ON SOME MATRIX MEAN INEQUALITIES WITH KANTOROVICH CONSTANT

DINH TRUNG HOA, DU THI HOA BINH, AND TOAN MINH HO

Received December 18, 2014 ; revised September 7, 2015

Abstract. Let A and B be positive definite matrices with $0 < m \le A, B \le M$ for some scalar $0 < m \le M$, and σ, τ two arbitrary means between the harmonic and the arithmetic means. Put $h = \frac{M}{m}$. Then for every unital positive linear map Φ,

$$\begin{aligned} \Phi^2(A\sigma B) &\le K^2(h)\Phi^2(A\tau B), \\ \Phi^2(A\sigma B) &\le K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2, \\ (\Phi(A)\sigma\Phi(B))^2 &\le K^2(h)\Phi^2(A\tau B), \\ (\Phi(A)\sigma\Phi(B))^2 &\le K^2(h)(\Phi(A)\tau\Phi(B))^2, \end{aligned}$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ is the Kantorovich constant.

We also give a new characterization of the trace property and operator monotonicity by the squared Cauchy inequality.

Keywords: Matrix means, unital positive linear maps, Kantorovich inequality, trace, operator monotonicity.

1. Introduction

Throughout this paper, $\mathbb{M}_n$ stands for the algebra of all $n \times n$ matrices over the field of complex numbers. A continuous function f on an interval J ($\subset \mathbb{R}$) is said to be *operator monotone if*

$$A \le B \implies f(A) \le f(B) \tag{1}$$

for any pair of self-adjoint bounded operators A, B on a separable infinite dimensional Hilbert space H with spectra $\sigma(A), \sigma(B) \subset J$. We

2010 *Mathematics Subject Classification.* 47A63.

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2014.40.

know that the function f is operator monotone if and only if the inequality (1) holds for every self-adjoint matrices A, B of order n for every $n \in \mathbb{N}$.

We reserve M, m for scalars and I (in $\mathbb{M}_n$ or in $B(H)$) for the identity operator. The axiomatic theory for connections and means for pairs of positive operators on H have been studied by Kubo and Ando [6]. A binary operation σ defined on the cone of positive operators is called a connection if

(i) $A \le C, B \le D$ implies $A\sigma B \le B\sigma D$;
(ii) $C^*(A\sigma B)C \le (C^*AC)\sigma(C^*BC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then σ is called a mean. This definition can also be defined for positive operators on a finite dimensional Hilbert space. Operators on an n-dimensional space are identified with complex matrices of order n, hence we usually call connections/means in this case *matrix connections/means* of order n (see [1]). The fact is that an operator connection/mean is a matrix connection/mean of every order. However, throughout this paper operator means/connections will be used even the main theorem (and some other consequences) still hold for matrix means/connections.

Many authors study matrix inequalities containing means and unital positive linear maps on the matrix algebras. Such inequalities are interesting by themselves and have many applications in quantum information theory. One of the most important inequalities is the non-commutative AM-GM inequality which states that, for positive semidefinite matrices A, B,

$$A\nabla B = \frac{A+B}{2} \ge A\sharp B, \tag{2}$$

where, for positive definite matrices A, B,

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}$$

and, for positive semidefinite matrices A, B,

$$A\sharp B = \lim_{\varepsilon\to 0^+}(A+\varepsilon I)\sharp(B+\varepsilon I).$$

Moreover, by [11, Lemma 4.2] the left hand side and the right hand side of Cauchy inequality cover all ordered pairs of positive semidefinite matrices $X \geq Y$. However, this property does not hold for squares because the following inequality fails in general

$$(\frac{A+B}{2})^2 \geq (A\sharp B)^2. \tag{3}$$

Indeed, take the following matrices

$$X = \begin{pmatrix} 5/6 & 2 \\ 2 & 5 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}. \tag{4}$$

By help of Matlab, we can see that

$$det((X+Y)^2/4 - (X\sharp Y)^2) = -1.2301 < 0.$$

In [2], Lin proved the following Theorem.

Theorem 1.1 ([2]). *Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and put $h = \frac{M}{m}$. Then for every unital positive linear map Φ on $\mathbb{M}_n$,*

$$\Phi^2(A\nabla B) \leq K^2(h)\Phi^2(A\sharp B), \tag{5}$$

and

$$\Phi^2(A\nabla B) \leq K^2(h)(\Phi(A)\sharp\Phi(B))^2, \tag{6}$$

where $K(h) = \frac{(h+1)^2}{4h}$ is the Kantorovich constant.

It is well-known that the arithmetic mean ∇ is the biggest one among symmetric means (see [6]). A natural question is that: *Is the theorem above still true if we replace the biggest mean by a smaller one?* In this paper, we consider such inequalities for two different means with Kantorovich constant. In applications, we give an analogous result of Uchiyama and Yamazaki in [9] and the reverse of Minkovskii type inequality in [10].

It is well-known that for a monotone increasing function f on $\mathbb{R}^+$: $Tr(f(A)) \leq Tr(f(B))$ whenever $0 \leq A \leq B$. Consequently, for any two positive semidefinite matrices A, B and $p \geq 0$ we have $Tr((A\sharp B)^p) \leq Tr((A\nabla B)^p)$ even the inequality (3) does not hold.

Also, we can characterize positive linear functionals φ on $\mathbb{M}_n$ and operator monotone functions satisfying the following inequality: $\varphi(f(A\sharp B)) \leq \varphi(f(A\nabla B))$.

2. Main results

Lemma 2.1. *Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and σ, τ two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map Φ,*

$$\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \leq M + m, \tag{7}$$

and

$$\Phi(A)\sigma\Phi(B) + Mm\Phi^{-1}(A\tau B) \leq M + m. \tag{8}$$

Proof. It is easy to see that

$$(M - A)(m - A)A^{-1} \leq 0,$$

or

$$mMA^{-1} + A \leq M + m.$$

Consequently,

$$\Phi(A) + mM\Phi(A^{-1}) \leq M + m.$$

Similarly,

$$\Phi(B) + mM\Phi(B^{-1}) \leq M + m.$$

Summing up two above inequalities, we get

$$\Phi(A \bigtriangledown B) + mM\Phi((A!B)^{-1}) \leq M + m.$$

Besides, by the hypothesis $\bigtriangledown \geq \sigma$ and $\tau \geq !$, we get

$$\begin{aligned}\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B) &\leq \Phi(A\sigma B) + mM\Phi((A\tau B)^{-1}) \\ &\leq \Phi(A \bigtriangledown B) + mM\Phi((A!B)^{-1}) \\ &\leq M + m.\end{aligned}$$

By a similar argument, we can get the inequality (8) using the fact that

$$\Phi(A)\sigma\Phi(B) \leq \Phi(A) \bigtriangledown \Phi(B) = \Phi(A \bigtriangledown B).$$

□

The following main theorem of this paper is a generalization of Lin's result (Theorem 1.1).

Theorem 2.1. *Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and σ, τ two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map Φ,*

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{9}$$

$$\Phi^2(A\sigma B) \leq K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2, \tag{10}$$

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)\Phi^2(A\tau B), \tag{11}$$

and

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)(\Phi(A)\tau\Phi(B))^2, \tag{12}$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$.

Proof. We prove the inequality (9). The inequality (9) is equivalent to the following

$$\Phi^{-1}(A\tau B)\Phi^2(A\sigma B)\Phi^{-1}(A\tau B) \leq K^2(h),$$

or

$$||\Phi(A\sigma B)\Phi^{-1}(A\tau B)|| \leq K(h).$$

On the other hand, it is well known that [7, Theorem 1] for $A, B \geq 0$,

$$||AB|| \leq \frac{1}{4}||A+B||^2.$$

So, it is necessary to prove that

$$\frac{1}{4mM}||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)||^2 \leq \frac{(M+m)^2}{4Mm},$$

or,

$$||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)|| \leq M + m.$$

The last inequality follows from Lemma 2.1.

The remain inequalities in this theorem can be proved analogously. □

From the operator monotonicity of the function $f(t) = t^{1/2}$ on $[0, \infty)$ it obviously implies the following proposition.

Proposition 2.1. *Let $0 < m \leq A, B \leq M$ and σ, τ are two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map Φ,*

$$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$$

$$\Phi(A\sigma B) \leq K(h)\left(\Phi(A)\tau\Phi(B)\right),$$

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B)$$

and

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ with $h = \dfrac{M}{m}$.

Remark 1. From the following well-known fact:

$$A\tau B \leq A\nabla B \leq K(h)A!B \leq K(h)A\sigma B,$$

it implies immediately Proposition 2.1. On the other hand, it is well-known in [5] that if σ is a symmetric mean, then

$$\frac{m\sigma M}{m\nabla M}A\nabla B \leq A\sigma B.$$

Therefore, we have

$$\frac{m\sigma M}{m\nabla M}A\tau B \leq A\sigma B. \tag{13}$$

Also, Theorem 13 in [5] says that if $0 \leq m \leq A, B \leq M$, then

$$\frac{2\sqrt{mM}}{m+M}A\nabla B \leq A\sharp B \leq \frac{M+m}{2\sqrt{mM}}A!B. \tag{14}$$

Now we will show that the inequality (13) could not be squared when $\sigma = \sharp, \tau = \nabla$. Indeed, let's take $m = 0.02, M = 11$ and the matrices X, Y as in the equation (4). It is obvious that $m \leq X, Y \leq M$. With a help of Matlab we get

$$det(K(h)(X\sharp Y)^2 - (X\nabla Y)^2) = -4.1122,$$

and

$$det(K(h)(X!Y)^2 - (X\sharp Y)^2) = -1.7545.$$

Hence, the following inequalities

$$K(h)(X\sharp Y)^2 \geq (X\nabla Y)^2, \quad (X\sharp Y)^2 \geq K(h)(X!Y)^2$$

do not hold.

Corollary 2.1. *Let f, g be symmetric operator monotone functions on $[0, \infty)$. Then for any pair $0 < m < M$,*

$$max\{\frac{f(t)}{g(t)}, \frac{g(t)}{f(t)}\} \leq \frac{(m+M)^2}{4mM}, \quad t \in [m, M]. \tag{15}$$

Proof. It is necessary to apply [6, Theorem 3.2] and the above Proposition 2.1 for the symmetric means σ and τ corresponding to the functions f and g, and definition of means via their representation functions. □

The inequality (15) is interesting by itself, and the authors do not know any elementary proof even in the case when $f(t) = \sqrt{t}$.

As an application, now we give a similar result as in [9]. Uchiyama and Yamazaki showed that for an operator monotone function f on $[0, \infty)$ if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leq I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$. By applying Proposition 2.1, we get a similar result for any symmetric mean.

Corollary 2.2. *Let f be an operator monotone function on $[0, \infty)$ with $f(1) = 1$ and σ an arbitrary mean between the harmonic and the arithmetic ones. Let $0 < m < 1 < M$ and A, B positive definite matrices such that $0 < m \leq A, B \leq M$. If for all sufficiently small $\lambda > 0$*

$$f(\lambda B + I)^{-1} \sigma f(\lambda A + I) \leq K^{-1} \quad (\textit{where } K = \frac{(m+M)^2}{4Mm}), \tag{16}$$

then

$$f(\lambda A + I) \leq f(\lambda B + I) \quad \textit{and} \quad A \leq B.$$

Proof. From the continuity of the function f and assumptions, it follows that for all sufficiently small $\lambda > 0$

$$m \leq f(\lambda B + I)^{-1}, f(\lambda A + I) \leq M.$$

On account of Proposition 2.1 and (16), we obtain

$$\begin{aligned} f(\lambda B + I)^{-1} \sharp f(\lambda A + I) &\leq K(f(\lambda B + I)^{-1} \sigma f(\lambda A + I)) \\ &\leq I. \end{aligned}$$

By [9, Theorem 1], we get

$$f(\lambda A + I) \leq f(\lambda B + I) \quad \text{and} \quad A \leq B.$$

□

The famous Minkovskii determinantal inequality is

$$det^{1/n}(A+B) \geq det^{1/n}A + det^{1/n}B,$$

for any positive semidefinite matrices of order n A, B. In [10] Bourin and Hiai obtained the Minkovskii type inequality as follows: Let σ be an operator mean whose representing function f is geometrically convex, i.e., $f(\sqrt{xy}) \geq \sqrt{f(x)f(y)}$. Then, for every positive semidefinite matrices of order n A, B,

$$det^{1/n}(A\sigma B) \geq (det^{1/n}A)\sigma(det^{1/n}B),$$

and the reverse inequality holds if the representing function is geometrically concave.

Combining the reverse inequalities in Proposition 2.1, we give the lower bound and upper bound of the value $det(A\sigma B)$ for any operator mean σ between the arithmetic and the harmonic ones.

Corollary 2.3. *Let σ be a symmetric mean. If A, B are positive definite matrices such that $0 < m \leq A, B \leq M$, then*

$$K^{-1}(det^{1/n}(A)\nabla det^{1/n}(B)) \leq det^{1/n}(A\sigma B) \leq K(det^{1/n}(A)!det^{1/n}(B)),$$

where $K = \frac{(M+m)^2}{4Mm}$.

Proof. By Proposition 2.1, we have

$$A\nabla B \leq KA\sigma B.$$

We also know that $det^{1/n}$ preserves the order of matrices by the famous Minkovskii determinantal inequality. Consequently,

$$\begin{aligned} K^{-1}det^{1/n}(A)\nabla det^{1/n}(B) &\leq K^{-1}det^{1/n}(A\nabla B) \\ &\leq det^{1/n}(A\sigma B). \end{aligned}$$

Now we prove the second inequality of the corollary. By Proposition 2.1, we have $A\sigma B \leq KA!B$. Hence,

$$det^{1/n}(A\sigma B) \leq K\det^{1/n}(A!B).$$

Moreover, the function $f(t) = 1!t = \frac{2t}{1+t}$ corresponding the harmonic mean is geometrically concave, by the result in [10] mentioned above, we have

$$det^{1/n}(A!B) \leq det^{1/n}(A)!det^{1/n}(B)$$

and then the second inequality is obtained.

□

Now let us consider the problem of characterization of the trace property which is closed to the characterization of the operator monotonicity. It is well-known that, for a monotone increasing function f on $\mathbb{R}^+$, from the assumption $0 < A \leq B$ it follows that

$$Tr(f(A)) \leq Tr(f(B)).$$

Consequently, for any two positive definite matrices A, B we have

$$Tr(f(A\sharp B)) \leq Tr(f(A\nabla B)).$$

In [12] O. E. Tikhonov and A. M. Bikchentaev showed that for a positive linear functional φ on $\mathbb{M}_n$ and any given $p > 1$ if the inequality

$$\varphi(A^p) \leq \varphi(B^p)$$

holds for any pair of positive definite matrices $A \leq B$, then φ should be a scalar of the canonical trace. Note that the function $f(t) = t^p$ for $p > 1$ is not operator monotone on $[0, \infty)$. In the following proposition, replacing A, B by the geometric and the arithmetic means we can get the characterization of the trace.

Proposition 2.2. *For a positive linear functional φ on $\mathbb{M}_n$ and a given $p > 1$. If the following inequality*

$$\varphi((A\nabla B)^p) \geq \varphi((A\sharp B)^p)$$

holds whenever positive definite matrices $A \leq B$, then φ is a scalar of the canonical trace.

Proof. It is well-know that for arbitrary $0 < A \leq B$ we can find positive definite matrices X, Y such that $B = X\nabla Y, A = X\sharp Y$ (see [11, Lemma 4.2]). In fact, put $X = B - B\sharp(B - AB^{-1}A)$ and $Y = B + B\sharp(B - AB^{-1}A)$. By the assumption,

$$\varphi((X\nabla Y)^p) \geq \varphi((X\sharp Y)^p)$$

or

$$\varphi(A^p) \leq \varphi(B^p).$$

By the characterization mentioned above, φ is a scalar of the trace. □

The following corollary is just an immediate consequence of [13, Theorem 1]. However, we can give here a direct proof.

Corollary 2.4. *Let H be an infinite dimensional, separable Hilbert space and φ a normal state on $B(H)$ such that its corresponding density operator is not finite rank. Let f be a function defined on $(0,\infty)$. Then the following statements are equivalent:*

(i) *f is operator monotone;*
(ii) *the following inequality*

$$\varphi(f(A\nabla B)) \geq \varphi(f(A\sharp B))$$

holds for any positive operators $A, B \in B(H)$ satisfying the condition $\sigma(A\nabla B), \sigma(A\sharp B) \subset (0,\infty)$.

Proof. $(i) \Rightarrow (ii)$ is obvious. Conversely, for arbitrary positive invertible operators $X \leq Y$, we can find $A, B \geq 0$ such that $Y = A\nabla B, X = A\sharp B$ (see [11, Lemma 4.2]). In fact, put $A = Y - Y\sharp(Y - XY^{-1}X)$ and $B = Y + Y\sharp(Y - XY^{-1}X)$. By the assumption,

$$\varphi(f(Y)) = \varphi(f(A\nabla B)) \geq \varphi(f(A\sharp B)) = \varphi(f(X)).$$

By [13, Theorem 1], it follows that the function f is operator monotone. □

3. Some comments on the Kantorovich inequality

The Kantorovich inequality [14] states that for any $0 < m \leq A \leq M$. and any unital positive linear map Φ,

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm}\Phi(A)^{-1}. \tag{17}$$

From the Kantorovich inequality it is easy to get the following:

Corollary 3.1 ([15])**.** *Let $0 < m \leq A \leq M$. Then for every unital positive linear map Φ,*

$$\Phi(A^{-1})\sharp\Phi(A) \leq \frac{M+m}{2\sqrt{Mm}}. \tag{18}$$

Question: If we replace the geometric mean by an arbitrary mean, does the inequality (18) hold?

Counterexample: Let σ be an operator mean corresponding to the function $f(t) = t$. Assume that the inequality (18) holds for any $0 < m \le A \le M$, then it should hold for $A = M$. That means,

$$M^{-1}\sigma M \le \frac{M+m}{2\sqrt{Mm}},$$

or

$$M \le \frac{M+m}{2\sqrt{Mm}}.$$

Substitute $M = 2$ and $m = 1$ into the latter inequality, we get a contradiction. Even for symmetric means, for example, the arithmetic mean, the inequality (18) does not hold.

However, it is easy to see that for a unital positive linear map Φ and any mean σ,

$$\Phi(A^{-1})\sigma\Phi(B) \le m^{-1}\sigma M,$$

for $0 < m \le A, B \le M$, and the latter inequality can be squared.

Back to the above question, if we restrict our attention to the class of symmetric means, the inequality (18) holds true as follows.

Proposition 3.1. *Let $0 < m \le A \le M$. Then for any symmetric mean σ,*

$$A^{-1}\sigma A \le \sqrt{K(h')}, \tag{19}$$

where $K(h') = \frac{(M'+m')^2}{4M'm'}$ and M' and m' are the maximum and the minimum of the set $\{M, m, 1/m, 1/M\}$.

Proof. Let f be the symmetric operator monotone function corresponding to σ. Then the function $g(t) = \frac{t}{f(t)}$ is symmetric. From Corollary 2.1 and by direct calculation we get

$$\begin{aligned} A^{-1}\sigma A &= A^{-1}f(A^2) \\ &\le K(h')A^{-1}g(A^2) \\ &= K(h')Af^{-1}(A^2) \\ &= K(h')(A^{-1}\sigma A)^{-1}. \end{aligned}$$

Then we obtain (19). ∎

Let $0 < m \leq A \leq M$. Then for any symmetric mean σ and any unital positive linear map Φ,

$$\Phi^{-1}(A)\sigma\Phi(A) \leq \sqrt{K(h)}.$$

Acknowledgements

The authors would like to thank the anonymous reviewer for his comments and suggestions to improve the final version of this work.

This work was completed when the third author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank the VIASM for financial support and hospitality.

References

[1] Dinh Trung Hoa, Toan M. Ho and H. Osaka, Interpolation classes and matrix means, Banach J. Math. Anal. **9** (2015), No.3, 127-139.

[2] M. Lin. Squaring a reverse AM-GM inequality. Studia Mathematica **215** (2) (2013), 187-194.

[3] M. Lin. On an operator Kantorovich inequality for positive linear maps. J. Math. Anal. Appl. **402** (2013), 127-132.

[4] M. Fujii, M. Nakamura, Kadisons Schwarz inequality and noncommutative Kantorovich inequality. Sci. Math. Japon. Online, e-2005, 605-606.

[5] J. I. Fujii, M. Nakamura, J. Pečarić, Y. Seo. Bounds for the ratio and difference between parallel sum and series via Mond-Pecaric method. Math. Inequal. Appl., **9** (4) (2006), 749759.

[6] K. Kubo, T. Ando, Means of positive linear operators. Math. Ann. **246** (1979/80), 205-224.

[7] R. Bhatia, F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities. Linear Algebra Appl. **308** (2000), 203-211.

[8] R. Kaura, M. Singh, J. S. Aujla, Generalized matrix version of reverse Holder inequality. Linear Algebra Appl. **434** (2011), 636-640.

[9] M. Uchiyama, T. Yamazaki, A converse of Loewner-Heinz inequality and applications to operator means. J. Math. Anal. Appl. **413**(1) (2014), 422-429.

[10] J. C. Bourin, F. Hiai, Jensen and Minkowski inequalities for operator means and anti-norms. Linear Algebra Appl. **456** (2014), 22-53.

[11] F. Hiai, T. Ando, Operator log-convex functions and operator means. Math. Ann. **350**(3) (2011), 611-630.

[12] O. E. Tikhonov, A. M. Bikchentaev. Characterization of the trace by monotonicity inequalities. Linear Algebra Appl. **422**(1) (2007), 274-278.

[13] Dinh Trung Hoa, O. E. Tikhonov. To the theory of operator monotone and operator convex functions. Russian Maths. **54**(3) (2010) 7-11.

[14] B. Mond, J. Pečarić, Converses of Jensens inequality for positive linear maps. Analele Uni-versit. din Timisoara Seria Math.-Inform. **XXXI 2** (1993), 223228.

[15] R. Nakamoto, M. Nakamura, Operator means and Kantorovich inequality. Math. Japon., **44** (1996), 495-498.

Communicated by *Hiroyuki Osaka*

Department of Mathematics and Statistics, 200 Parker Hall, Auburn University, AL 36849-5310, USA, and Department of Fundamental Sciences, Ho Chi Minh City University of Food Industry, Ho Chi Minh City, Vietnam

E-mail address: dzt0022@auburn.edu or trunghoa.math@gmail.com

Ha Tay College of Pedagogy, Ha Noi, Vietnam, and Quy Nhon University, Vietnam

E-mail address: hoabinhcdsp@gmail.com

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, Ha Noi, Vietnam

E-mail address: hmtoan@math.ac.vn