COMBINATION OF OPTIMAL STOPPING ALGORITHMS

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Abstract. In this paper we investigate the possibility of combination two optimal stopping algorithms: Odds algorithm and Elimination algorithm. We show how reduce a problem to monotone problem and after this step find the optimal strategy which will be valid also in the original problem.

1 Introduction Bruss (2000) in [3] developed Odds algorithm which is very simple tool used to solve optimal stopping problems. In this model observe sequence of independent indicators and want to stop on the last (if any) success. Extension of this idea was presented in [4] and [9]. Different approaches are presented in work of Dendievel [6]. The result of Bruss' can be obtain in another way if we focus on monotonicity of a problem of selecting last success in sequence of events. However there are some problems which are not monotone and therefore Odds algorithm can give us strategy that is not optimal. Sonin (1999) in [13] presented so called Elimination Algorithm (EA) for solving optimal stopping problems (OSP). The idea is to combine this two algorithms by reducing original problem to monotone problem using EA and then find the optimal strategy by One-Step-Look-Ahead (1-SLA) method. Similar work was done by Ferguson [8]. This problem was also considered by Ano [1].

2 Optimal stopping for unobservable event Let a probability space $(\Omega, \mathcal{G}, P)$ be given and let $\{X_k\}_{k=1}^{\infty}$ be a sequence of random variables whose joint distribution is known. Let $\mathcal{F}_k = \sigma(X_1, ..., X_k)$ be a sigma field generated by $X_1, ..., X_k$ (natural filtration). In many cases we deal with Markov chain. We assume that we have finite horizon n. Define function $g_k((X_1, ...X_k))$ and call it reward function. g_k is $\mathcal{F}_k$ measurable. Further we will denote $g_k((X_1, ...X_k))$ as G_k. We observe X_k sequentially. The goal is to stop observation on index i for which reward function reach the maximum value. The triplet (space, filtration, function) we will call an optimal stopping problem (OSP).

Definition 1. *Let A_k denote a set $\{G_k \geq E[G_{k+1}|\mathcal{F}_k]\}$. We say that the stopping rule problem is monotone if*

$$A_0 \subset A_1 \subset A_2 \subset ... \ a.s. \tag{1}$$

One of the simplest stopping rule is known as One-Step-Look-Ahead (OSLA or 1-SLA). The 1-SLA is the rule which calls for stopping on the first k for which the return for stopping is greater or equal as the expected return of continuing one step and then stopping.

Definition 2. *1-SLA is described by the stopping time*

$$\nu_1 = \min\{k \geq 0 : G_k \geq E[G_{k+1}|\mathcal{F}_k]\}.$$

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Theorem 1. *In a finite horizon monotone stopping rule problem, the 1-SLA rule is optimal.*

The proof of this fact is here omitted. It can be found in [7]

3 Odds theorem Idea is that we consider n independent indicators $I_k, 1 \leq k \leq n$ observed sequentially. If the indicator on place k has value 1 we say that the success occur. If 0 then we say that the failure occur. The aim is to stop on last 1.

Let $(\Omega, \mathcal{G}, P)$ be a probability space. On this space we define sequence of independent events $\{A_k\}_{k=1}^n$. We observe sequence of indicators of this events $\{I_k\}_{k=1}^n$. Let us denote by $\mathcal{F}_k = \sigma(I_1, ... I_k)$ sequence of sigma fields generated by indicators and let $\mathcal{T}$ be the set of all stopping moments τ wrt σ -fields $\mathcal{F}_k, k = 1, ..., n$. We want to stop on such time τ^* that will maximize $P(I_t = 1, I_{t+1} = ... = I_n = 0)$ over all $t \in \mathcal{T}$.

Theorem 2. *(Bruss 2000)*
Let $I_1, I_2, ..., I_n$ be a sequence of independent indicator functions with $p_j = E[I_j]$. Let $q_j = 1 - p_j$ and $r_j = \frac{p_j}{q_j}$. Then an optimal rule τ_n for stopping on the last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s$ where

$$s = \sup\{1, \sup\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\}\}$$

with $\sup\{\emptyset\} = -\infty$. *The optimal reward (win probability) is given by*

$$V(n) = \prod_{j=s}^{n} q_j \sum_{j=s}^{n} r_j.$$

Proof presented by Bruss in [3] is based on probability generating function. We present different approach.

Proof. Define a process ξ_t in the following way

$$\xi_t = \inf\{k \geq \xi_{t-1} : I_k = 1\}$$

with initial point $\xi_0 = 1$. Calculate transition probabilities

$$p_{i,s} = P(\xi_{k+1} = s | \xi_k = i) = \frac{P(\xi_{k+1} = s, \xi_k = i)}{P(\xi_k = i)} = \tag{2}$$
$$= \frac{P(I_i = 1, I_{i+1} = ... = I_{s-1} = 0, I_s = 1)}{P(I_i = 1)} = p_s \prod_{j=i+1}^{s-1} q_j.$$

Define a gain function g in the following way

$$g(i) = P(I_{i+1} = ... = I_n = 0) = \prod_{j=i+1}^{n} q_j. \tag{3}$$

Definition 3. *An operator $T(\cdot)$ defined as follows*

$$Tf(x) = \sum_{y} p(x, y) f(y)$$

is called the averaging operator.

Using averaging operator calculate the expected pay-off in next step.

$$
\begin{aligned}
Tg(i) &= \sum_{s=i+1}^{n} p_{i,s} g(s) = \sum_{s=i+1}^{n} p_s \prod_{j=i+1}^{s-1} q_j \prod_{j=s+1}^{n} q_j = \\
&= \sum_{s=i+1}^{n} p_s \prod_{j=i+1}^{s-1} q_j \prod_{j=s+1}^{n} q_j \frac{q_s}{q_s} = \\
&= \prod_{j=i+1}^{n} q_j \sum_{s=i+1}^{n} r_s.
\end{aligned} \tag{4}
$$

To find an optimal stopping rule we check when $Tg \leq g$, i.e. when the expected value of doing one step more is less or equal to pay-off in current state. We get condition that stopping rule is

$$s = \min\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \leq 1\}. \tag{5}$$

We show that it is optimal. In Bruss' theorem we can see that problem is monotone, because sets $A_k = \{Tg(k) \leq g(k)\}$ satisfies condition (1). Therefore we know that method 1-SLA is optimal. In this case, because we deal with independent events 1-SLA is described as follows

$$\nu_0 = \min\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \leq 1\}. \tag{6}$$

So it is exactly the same rule as in (5). Therefore we get the thesis. Win probability is calculated as follows

$$V(n) = Eg(\nu_0) = \prod_{j=\nu_0}^{n} q_j \sum_{s=\nu_0}^{n} r_s. \tag{7}$$

□

3.1 Extension of Bruss' theorem ¿From Odds theorem we can find the moment of last success in n trials. The obvious question is how to find the moment of last l-th success in n independent trials. Idea is to find such a stopping time τ_l^* that will maximize $P(I_t = 1, I_{t+1} + ... + I_n = l)$ and its value. The following theorem gives us the answer of this question.

Theorem 3. *(Bruss, Paindaveine 2000)*
Let $I_1, I_2, ..., I_n$ be a sequence of independent indicator functions with $p_j = E[I_j]$. Let $q_j = 1 - p_j$ and $r_j = \frac{p_j}{q_j}$. Then an optimal rule τ_n for stopping on the l-th last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s_l$ where

$$s_l = \sup\{1, \sup\{1 \leq k \leq n - l + 1 : R_{l,k} \geq l R_{l-1,k} \text{ and } \pi_k \geq l\}\}$$

where

$$R_{l,k} = \sum_{j_1, ... j_l = k, all \neq}^{n} r_{j_1} ... r_{j_l}$$

$$\pi_k = \#\{j \geq k | r_j > 0\}$$

with $\sup\{\emptyset\} = -\infty$. *The optimal reward (win probability) is given by:*

$$V(l,n) = \prod_{j=s_l}^{n} q_j \frac{R_{l,s_l}}{l!}.$$

The proof of this fact can be found in [4].
Another similar problem is to stop on any of last l-th success. The following theorem gives the solution of it.

Theorem 4. *(Tamaki 2010)*
Let $I_1, I_2, ..., I_n$ *be a sequence of independent indicator functions with* $p_j = E[I_j]$. *Let* $q_j = 1 - p_j$ *and* $r_j = \frac{p_j}{q_j}$. *Then an optimal rule* τ_n *for stopping on any of the* l*-th last success exists and is to stop on the first index (if any)* k *with* $I_k = 1$ *and* $k \geq s_l$ *where*

$$s_l = \sup\{1, \sup\{1 \leq k \leq n : \widehat{R}_{l,k+1} \geq 1\}\}$$

where

$$\widehat{R}_{l,k} = \sum_{k \leq j_1 < ... < j_l \leq n} r_{j_1} ... r_{j_l}$$

with $\sup\{\emptyset\} = -\infty$. *The optimal reward (win probability) is given by*

$$V(l,n) = \prod_{j=s_l}^{n} q_j \Big(\sum_{j=1}^{l} \widehat{R}_{j,s_l} \Big).$$

The proof of this fact can be found in [16].

4 Eliminate *and* Stop. Theorem 2 provides a simple rule for stopping on problems which can be described via simple indicator functions. As an example we consider Classical Secretary problem:

4.1 Example 1 - Selecting the best object. Consider the classical secretary problem. Let X_k be the absolute rank of the k-th candidate. We define

$$Y_k = \#\{1 \leq i \leq k : X_i \leq X_k\}.$$

The random variable Y_k is called the relative rank of k-th candidate.
Let $(\Omega, \mathcal{F}, P)$ be the probability space, where elementary events are permutations of the elements from $\{1, ..., n\}$ and the probability measure P is the uniform distribution on Ω. For $k = 1, ..., n$ let $\mathcal{F}_k = \sigma\{Y_1, ..., Y_k\}$ be a sequence of σ -fields. It can be proved that Y_k are independent and $P(Y_k = i) = \frac{1}{k}, i = 1, ..., k$. Set a function

$$I_k := I_{\{Y_k = 1\}}.$$

Then we get that $p_k = E[I_k] = P(Y_k = 1) = \frac{1}{k}$ and $q_k = \frac{k-1}{k}, r_k = \frac{1}{k-1}$. The optimal stopping rule is therefore

$$s = \min\{1 \leq k \leq n : \sum_{i=k}^{n} \frac{1}{i-1} \leq 1\}.$$

The gain is

$$V(n) = \frac{s-1}{n} \sum_{i=s}^{n} \frac{1}{i-1}.$$

4.2 Example 2A - Selecting the second best object. There are problems that can be described similarly as in Odds theorem: we want to maximize the probability of unobservable event describing them via Indicator functions. But because of non-monotonicity of the problem there does not exist a simple rule as the above. As an example consider secretary problem with choosing the second best applicant.
Let $A = \{X_k = 2\}$ denote an event that k-th absolute rank is equal to 2.

$$\{X_k = 2\} =$$
$$= \bigcup_{s=k+1}^{n} \{Y_k = 1, Y_{k+1} > 1, ..., Y_s = 1, Y_{s+1} > 2, ..., Y_n > 2\} \cup \{Y_k = 2, Y_{k+1} > 2, ..., Y_n > 2\} :=$$
$$:= \bigcup_{s=k+1}^{n} B_1^{(s)} \cup B_2.$$

The sets $B_1^{(s)}, B_2$ for all indexes s are disjoint. We have that

$$P(A) = P(X_k = 2) = P(\bigcup_{s=k+1}^{n} B_1^{(s)} \cup B_2) = \sum_{s=k+1}^{n} P(B_1^{(s)}) + P(B_2). \tag{8}$$

First calculate $P(B_2)$. Let us introduce function G

$$G(Y_i) = \begin{cases} I_{\{Y_i=2\}} & \text{for } i = k \\ I_{\{Y_i \in \{1,2\}\}} & \text{for } k+1 \le i \le n. \end{cases} \tag{9}$$

$$P(B_2) = P(G(Y_k) = 1, G(Y_{k+1}) = 0, ..., G(Y_n) = 0) = P(\sum_{i=k}^{n} G(Y_i) = 1). \tag{10}$$

Now we calculate $P(B_1^{(s)})$. Let us introduce function $F_{(s)}$

$$F_{(s)}(Y_i) = \begin{cases} I_{\{Y_i=1\}} & \text{for } k \le i \le s \\ I_{\{Y_i \in \{1,2\}\}} & \text{for } s < i \le n \end{cases} \tag{11}$$

$$P(\bigcup_{s=k+1}^{n} B_1^{(s)}) = \sum_{s=k+1}^{n} P(B_1^{(s)}) =$$
$$= \sum_{s=k+1}^{n} P(F_s(Y_k) = 1, F_s(Y_{k+1}) = 0, ..., F_s(Y_s) = 1, F_s(Y_{s+1}) = 0, ..., F_s(Y_n) = 0) =$$

$$= \sum_{s=k+1}^{n} P(\sum_{i=k}^{n} F_s(Y_i) = 2). \tag{12}$$

From 8, 10 and 12 we get that

$$\sum_{s=k+1}^{n} P(\sum_{i=k}^{n} F_s(Y_i) = 2) + P(\sum_{i=k}^{n} G(Y_i) = 1). \tag{13}$$

We are looking for such $\tau^* \in \mathcal{T}$ that $P(A)$ is the greatest, i.e.

$$\tau^* = \arg\sup_{\tau \in \mathcal{T}} P(A).$$

From Theorem 2 we can find a stopping time $\tau_2 \in \mathcal{T}$ that:

$$\tau_2 = \arg\sup_{\tau \in \mathcal{T}} P(\sum_{i=\tau}^{n} G(Y_i) = 1).$$

We have

$$p_i = P(G(Y_i) = 1) = \begin{cases} P(Y_i = 2) = \frac{1}{k} & \text{for } i = k \\ P(Y_i \in \{1,2\}) = \frac{2}{i} & \text{for } k+1 \le i \le n \end{cases}$$

and

$$q_i = \begin{cases} \frac{k-1}{k} & \text{for } i = k \\ \frac{i-2}{i} & \text{for } k+1 \le i \le n \end{cases}$$

$$r_i = \begin{cases} \frac{1}{k-1} & \text{for } i = k \\ \frac{2}{i-2} & \text{for } k+1 \le i \le n \end{cases}$$

For $i = 1, p_1 = 0, q_1 = 1, r_1 = 1$. We get that

$$\tau_2 = \sup\{1, \sup\{1 \le k \le n : \frac{1}{k-1} + \sum_{i=k+1}^{n} \frac{2}{i-2} \ge 1\}\}.$$

$$\tau_2 = \sup\{1, \sup\{1 \le k \le n : \sum_{i=k}^{n-1} \frac{1}{i-1} \ge \frac{k-2}{2k-2}\}\}.$$

The win probability is $V(n) = \frac{(k-1)^2}{n(n-1)}(\frac{1}{k-1} + \sum_{i=k+1}^{n} \frac{2}{i-2})$.

From Theorem 3 we can find a stopping time $\tau_1^{(s)} \in \mathcal{T}$ that

$$\tau_1^{(s)} = \arg\sup_{\tau \in \mathcal{T}} P(\sum_{i=\tau}^{n} F_s(Y_i) = 2).$$

We have

$$p_i = P(F_s(Y_i) = 1) = \begin{cases} P(Y_i = 1) = \frac{1}{i} & \text{for } k \le i \le s \\ P(Y_i \in \{1,2\}) = \frac{2}{i} & \text{for } s < i \le n \end{cases}$$

and

$$q_i = \begin{cases} \frac{i-1}{i} & \text{for } k \le i \le s \\ \frac{i-2}{i} & \text{for } s < i \le n \end{cases}$$

$$r_i = \begin{cases} \frac{1}{i-1} & \text{for } k \le i \le s \\ \frac{2}{i-2} & \text{for } s < i \le n \end{cases}$$

Let us consider the following inequality

$$\sum_{i,j=k, i \ne j}^{n} r_i r_j \ge 2 \sum_{j=k}^{n} r_j.$$

$$LHS = \sum_{i,j=k,i\neq j}^{n} r_i r_j = ((\sum_{i=k}^{n} r_i)^2 - \sum_{i=k}^{n} r_i^2) =$$
$$=((\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2})^2 - \sum_{i=k}^{s} \frac{1}{(i-1)^2} - \sum_{i=s+1}^{n} \frac{4}{(i-2)^2}).$$
$$RHS = 2\sum_{i=k}^{n} r_j = 2(\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2}).$$

We get that

$$\tau_1^{(s)} = \sup\{1, \sup\{1 \le k \le n-1 : (\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2})^2 - \sum_{i=k}^{s} \frac{1}{(i-1)^2} - \sum_{i=s+1}^{n} \frac{4}{(i-2)^2} \ge 2(\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2}) \quad \text{and} \quad \pi_k \ge 2\}\}.$$

Which after some simplifications gives us

$$\tau_1^{(s)} = \sup\{1, \sup\{1 \le k \le n-1 : \sum_{i=k}^{s} (\frac{i}{i-1})^2 + \sum_{i=s+1}^{n} (\frac{i}{i-2})^2 - (\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2})^2 \le n-k+1 \quad \text{and} \quad \pi_k \ge 2\}\}.$$

The value of the problem is (according to Theorem 3)

$$V(n) = \frac{(k-1)(s-1)}{n(n-1)}((\sum_{i=k}^{s} \frac{1}{i-1} + \sum_{i=s+1}^{n} \frac{2}{i-2})^2 - \sum_{i=k}^{s} \frac{1}{(i-1)^2} - \sum_{i=s+1}^{n} \frac{4}{(i-2)^2}).$$

Remark 5. *Exact results for stopping on second best object can be found in [11]. The above probabilities are conditional probabilities that selected relatively best object is the second one from the end. Denote as k^* the first moment after k when relatively first occurs and let $S_j := I_j + ... + I_n$. Then we have the following approximation*

$$\begin{aligned}
&P(X_{k^*} = 2|S_k = 2) = \\
&= \sum_{s=k+1}^{n} \frac{k}{s(s-1)} \Big(\sum_{l=s+1}^{n} \frac{s}{l(l-1)} (1 - 2 \sum_{j=l+1}^{n} \frac{l(l-1)}{j(j-1)(j-2)}) \Big) \to \\
&\to x \int_x^1 \frac{1}{t^2} \big(t \int_t^1 \frac{1}{u^2} (1 - 2 \int_u^1 \frac{u^2}{z^3} dz) du \big) dt = \\
&= x \int_x^1 \frac{t(1-t)}{t^2} dt = x(x - 1 - \log(x)) := v(x).
\end{aligned} \tag{14}$$

We have that

$$k^* = s_2^*, \quad x^* := \frac{s_2^*}{n} \to e^{-2} \approx 0.13534 \text{ as } n \to \infty. \tag{15}$$

Approximated reward (probability of stopping on relative rank 1, such that $S_k = 2$) is

$$V(2, n) \to \frac{2^2}{2! e^2} = \frac{2}{e^2} \approx 0.27067 \text{ as } n \to \infty. \tag{16}$$

But approximating win probability of $P(X_{k^*} = 2)$ we get that $P(X_{k^*} = 2) = v(x)$. Substituting $x^* = e^{-2}$ to this formula we get

$$v(e^{-2}) = e^{-2} + e^{-4} \approx 0.15361. \tag{17}$$

4.3 Reduction of states We want to consider the above example as a stopping problem of some Markov chain. It is obvious that the problem is not monotone. Thus we can not use 1-SLA method. In similar problems we would like to find the most simple optimal stopping rule. But the simplest rule is provided by monotone problems. Idea is to eliminate those states that spoils monotonicity and afterwards use 1-SLA.

State reduction approach (SRA). Let us assume that the model $(\mathbb{X}_1, P_1)$, where $\mathbb{X}_1$ is a state space and P_1 is a transition matrix is given. Let Z_n be a Markov chain in this model and let $\tau_1, ..., \tau_n$ be the sequence of the moments of first,..., n-th exit of Z_n from set $D \subset \mathbb{X}_1$. Consider the chain $Z'_n = Z_{\tau_n}$. Denote by $\mathbb{X}_2 = \mathbb{X}_1 \setminus D$. Let us denote by $u_1(z, \mathbb{X}_2, \cdot)$ the distribution of the Markov chain Z_n for the initial model at the moment τ_1 of first exit from D starting at $z, z \in D$.

The sequence Z'_n is a Markov chain in model $(\mathbb{X}_2, P_2)$, where the transition matrix is given by the formula

$$p_2(x, y) = p_1(x, y) + \sum_{z \in D} p_1(x, z) u_1(z, \mathbb{X}_2, y), \quad x, y \in \mathbb{X}_2. \tag{18}$$

In case when $D = \{\widehat{z}\}$ and it is not absorbing point we get simpler formula

$$p_2(x, y) = p_1(x, y) + \frac{p_1(x, \widehat{z}) p_1(\widehat{z}, y)}{1 - p_1(\widehat{z}, \widehat{z})}. \tag{19}$$

New model is called D-reduced model. Z_n and Z'_n are different chains, with different state spaces and transition probabilities, but there are some characteristics that are common for them. We formulate one result that will be used later.

Lemma 1. *Let us assume that we have two models $(\mathbb{X}_1, P_1)$ and $(\mathbb{X}_2, P_2)$ defined as above, $U \subset \mathbb{X}_2$ and $\tau_U, (\tau'_U)$ be the moment of first visit to U in the first (second) model. Then*

$$\forall x \in \mathbb{X}_2 \quad u_1(x, U, y) = u_2(x, U, y), \quad (x \in \mathbb{X}_2, y \in U).$$

Proof of this lemma can be found in [13]. In a finite model we can use procedure of eliminating states recursively by eliminating on each step one state. This is very simple implication from the Lemma 1.

Elimination theorem. Let us assume that we have Markov model $M = (\mathbb{X}, P_1, g)$, where $\mathbb{X}$ is a state space, and P_1 is a transition matrix and g is reward function. Let Z_n be a Markov chain specified on this model with initial point z. We denote by P_z, E_z the probability measure and expectation of the Markov chain with the initial point z. We introduce natural filtration and with respect to it we define stopping times. Denote by $\mathcal{T}$ the set of all stopping times.

Let v be the value function, i.e. $v(z) = \sup_{\tau \in \mathcal{T}} E_z g(Z_\tau)$. Let T be an averaging operator. By D let us denote a subset of $\mathbb{X}$ and by τ_D we denote moment of first visit of the chain in set D, i.e: $\tau_D = \min\{k \geq 1 : Z_k \in D\}$.

Definition 4. *We call a set S an optimal stopping set if*

$$S = \{x : v(x) = g(x)\} \text{ and } P(\tau_S < \infty) = 1.$$

The idea of state elimination approach is to eliminate states where is not optimal to stop. We want to eliminate those states, where doing one step more is optimal. In this case we want to satisfy the condition

$$Tg(x) > g(x). \tag{20}$$

Theorem 6. *(Sonin 1995)*
Let $M_1 = (\mathbb{X}_1, P_1, g)$ be an OSP,$D \subseteq \{z \in \mathbb{X}_1 : T_1 g(z) > g(z)\}$ and $P_{1,x}(\tau_{\mathbb{X}_1 \setminus D} < \infty) = 1$ for all $x \in D$. Consider an OSP $M_2 = (\mathbb{X}_2, P_2, g)$ with $\mathbb{X}_2 = \mathbb{X}_1 \setminus D, p_2(x, y)$ defined by (18). Let S be the optimal stopping set in M_2. Then S is the optimal stopping set in the problem M_1 also and $v_1(x) = v_2(x),\ \forall x \in \mathbb{X}_2$.

Second theorem from [13] deals with situation when the problem can be divided into disjoint classes with two properties:

- for any class the transition probability from each state in one class to another class are the same for all states in first class
- the reward function is a constant inside of each of these classes.

Theorem 7. *Let $M_1 = (\mathbb{X}_1, P_1, g)$ and $M_2 = (\mathbb{X}_2, P_2, g)$ be two optimal stopping problems and let $f : \mathbb{X}_1 \to \mathbb{M}_2$ be surjection such that*

- $P_1(x, f^{-1}(y)) = p_2(f(x), y)\ \forall x \in \mathbb{X}_1, y \in \mathbb{X}_2$
- $g(x) = g(f(x))\ \forall x \in \mathbb{X}_1$.

Then

1. $v_1(x) = v_2(f(x)),\ \forall x \in \mathbb{X}_1$
2. *if S_2 is an optimal stopping set for the problem $\mathbb{X}_2$ then $S_1 = \{f^{-1}(S_2)\}$ is an optimal stopping set for the problem $\mathbb{M}_1$.*

Proof. 1. Denote $f(z) = y$. Then

$$\begin{aligned} Tg_1(x) &= \sum_z p_1(x,z) g_1(z) = \\ &= \sum_{f^{-1}(y)} p_1(x, f^{-1}(y)) g_1(f^{-1}(y)) = \\ &= \sum_y p_2(f(x), y) g_2(f(f^{-1}(y))) = Tg_2(f(x)). \end{aligned}$$

Thus

$$v_1(x) = \max\{g_1(x), Tv_1(x)\} = \max\{g_2(f(x)), Tv_2(f(x))\} = v_2(f(x)).$$

2.

$$S_2 = \{y : g_2(y) = v_2(y)\}.$$

$$\begin{aligned} f^{-1}S_2 = f^{-1}\{y : g_2(y) = v_2(y)\} &= \{x : g_2(f(x)) = v_2(f(x))\} = \\ &= \{x : g_1(x) = v_1(x)\} = S_1. \end{aligned}$$

□

4.4 The monotonicity of the model after the state reduction Consider a Markov model $(\mathbb{X}_1, P_1, g)$, where $\mathbb{X}_1$ is a state space and P_1 is a transition matrix. Let Z_n be a Markov chain in this model with special absorbing state 0. Denote $G_k = g_k(Z_1, .., Z_k)$ Consider sets

$$D^{(1)} = \{z_k \in \mathbb{X}_1 : G_k < E[G_{k+1}|\mathcal{F}_k]\}.$$

We denote by T_i an averaging operator in model $\mathbb{X}_i$.
Idea is to eliminate all states from set D. We do it sequentially till we get such a model $(\mathbb{X}_j, P_j, g)$, that $T_j g(z) \le g(z)$. It means that

$$D^{(j)} = \{z \in \mathbb{X}_j : G_k < E[G_{k+1}|\mathcal{F}_k]\} = \emptyset$$

and therefore

$$\forall\ z \in \mathbb{X}_j : T_j g(z) \le g(z). \tag{21}$$

We get that new Markov chain $Z_k^{(j)}$. For every index k we have that

$$G_k^{(j)} \ge E[G_{k+1}^{(j)}|\mathcal{F}_k^{(j)}].$$

Denote this set by A_k^j. It is easy to see that in this model condition (1) is satisfied. Thus we get a monotone stopping problem.
In this new problem we want to find an optimal stopping rule. But according to Theorem 1 1-SLA is optimal for this problem.

Lemma 2. *Suppose that we have Markov model $(\mathbb{X}_1, P_1, g)$ and reduced model $(\mathbb{X}_2, P_2, g)$ such that condition (21) is satisfied. Then 1-SLA stopping rule optimal in model $\mathbb{X}_2$ is also optimal is $\mathbb{X}_1$.*

Proof. Suppose that in reduced model $\mathbb{X}_1$. From SRA we can reduce this model to $\mathbb{X}_2$. We do it sequentially till condition (21) is satisfied. Therefore stopping set is

$$\mathbb{X}_2 = \{z : g_k(z'_1, ..., z'_k) \ge E[g_k(z'_1, ..., z'_k, Z'_{k+1})|z'_1, ..., z'_k])\}$$

where Z'_i is a Markov chain in reduced model. Consider set $A'_k = \{G_k \ge E[G_{k+1}|\mathcal{F}'_k]\}$, where $\mathcal{F}'_k$ is sigma-field generated by $Z'_1, ..., Z'_k$. We show that $A'_k \subset A'_{k+1}$.
Take an arbitrary elementary event $\omega \in A'_k$. Then we have

$$G_{k+1} = g_{k+1}(Z'_1(\omega), ..., Z'_k(\omega), Z'_{k+1}(\omega)) \quad (*)$$

Since $Z'_{k+1}(\omega) \in \mathbb{X}_2$ thus we have:

$$(*) \ge E[g_{k+1}(Z'_1(\omega), ..., Z'_{k+1}(\omega), Z'_{k+2})|Z'_1(\omega), ..., Z'_{k+1}(\omega)])$$

Therefore $\omega \in A'_{k+1}$. Because ω and k was arbitrary we have that

$$\omega \in A'_k \Rightarrow \omega \in A'_{k+1},$$

$$A'_k \subset A'_{k+1}.$$

So we have that 1-SLA is optimal in model $\mathbb{X}_2$. From Theorem 6 we have the that the same stopping rule is valid in model $\mathbb{X}_1$. $\square$

4.5 General model for monotone problems One of the most important modifications of Odds theorem provided in [8] was finding the connection between Bruss' result and 1-SLA method. Let $Z_1, Z_2, ...$ be a stochastic process on an arbitrary space with special absorbing state which will be denoted as 0. Z_k denote the set of random variables observed after $k-1$ success up to and including success k. If there are less than k successes then $Z_k = 0$. Assume that the process will be absorbed with probability one. We want to predict when the process will first hit state 0. If we predict correctly then we win 1, if we predict incorrectly we win nothing, if the process hits 0 before our prediction then we win $\omega < 1$. Therefore the pay-off function is given by

$$G_n = \omega I(Z_n = 0) + I(Z_n \neq 0) P(Z_{n+1} = 0 | \mathcal{G}_n) \\ G_\infty = \omega. \tag{22}$$

where $\mathcal{G}_n = \sigma(Z_1, ..., Z_n)$.
This problem is solved by 1-SLA described in Definition 2. The optimal stopping rule is given by

$$\nu_1 = \min\{k \geq 1 : Z_k = 0 \text{ or } (Z_k \neq 0 \text{ and } \frac{W_k}{V_k} \leq 1 - \omega)\} \tag{23}$$

where

$$V_k = P(Z_{k+1} = 0 | \mathcal{G}_k) \\ W_k = P(Z_{k+1} \neq 0, Z_{k+2} = 0 | \mathcal{G}_k).$$

¿From the condition in Definition 1 it is easy to see that the sufficient condition for the problem to be monotone is

$$\frac{W_k}{V_k} \text{ is } a.s \text{ non-increasing in } k. \tag{24}$$

Theorem 8. *(Ferguson 2008)*
Suppose that process $Z_1, Z_2, ...$ has an absorbing state 0 such that probability that the process is absorbed is 1 and that the stopping problem with reward sequence (22) satisfies the condition (24). Then the 1-SLA is optimal.

The problem for the Bruss' theorem deals with situation where we observe independent indicators and natural filtration generated by this indicators. Nevertheless this method can be also applied to possibly dependent indicators. Then we have that

$$V_k = P(I_{k+1} = I_{k+2} = ... = 0 | \mathcal{G}_k) \\ W_k = \sum_{j=k+1}^{\infty} P(I_{k+1} = I_{k+2} = ... = I_{j-1} = 0, I_j = 1, I_{j+1} = I_{j+2} = ... = 0 | \mathcal{G}_k).$$

In Bruss' result we have also $\omega = 0$. From Theorem 8 we get the following corollary.

Corollary 1. *Suppose the Bernoulli variables $I_1, I_2, ...$ satisfy the condition that there are finite number of successes with probability one. Let $\mathcal{G}_1, \mathcal{G}_2, ...$ be an increasing sequence of sigma-fields such that $\{I_k = 1\}$ is in $\mathcal{G}_k$ for any $k = 1, 2, ...$. Then among stopping rules adapted to the sequence $\{\mathcal{G}_k\}$, the rule (23) is an optimal stopping rule provided condition (24) is satisfied.*

It is easy to see that this corollary implies the Bruss' theorem. In the theorem of Bruss indicators I_k are independent so the ratio $\frac{W_k}{V_k}$ in (23) may be written as $\frac{W_k}{V_k} = \sum_{j=k+1}^{\infty} \frac{p_j}{1-p_j}$. All conditions for monotonicity of the problem are satisfied. Thus problem is monotone and 1-SLA is optimal. This also proves the Bruss' result in the infinite horizon case. Using this approach we can easily find 1-SLA rule in reduced model from Lemma 2. Therefore it is also optimal stopping rule in non-reduced model.

4.6 Example 2B - Selecting the second best object We want to find optimal stopping set for event $\{X_k = 2\}$. Gain function is given by:

$$g((n,k)) = E[I_{\{X_n=2\}}|Y_n = k], \qquad n = 1, ..., N; k = 1, ..., n.$$

Because absolute rank 2 we can obtain only if we focus on relative ranks 1 or 2 then we get that

$$g((n,l)) = 0, \ \forall\, l \geq 3.$$

$$\begin{aligned} g((n,1)) &= E[I_{\{X_n=2\}}|Y_n = 1] = \\ &= P(X_n = 2|Y_n = 1) = \frac{\binom{1}{0}\binom{N-2}{n-1}}{\binom{N}{n}} = \\ &= \frac{(N-2)!}{(n-1)!(N-n-1)!} \cdot \frac{n!(N-n)!}{N!} = \frac{n(N-n)}{N(N-1)}. \end{aligned} \tag{25}$$

$$\begin{aligned} g((n,2)) &= E[I_{\{X_n=2\}}|Y_n = 2] = \\ &= P(X_n = 2|Y_n = 2) = \frac{\binom{1}{1}\binom{N-2}{n-2}}{\binom{N}{n}} = \\ &= \frac{(N-2)!}{(n-2)!(N-n)!} \cdot \frac{n!(N-n)!}{N!} = \frac{n(n-1)}{N(N-1)}. \end{aligned} \tag{26}$$

Define mapping

$$f((Y_1, ..., Y_k)) = \begin{cases} (k,2) & \text{for } Y_k = 2 \\ (k,1) & \text{for } Y_k = 1 \\ (k,0) & \text{otherwise} \end{cases}$$

New transition probabilities are given by $p_2((k-1,j),(k,1)) = p_2((k-1,j),(k,2)) = \frac{1}{k}$ and $p_2((k-1,j),(k,0)) = \frac{k-2}{k}$. We want to create a simpler model M_3 and eliminate states in which is not optimal to stop. First notice that all states (n,l) where $l \geq 3$ are eliminated, because

$$Tg(n,l) > 0 = g(n,l).$$

Thus we get new model M_3:

1. $\mathbb{X}_3$ is set of all pairs (n,k), where $1 \leq n \leq N$ and $k = 1, 2$
2. transition matrix is defines as

$$p_3((n,k),(m,j)) = \frac{n(n-1)}{m(m-1)(m-2)}, \quad 2 \leq n < m \leq N,$$

$$p_3((1,1),(2,j)) = \frac{1}{2}, \quad j = 1, 2$$

and satisfies monotonicity property, i.e. for $m \leq n$, $p_3((n,k),(m,j)) = 0$.

3. Z_n be a Markov chain with initial point $z = (1, 1)$.

There are also some states with relative ranks 1 and 2 that should be eliminated. We will find condition for that. First calculate $Tg(n, j), j = 1, 2$.

$$\begin{aligned}
Tg(n,1) &= \sum p((n,1),(m,k))g((m,k)) = \\
&= \sum_{m=n+1}^{N} p((n,1),(m,1))g((m,1)) + p((n,1),(m,2))g((m,2)) = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{m(m-1)(m-2)} \frac{m(N-m)}{N(N-1)} + \frac{n(n-1)}{m(m-1)(m-2)} \frac{m(m-1)}{N(N-1)} = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{N(N-1)(m-2)} \left(\frac{N-m}{m-1} + 1 \right) = \\
&= \frac{n(n-1)}{N(N-1)} \sum_{m=n+1}^{N} \frac{1}{m-2} \left(\frac{N-m+m-1}{m-1} \right) = \\
&= \frac{n(n-1)}{N} \sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)}.
\end{aligned} \tag{27}$$

Similarly

$$\begin{aligned}
Tg(n,2) &= \sum p((n,2),(m,k))g((m,k)) = \\
&= \sum_{m=n+1}^{N} p((n,2),(m,1))g((m,1)) + p((n,2),(m,2))g((m,2)) = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{m(m-1)(m-2)} \frac{m(N-m)}{N(N-1)} + \frac{n(n-1)}{m(m-1)(m-2)} \frac{m(m-1)}{N(N-1)} = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{N(N-1)(m-2)} \left(\frac{N-m}{m-1} + 1 \right) = \\
&= \frac{n(n-1)}{N(N-1)} \sum_{m=n+1}^{N} \frac{1}{m-2} \left(\frac{N-m+m-1}{m-1} \right) = \\
&= \frac{n(n-1)}{N} \sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)}.
\end{aligned} \tag{28}$$

We see that $Tg((n,1)) = Tg((n,2))$. From (20), (25) and (27) we get

$$\sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)} > \frac{N-n}{(n-1)(N-1)} \tag{29}$$

and from (20), (26) and (28)

$$\sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)} > \frac{1}{N-1}. \tag{30}$$

Then we eliminate states for which conditions (29) and (30) are satisfied and recalculate transition probabilities using (29). We get simpler model M_4 and from Theorem 6 we know that optimal stopping set in M_4 is also optimal stopping set in M_1.

From calculus we know that

$$\sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)} = \frac{N-n}{(n-1)(N-1)}. \tag{31}$$

It means that we do not eliminate any state $(n,1)$ and eliminate states $(n,2)$ such that

$$\begin{aligned} \frac{N-n}{(n-1)(N-1)} &> \frac{1}{N-1} \\ \frac{N-n}{n-1} &> 1 \\ n &< \frac{N+1}{2}. \end{aligned} \tag{32}$$

Denote: $K = \lfloor \frac{N}{2} \rfloor$. According to the Lemma 1 we can eliminate the states recursively using formula (18). Therefore the new transition probabilities are

$$\begin{aligned} p_4((n,1),(m,1)) &= \frac{n}{m(m-1)}, \quad 1 \le n < m \le K \\ p_4((n,1),(m,j)) &= \frac{n(K-1)}{m(m-1)(m-2)}, \quad n \le K < m \\ p_4((n,k),(m,j)) &= \frac{n(n-1)}{m(m-1)(m-2)}, \quad K < n < m \end{aligned} \tag{33}$$

Continuing this procedure of course should give us the minimal optimal stopping set and transition probabilities. Once again calculate $Tg(n,j), j = 1,2$.
For $n < K$

(34)

$$\begin{aligned} &Tg(n,1) = \\ &= \sum_{m=n+1}^{K} \frac{n}{m(m-1)} \cdot \frac{m(N-m)}{N(N-1)} + \sum_{m=K+1}^{N} \frac{n(K-1)}{m(m-1)(m-2)} \frac{m(N-m)+m(m-1)}{N(N-1)} = \\ &= \frac{n}{N(N-1)} \Big((N-1) \sum_{m=n+1}^{K} \frac{1}{m-1} - K + n + N - K \Big) = \\ &= \frac{n}{N} \sum_{m=n+1}^{K} \frac{1}{m-1} + \frac{n}{N(N-1)} (N + n - 2K). \end{aligned}$$

Using 20 we get

$$\sum_{m=n+1}^{K} \frac{1}{m-1} > \frac{2(K-1)}{N-1}. \tag{35}$$

From this we find an index k^* such that the above condition is satisfied. Of course neither for $n \ge K$ states $(n,1)$ and states $(n,2)$ are eliminated.

It is easy to check, that there are no more states that can be eliminated. Thus the optimal stopping rule is

$$N^* = \min\{1 \leq n \leq N : (Y_n = 1 \text{ and } \sum_{m=n+1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{m-1} \leq \frac{2(\lfloor \frac{N}{2} \rfloor - 1)}{N-1}) \\ \text{or } (Y_n \in \{1, 2\} \text{ and } n > \lfloor \frac{N}{2} \rfloor)\}.$$

Now from Lemma 2 we know that the same optimal stopping rule holds for initial model.

5 Conclusion We have shown two important results: one is that Odds Theorem comes from problem of optimal stopping of Markov chains. Second is that optimal stopping problem of Markov chain can be reduced to monotone stopping problem. The procedure is the following: eliminate those states which is not optimal to stop on, apply 1-SLA method to find the optimal stopping rule and calculate the expected reward. This explains why the procedure was called 'Eliminate and stop'. This algorithm can be used to solve many problems. One of them is 'secretary problem'.

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