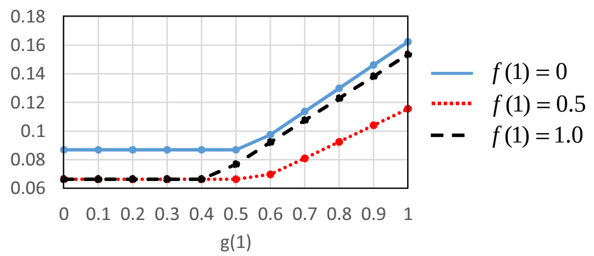

Figure 7: Value of the game for varying $g(1)$

As the last analysis, we evaluate the value of the information about the target's initial state. The value of information can be estimated from the difference between two values of the games, which are the games with private information or common knowledge about the initial state. The value of the game with common knowledge is computed by Hohzaki et al. [20]. The difference indicates how much detection probability the searcher can increase by knowing the target's initial state with certainty. Figure 8 shows the value of information for varying $g(1)$ and $f(1)$. The value reaches its maximum 0.0557 around the point of $f(1) = 0.5$ and $g(1) = 0.6$. Since the target's initial position and energy are the vaguest for the searcher around the point, the information has the maximum value if obtained.

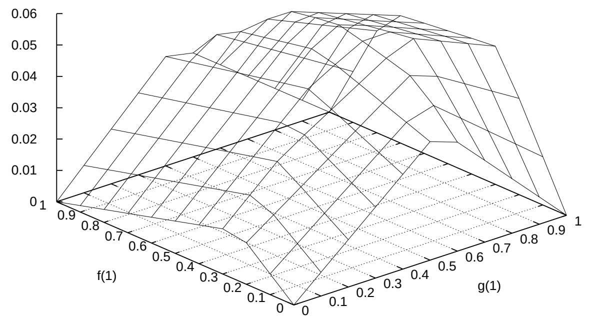

Figure 8: Value of information about target initial state

6 Conclusions In this study, we have discussed a SAG considering the targets' initial state consisting of its initial position and initial energy as a private information of the target which the searcher doesn't know. In rescue operations or military operations, however, the searcher does not start his search without any information of target's state, being concerned about the waste of search efforts in a wide operational area. The information about the target's initial state is very important for the searcher to restrict the target's possible area and make an effective use of searching resource in the restricted area.

Under the above background, we aimed to quantify the importance of the information about the target's initial state by deriving searcher's and target optimal strategies in the search game. We proposed two methods to derive an equilibrium point for the game. We also analyzed the equilibrium by some numerical examples.

¿From our analysis, we found that the target's initial energy has a great impact on the efficiency of the search. When a high-energy target is predicted to appear, the searcher has to distribute his searching resource widely, which makes the search less efficient. On the other hand, when the target is predicted to have low energy, the searcher would be able to concentrate his search efforts into comparatively small areas to make the search more efficient and bring large detection probability.

We also found three features of the targets' optimal strategy: diffusiveness, uniformity and preference to ineffective cell of search. Although the target tends to go to ineffective cells of search, he does not stay at those cells but he keeps going to expand his existence area while maintaining his probability distribution as uniform as possible over the areas with the same parameter of detection efficiency.

As for the value of the information about the target's initial state, we clarified that the value reaches its maximum when the searcher's anticipation about the target state is around the vaguest from the quantitative point of view. These results we have from our analysis are compatible with our common sense and could become precepts for practical searches.

Lastly we would like to mention our future works. In this study, we modeled our search game into a one-shot SAG. That is why the players are assumed to acquire some information about their opponents just at the beginning of the game but not to obtain any information in the middle of playing the game. In many realistic operations, however, the searcher makes efforts to update the information time by time. To examine such a situation, we need to extend our model to a multi-stage game with the change of the strategy by updated information. In some search operations such as anti-submarine warfare, it is conceivable that an evading target refuels its energy to maintain its mobility like a submarine. To investigate this problem, we require a model with an additional target strategy of energy refueling.

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