ON RELATIONS BETWEEN OPERATOR VALUED α-DIVERGENCE AND RELATIVE OPERATOR ENTROPIES

Hiroshi Isa(1), Masatoshi Ito(2), Eizaburo Kamei(3), Hiroaki Tohyama(4) and Masayuki Watanabe(5)

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Abstract. Let A and B be two strictly positive operators, and $\alpha \in (0,1)$. The operator valued α-divergence is defined by

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right),$$

where $A \nabla_\alpha B = (1-\alpha)A + \alpha B$ and $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. In this paper, firstly, we show some fundamental relations between operator valued α-divergence and relative operator entropies (relative operator entropy, Tsallis relative operator entropy etc.). Next, we introduce noncommutative ratio $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ on the path $A \natural_w B$, and we discuss noncommutative ratio translation. Moreover, we discuss α-divergence for operator distributions.

1 Introduction.

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. An operator T on H is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

A relative operator entropy is introduced by Fujii and Kamei [3] as follows: For strictly positive operators A and B,

$$S(A|B) \equiv A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

Moreover, for $u \in \mathbf{R}$, Furuta [8] introduced

$$S_u(A|B) \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^u \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

as an extension of $S(A|B)$, and Yanagi, Kuriyama and Furuichi [16] call it generalized relative operator entropy.

For $w \in \mathbf{R}$, we consider a path $A \natural_w B$ through A and B defined by [4], [5], [12] etc.:

$$A \natural_w B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^w A^{\frac{1}{2}}.$$

A path through A and B is an extended notion of weighted geometric mean $A \sharp_\alpha B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ defined for $\alpha \in [0,1]$. $S_u(A|B)$ can be regarded as a tangent vector at u on the path, and from this viewpoint, we showed several relations between $S(A|B)$ and $S_u(A|B)$ in [9].

Yanagi, Kuriyama and Furuichi [16] introduced Tsallis relative operator entropy as follows:

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For strictly positive operators A and B,

$$T_\alpha(A|B) \equiv \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} - A}{\alpha} = \frac{A \sharp_\alpha B - A}{\alpha},\ \alpha \in (0,1].$$

Since $\lim_{\alpha\to 0}\frac{x^\alpha-1}{\alpha} = \log x$ holds for $x > 0$, we have $T_0(A|B) \equiv \lim_{\alpha\to 0} T_\alpha(A|B) = S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha \in \mathbf{R}$. In [9], we showed the following essential relation between relative operator entropies: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$(*) \qquad S(A|B) \le T_\alpha(A|B) \le S_\alpha(A|B) \le -T_{1-\alpha}(B|A) \le -S(B|A) = S_1(A|B).$$

In the information geometry, α-divergence defined by Amari [1] plays an important role as a notion to measure the difference between two probability distributions. Fujii [2] defined an operator version of α-divergence as follows: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right),$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is weighted arithmetic mean. In section 2, we show some fundamental relations between operator valued α-divergences and relative operator entropies.

In section 3, we show the following equality for u, $v \in \mathbf{R}$:

$$(\diamondsuit) \qquad (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) = S_{u+v}(A|B).$$

We call $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ *noncommutative ratio* on the path $A \natural_w B$, and show a preservation on this ratio. We call to multiply $S_u(A|B)$ by $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ like the equality $(\diamondsuit)$ *noncommutative ratio translation* for generalized relative operator entropy. Applying noncommutative ratio translation to fundamental relations between operator valued α-divergences and relative operator entropies shown in section 2, we get similar results to the waving property in [9].

For discrete (positive) probability distributions $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$, Shannon inequality $0 \ge \sum_{i=1}^n p_i \log \dfrac{q_i}{p_i}$ holds. Furuta [8] showed operator Shannon inequality, that is, $0 \ge \sum_{i=1}^n S(A_i|B_i)$ for A_i, $B_i > 0$ $(1 \le i \le n)$ with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. We call an operator sequence $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ *an operator distribution* if $A_i > 0$ $(1 \le i \le n)$ and $\sum_{i=1}^n A_i = I$, since it can be regarded as an operator version of discrete probability distribution.

Let $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$ be operator distributions, and $\alpha \in (0,1)$. In [9] and [10], we introduced relative operator entropy for operator distributions $S(\mathbb{A}|\mathbb{B})$, Tsallis relative entropy for operator distributions $T_\alpha(\mathbb{A}|\mathbb{B})$, and generalized relative entropy for operator distributions $S_\alpha(\mathbb{A}|\mathbb{B})$ as follows:

$$S(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n S(A_i|B_i),\ T_\alpha(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n T_\alpha(A_i|B_i),\ S_\alpha(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n S_\alpha(A_i|B_i).$$

Yanagi, Kuriyama and Furuichi [16] improved the operator Shannon inequality:

$$0 \ge T_\alpha(\mathbb{A}|\mathbb{B}) \ge S(\mathbb{A}|\mathbb{B}),\ \alpha \in (0,1).$$

From the viewpoint of this improvement of Shannon inequality, in [9], we got

$$S(\mathbb{A}|\mathbb{B}) \le T_\alpha(\mathbb{A}|\mathbb{B}) \le S_\alpha(\mathbb{A}|\mathbb{B}) \le -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) \le -S(\mathbb{B}|\mathbb{A}) = S_1(\mathbb{A}|\mathbb{B})$$

by $(*)$ and showed related inequalities. Moreover, in [10], we discussed generalizations of these inequalities. In section 4, we define α-divergence for operator distributions, and show its fundamental properties.

2 Operator valued α-divergence and fundamental properties. Amari [1] defined α-divergence as a notion to measure the difference between two probability distributions as follows: For two discrete probability distributions $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$, that is, $p_i, q_i > 0$ $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, and for $\alpha \in \mathbf{R}$,

$$D_\alpha[\, p \,:\, q\,] \equiv \frac{4}{1-\alpha^2}\left(1 - \sum_{i=1}^{n} p_i^{\frac{1-\alpha}{2}} q_i^{\frac{1+\alpha}{2}}\right),\ \alpha \neq \pm 1.$$

If $\alpha = -1$, then $D_{-1}[p:q] \equiv \lim_{\alpha \to -1} D_\alpha[p:q] = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$, and if $\alpha = 1$, then $D_1[p:q] \equiv \lim_{\alpha \to 1} D_\alpha[p:q] = D_{-1}[q:p]$. We call this quantity $D_{-1}[p:q]$ the relative entropy (Kullback-Leibler divergence, Kullback-Leibler distance), and denote it by $D_{KL}(\,p\,|\,q\,)$ ([13], [14]). If we put $t = \frac{1+\alpha}{2}$, then α-divergence can be expressed as follows:

$$D_t(\,p\,|\,q\,) \equiv D_{2t-1}[\,p\,:\,q\,] = \frac{1}{t(1-t)} \sum_{i=1}^{n} \left\{(1-t)p_i + tq_i - p_i^{1-t} q_i^{t}\right\},\ t \neq 0, 1.$$

Based on this expression, Fujii [2] defined an operator valued α-divergence as follows.

Definition 2.1. For strictly positive operators A and B, and for $\alpha \in (0,1)$, operator valued α-divergence is defined as follows ([2], [6], [7]):

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \,\nabla_\alpha\, B - A \,\sharp_\alpha\, B\right),$$

where $A \,\nabla_\alpha\, B \equiv (1-\alpha)A + \alpha B$ and $A \,\sharp_\alpha\, B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

In this section, we show some fundamental properties of operator valued α-divergences. Petz [15] introduced the operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Fujii et al. showed the following relation between $D_{FK}(A|B)$ and operator valued α-divergences at end points for interval $(0,1)$.

Proposition 2.2. (Fujii-Mićić-Pečarić-Seo, [6], [7]) *Let A and B be strictly positive operators. Then,*

(1) $$D_0(A|B) \equiv \lim_{\alpha \to 0} D_\alpha(A|B) = D_{FK}(A|B) = B - A - S(A|B),$$

(2) $$D_1(A|B) \equiv \lim_{\alpha \to 1} D_\alpha(A|B) = D_{FK}(B|A) = A - B + S_1(A|B)$$

hold.

The following (1) in Proposition 2.3 interpolates (1) and (2) in Proposition 2.2 since $T_0(A|B) = S(A|B)$ and $-S(B|A) = S_1(A|B)$ by $(*)$.

Proposition 2.3. *Let A and B be strictly positive operators. Then,*

(1) $$D_\alpha(A|B) = \frac{1}{1-\alpha}(B - A - T_\alpha(A|B)) = \frac{1}{\alpha}(A - B - T_{1-\alpha}(B|A)),\ \textit{for } \alpha \in (0,1),$$

(2) $$D_{1-\alpha}(B|A) = D_\alpha(A|B),\ \textit{for } \alpha \in [0,1]$$

hold.

Proof. **(1)** This can be shown as follows:

$$\begin{aligned}(1-\alpha)D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha} = \frac{A \nabla_\alpha B - A}{\alpha} - \frac{A \sharp_\alpha B - A}{\alpha} \\ &= B - A - T_\alpha(A|B), \\ \alpha D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{1-\alpha} = \frac{A \nabla_\alpha B - B}{1-\alpha} - \frac{A \sharp_\alpha B - B}{1-\alpha} \\ &= A - B - \frac{B \sharp_{1-\alpha} A - B}{1-\alpha} = A - B - T_{1-\alpha}(B|A).\end{aligned}$$

(2) For $\alpha \in (0,1)$,

$$D_{1-\alpha}(B|A) = \frac{B \nabla_{1-\alpha} A - B \sharp_{1-\alpha} A}{(1-\alpha)\{1-(1-\alpha)\}} = \frac{A \nabla_\alpha B - A \sharp_\alpha B}{(1-\alpha)\alpha} = D_\alpha(A|B)$$

holds. In case of $\alpha = 0$ or $\alpha = 1$, this can be obtained by Proposition 2.2 and the relation $-S(B|A) = S_1(A|B)$ in $(*)$.

□

The following result gives bounds of operator value $D_\alpha(A|B)$.

Theorem 2.4. *Let A and B be strictly positive operators. Then,*

$$0 \le D_\alpha(A|B) \le \frac{1}{1-\alpha} D_0(A|B), \tag{1}$$

$$0 \le D_\alpha(A|B) \le \frac{1}{\alpha} D_1(A|B) \tag{2}$$

hold for $\alpha \in (0,1)$.

Proof. Since $A \nabla_\alpha B \ge A \sharp_\alpha B$ for any $\alpha \in (0,1)$, $D_\alpha(A|B) \ge 0$ holds. Moreover, by $(*)$ and (1) in Proposition 2.3, we have

$$\begin{aligned}D_\alpha(A|B) &= \frac{1}{1-\alpha}(B - A - T_\alpha(A|B)) \le \frac{1}{1-\alpha}(B - A - S(A|B)) = \frac{1}{1-\alpha} D_0(A|B), \\ D_\alpha(A|B) &= \frac{1}{\alpha}(A - B - T_{1-\alpha}(B|A)) \le \frac{1}{\alpha}(A - B + S_1(A|B)) = \frac{1}{\alpha} D_1(A|B).\end{aligned}$$

□

By the following Theorem 2.5, it is shown that an operator value $D_\alpha(A|B)$ can be represented by the sum of two operator values for Tsallis entropies.

Theorem 2.5. *Let A and B be strictly positive operators. Then,*

$$D_\alpha(A|B) = -\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\}$$

holds for $\alpha \in (0,1)$.

Proof. This theorem can be shown as follows:

$$\begin{aligned}D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \\ &= \frac{\{(1-\alpha)A + \alpha B\} - \{(1-\alpha)(A \sharp_\alpha B) + \alpha(A \sharp_\alpha B)\}}{\alpha(1-\alpha)} \\ &= -\left\{\frac{(1-\alpha)(A \sharp_\alpha B) - (1-\alpha)A}{\alpha(1-\alpha)} + \frac{\alpha(B \sharp_{1-\alpha} A) - \alpha B}{\alpha(1-\alpha)}\right\} \\ &= -\left\{\frac{A \sharp_\alpha B - A}{\alpha} + \frac{B \sharp_{1-\alpha} A - B}{1-\alpha}\right\} = -\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\}.\end{aligned}$$

□

Theorem 2.5 gives a new viewpoint for operator valued α-divergence. Tsallis relative operator entropy $T_\alpha(A|B)$ can be regarded as the slope of the line through points A and $A \natural_\alpha B$. Since $-T_{1-\alpha}(B|A) = -\frac{B \natural_{1-\alpha} A - B}{1-\alpha} = \frac{B - A \natural_\alpha B}{1-\alpha}$, we can regard this operator value as the slope of the line through points $A \natural_\alpha B$ and B. Therefore, we can regard $D_\alpha(A|B)$ as the difference between the slops of these two lines. We illustrate the quantity corresponding to $D_\alpha(A|B)$ by bold straight line in Figure 1.

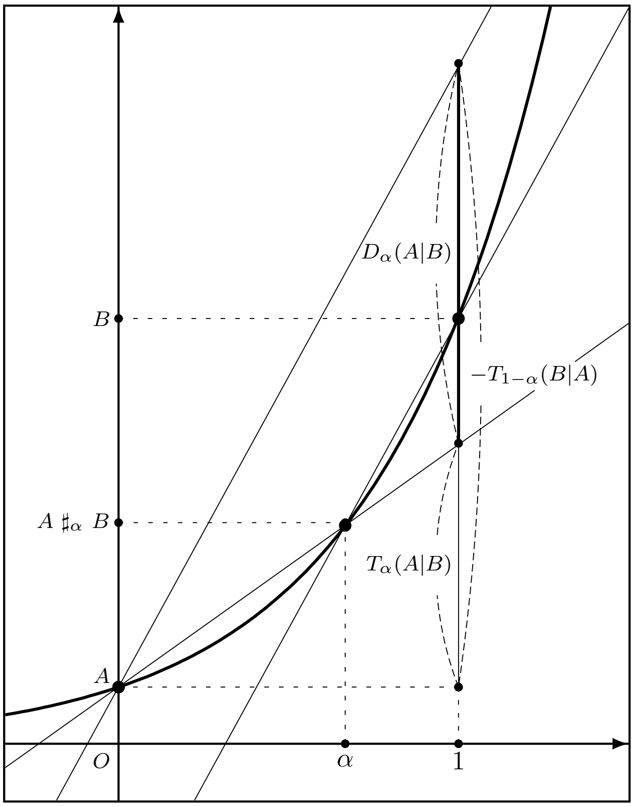

Figure 1: $D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B)$.

The following result can be shown by Theorem 2.5 and $(*)$ easily.

Corollary 2.6. *Let A and B be strictly positive operators. Then,*

$$D_\alpha(A|B) \leq S_1(A|B) - S(A|B)$$

holds for $\alpha \in (0,1)$.

3 Noncommutative ratio translation on the path. First, we show the following result on translation of generalized relative operator entropies.

Proposition 3.1. *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) = S_{u+v}(A|B)$$

holds for u, $v \in \mathbf{R}$.

Proof. This can be shown as follows:

$$\begin{aligned}
& (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) \\
= \;& A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} A^{\frac{1}{2}} A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{-u} A^{-\frac{1}{2}} \\
& \qquad \times A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \\
= \;& A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \\
= \;& S_{u+v}(A|B).
\end{aligned}$$

□

We can regard $S_u(A|B)$ and $S_{u+v}(A|B)$ as tangent vectors at u and $u+v$ on the path $A \natural_w B$, respectively. Then, Proposition 3.1 means that $S_{u+v}(A|B)$ is parallelly transferring $S_u(A|B)$ by v along the path.

Here, we define the following noncommutative ratio on the path $A \natural_w B$, and give a new viewpoint for the equality in Proposition 3.1.

Definition 3.2. For strictly positive operators A and B, and for u, $v \in \mathbf{R}$, *noncommutative ratio* on the path $A \natural_w B$ is defined as follows:

$$\mathcal{R}(u, v; A, B) \equiv (A \natural_{u+v} B)(A \natural_u B)^{-1}.$$

We have the following property of noncommutative ratio.

Proposition 3.3. *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} = (A \natural_v B)A^{-1},$$

that is,

$$\mathcal{R}(u, v; A, B) = \mathcal{R}(0, v; A, B) = (A \natural_v B)A^{-1}$$

holds for u, $v \in \mathbf{R}$.

Proof. This can be shown as follows:

$$\begin{aligned}
\mathcal{R}(u, v; A, B) &= (A \natural_{u+v} B)(A \natural_u B)^{-1} \\
&= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} A^{\frac{1}{2}} A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{-u} A^{-\frac{1}{2}} \\
&= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{v} A^{-\frac{1}{2}} \\
&= (A \natural_v B)A^{-1} \\
&= \mathcal{R}(0, v; A, B).
\end{aligned}$$

□

By Proposition 3.3, $\mathcal{R}(u, v; A, B)$ does not depend on u. So, we denote $\mathcal{R}(u, v; A, B)$ by $\mathcal{R}(v; A, B)$, or simply $\mathcal{R}(v)$ in the rest of this section. We call multiplying by $\mathcal{R}(v)$ from the left side *noncommutative ratio translation.*

From Proposition 3.1 and Definition 3.2, we get the following immediately.

Corollary 3.4. *Let A and B be strictly positive operators. Then,*

$$\mathcal{R}(v)S_u(A|B) = S_{u+v}(A|B)$$

hold for u, $v \in \mathbf{R}$.

In particular, by putting $u = 0$ in Corollary 3.4 , we have

$$\mathcal{R}(v)S(A|B) = S_v(A|B).$$

Tsallis relative operator entropy can be extended as follows: For strictly positive operators A and B, and for $u \in \mathbf{R}$,

$$T_u(A|B) \equiv \frac{A \natural_u B - A}{u}.$$

From above definition and Proposition 3.3, we have

$$\mathcal{R}(v)T_u(A|B) = \frac{A \natural_{u+v} B - A \natural_v B}{u}.$$

Let n be an integer. Then, $\mathcal{R}(n) = (A \natural_n B)A^{-1} = (BA^{-1})^n$ holds. In [9], we showed a similar relation to $(*)$ as follows: For strictly positive operators A, B and $u \in (n, n+1)$,

$$(\star) \quad S_n(A|B) \leq \frac{A \natural_u B - A \natural_n B}{u-n} \leq S_u(A|B) \leq \frac{A \natural_{n+1} B - A \natural_u B}{n+1-u} \leq S_{n+1}(A|B),$$

or equivalently,

$$(\star\star) \qquad (BA^{-1})^n S(A|B) \leq (BA^{-1})^n T_{u-n}(A|B) \leq (BA^{-1})^n S_{u-n}(A|B)$$
$$\leq -(BA^{-1})^n T_{n+1-u}(B|A) \leq (BA^{-1})^n S_1(A|B).$$

The relation $(\star)$ can be expressed by $(\star\star)$ which is the transferred form of $(*)$ by n along the path. We call this the waving property in [9].

The relation $(\star\star)$ can be generalized as follows:

Corollary 3.5. *Let A and B be strictly positive operators and $u \in (v, v+1)$. Then,*

$$S_v(A|B) = \mathcal{R}(v)S(A|B) \leq \mathcal{R}(v)T_{u-v}(A|B) \leq S_u(A|B)$$
$$\leq -\mathcal{R}(v)T_{v+1-u}(B|A) = \mathcal{R}(u)T_{v+1-u}(A|B) \leq S_{v+1}(A|B)$$

hold for u, $v \in \mathbf{R}$.

Proof. We only show the relation $-\mathcal{R}(v)T_{v+1-u}(B|A) = \mathcal{R}(u)T_{v+1-u}(A|B)$ since the others can be obtained by the similar way to the proof in [9].
By Proposition 3.3, we have

$$\begin{aligned}
\mathcal{R}(v)T_{v+1-u}(B|A) &= (A \natural_v B)A^{-1}T_{v+1-u}(B|A) = (A \natural_v B)A^{-1}\frac{B \natural_{v+1-u} A - B}{v+1-u} \\
&= \frac{(A \natural_v B)A^{-1}(A \natural_{u-v} B) - (A \natural_v B)A^{-1}B}{v+1-u} \\
&= \frac{A \natural_u B - A \natural_{v+1} B}{v+1-u} \\
&= (A \natural_u B)A^{-1}\frac{A - A \natural_{v+1-u} B}{v+1-u} \\
&= -\mathcal{R}(u)T_{v+1-u}(A|B).
\end{aligned}$$

□

We apply noncommutative ratio translation to fundamental relations shown in section 2, and try to show the similar property to the waving property. To see this, we make some preparations.

Lemma 3.6. *Let A and B be strictly positive operators. Then,*

$$(A \natural_u B) \natural_w (A \natural_{u+v} B) = A \natural_{u+vw} B$$

holds for u, v, $w \in \mathbf{R}$.

Proof. By Lemma 4.2 in [11], $T^*(X \natural_u Y)T = (T^*XT) \natural_u (T^*YT)$ holds for any invertible operator T, for any positive invertible operators X, Y and for $u \in \mathbf{R}$. Therefore, we have

$$\begin{aligned}(A \natural_u B) \natural_w (A \natural_{u+v} B) &= \left\{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^u A^{\frac{1}{2}}\right\} \natural_w \left\{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+v} A^{\frac{1}{2}}\right\} \\ &= A^{\frac{1}{2}}\left\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^u \natural_w (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+v}\right\} A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+vw} A^{\frac{1}{2}} \\ &= A \natural_{u+vw} B.\end{aligned}$$

□

In [12], Kamei showed some kind of the additivity for entropy

$$S(A|A \sharp_t B) = tS(A|B)$$

for $t \in [0,1]$. The following is an extension of this result.

Proposition 3.7. *Let A and B be strictly positive operators. Then,*

$$S_u(A \natural_v B|A \natural_{v+w} B) = wS_{v+uw}(A|B)$$

holds for u, v, $w \in \mathbf{R}$.

Proof. Since $\lim_{t\to 0} \frac{x^{u+t}-x^u}{t} = x^u \lim_{t\to 0} \frac{x^t-1}{t} = x^u \log x$ holds for $x > 0$, we have

$$\lim_{t\to 0} \frac{X \natural_{u+t} Y - X \natural_u Y}{t} = S_u(X|Y)$$

for strictly positive operators X, Y and $w \in \mathbf{R}$. Therefore, by Lemma 3.6, we get

$$\begin{aligned}S_u(A \natural_v B|A \natural_{v+w} B) &= \lim_{t\to 0} \frac{(A \natural_v B) \natural_{u+t} (A \natural_{v+w} B) - (A \natural_v B) \natural_u (A \natural_{v+w} B)}{t} \\ &= w \lim_{wt\to 0} \frac{A \natural_{v+uw+wt} B - A \natural_{v+uw} B}{wt} \quad = \quad wS_{v+uw}(A|B).\end{aligned}$$

□

We give the special cases of Proposition 3.7 which are useful in our calculations.

Corollary 3.8. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} &(1) \qquad S(A \natural_v B|A \natural_{v+w} B) &=& \; wS_v(A|B), \\ &(2) \qquad S_u(A \natural_v B|A \natural_{v+1} B) &=& \; S_{u+v}(A|B) \end{aligned}$$

hold for v, $w \in \mathbf{R}$.

For the following two operator values which appear in ($\star$),

$$\begin{aligned} \frac{A \natural_u B - A \natural_n B}{u-n} &= \frac{(A \natural_n B) \sharp_{u-n} (A \natural_{n+1} B) - A \natural_n B}{u-n} \\ &= T_{u-n}(A \natural_n B|A \natural_{n+1} B), \\ \frac{A \natural_{u+1} B - A \natural_u B}{n+1-u} &= -\frac{(A \natural_n B) \sharp_{u-n} (A \natural_{n+1} B) - A \natural_{n+1} B}{n+1-u} \\ &= -\frac{(A \natural_{n+1} B) \sharp_{1-(u-n)} (A \natural_n B) - A \natural_{n+1} B}{1-(u-n)} \\ &= -T_{1-(u-n)}(A \natural_{n+1} B|A \natural_n B) \end{aligned}$$

hold.

From these facts and (2) in Corollary 3.8, the relation ($\star$) is equivalent to the following:

$$\begin{aligned} S(A \natural_n B|A \natural_{n+1} B) &\leq T_{u-n}(A \natural_n B|A \natural_{n+1} B) \leq S_{u-n}(A \natural_n B|A \natural_{n+1} B) \\ &\leq -T_{1-(u-n)}(A \natural_n B|A \natural_{n+1} B) \leq S_1(A \natural_n B|A \natural_{n+1} B). \end{aligned}$$

We show the similar phenomena for each operator value $S_u(A|B)$, $T_u(A|B)$, and $D_\alpha(A|B)$.

Theorem 3.9. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} &(1) \qquad \mathcal{R}(v)S_u(A|B) = S_u(A \natural_v B|A \natural_{v+1} B), \\ &(2) \qquad \mathcal{R}(v)T_u(A|B) = T_u(A \natural_v B|A \natural_{v+1} B) \end{aligned}$$

hold for u, $v \in \mathbf{R}$.

In particular, by putting $u = 0$ in Theorem 3.9, we have

$$\mathcal{R}(v)S(A|B) = S(A \natural_v B|A \natural_{v+1} B).$$

Proof. **(1)** By Corollary 3.4 and (2) in Corollary 3.8, we have

$$\mathcal{R}(v)S_u(A|B) = S_{u+v}(A|B) = S_u(A \natural_v B|A \natural_{v+1} B).$$

(2) By Proposition 3.3 and Lemma 3.6, we get

$$\begin{aligned} \mathcal{R}(v)T_u(A|B) &= (A \natural_v B)A^{-1}T_u(A|B) \\ &= \frac{(A \natural_v B)A^{-1}(A \natural_u B) - (A \natural_v B)A^{-1}A}{u} \\ &= \frac{A \natural_{u+v} B - A \natural_v B}{u} \\ &= \frac{(A \natural_v B) \natural_u (A \natural_{v+1} B) - A \natural_v B}{u} \\ &= T_u(A \natural_v B|A \natural_{v+1} B). \end{aligned}$$

□

Theorem 3.10. *Let A and B be strictly positive operators. Then,*

$$\mathcal{R}(v)D_\alpha(A|B) = D_\alpha(A \natural_v B|A \natural_{v+1} B)$$

holds for $\alpha \in (0,1)$ and $v \in \mathbf{R}$.

Proof. By Proposition 3.3 and Lemma 3.6, we have

$$\begin{aligned}
\mathcal{R}(v)D_\alpha(A|B) &= (A \natural_v B)A^{-1}D_\alpha(A|B) \\
&= (A \natural_v B)A^{-1}\frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \\
&= \frac{(1-\alpha)(A \natural_v B)A^{-1}A + \alpha(A \natural_v B)A^{-1}B - (A \natural_v B)A^{-1}(A \sharp_\alpha B)}{\alpha(1-\alpha)} \\
&= \frac{(1-\alpha)(A \natural_v B) + \alpha(A \natural_{v+1} B) - A \natural_{v+\alpha} B}{\alpha(1-\alpha)} \\
&= \frac{(A \natural_v B) \nabla_\alpha (A \natural_{v+1} B) - (A \natural_v B) \sharp_\alpha (A \natural_{v+1} B)}{\alpha(1-\alpha)} \\
&= D_\alpha(A \natural_v B|A \natural_{v+1} B).
\end{aligned}$$

□

Theorem 3.9 can be generalized as follows.

Theorem 3.11. *Let A and B be strictly positive operators. Then,*

(1) $$w\mathcal{R}(v)S_{uw}(A|B) = S_u(A \natural_v B|A \natural_{v+w} B),$$

(2) $$w\mathcal{R}(v)T_{uw}(A|B) = T_u(A \natural_v B|A \natural_{v+w} B)$$

hold for u, v, $w \in \mathbf{R}$.

Proof. **(1)** By Corollary 3.4 and Proposition 3.7, we have

$$w\mathcal{R}(v)S_{uw}(A|B) = wS_{v+uw}(A|B) = S_u(A \natural_v B|A \natural_{v+w} B).$$

(2) By Proposition 3.3 and Lemma 3.6, we get

$$\begin{aligned}
w\mathcal{R}(v)T_{uw}(A|B) &= w(A \natural_v B)A^{-1}T_{uw}(A|B) \\
&= w\frac{(A \natural_v B)A^{-1}(A \natural_{uw} B) - (A \natural_v B)A^{-1}A}{uw} \\
&= \frac{A \natural_{v+uw} B - A \natural_v B}{u} \\
&= \frac{(A \natural_v B) \natural_u (A \natural_{v+w} B) - A \natural_v B}{u} \\
&= T_u(A \natural_v B|A \natural_{v+w} B).
\end{aligned}$$

□

By using Theorem 3.10, we get the following properties by applying noncommutative ratio translation to fundamental relations between operator valued α-divergences and relative operator entropies shown in section 2.

Theorem 3.12. *Let A and B be strictly positive operators. Then,*

(1-a) $$\mathcal{R}(v)D_0(A|B) = D_0(A \natural_v B|A \natural_{v+1} B),$$

(1-b) $$\mathcal{R}(v)D_1(A|B) = D_1(A \natural_v B|A \natural_{v+1} B),$$

(2) $$\mathcal{R}(v)D_\alpha(A|B) = -\mathcal{R}(v)\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\},$$

(3-a) $$0 \le \mathcal{R}(v)D_\alpha(A|B) \le \frac{1}{1-\alpha}\mathcal{R}(v)D_0(A|B),$$

(3-b) $$0 \le \mathcal{R}(v)D_\alpha(A|B) \le \frac{1}{\alpha}\mathcal{R}(v)D_1(A|B),$$

(4) $$\mathcal{R}(v)D_\alpha(A|B) \le \mathcal{R}(v)\{S_1(A|B) - S(A|B)\}$$

hold for $\alpha \in (0,1)$ and $v \in \mathbf{R}$.

Proof. These can be obtained by applying Theorem 3.9 and Theorem 3.10 to Proposition 2.2, Theorem 2.5, Theorem 2.4, and Corollary 2.6. □

Remark 1. Although noncommutative ratio translation has been defined as multiplying each operator value by noncommutative ratio $\mathcal{R}(v)$ from the left side, this is equivalent to multiplying the operator value by $\mathcal{R}(v)^*$ from the right side. For instance, in Theorem 3.9,

(1) $$\mathcal{R}(v)S_u(A|B) = S_u(A|B)\mathcal{R}(v)^*,$$

(2) $$\mathcal{R}(v)T_u(A|B) = T_u(A|B)\mathcal{R}(v)^*$$

hold for u, $v \in \mathbf{R}$.

Remark 2. In [9], we introduced $D_r(A,B) \equiv A \natural_{r+1} B - A \natural_r B - S_r(A|B)$ for $r \in \mathbf{R}$ as a generalization of $D_{FK}(A|B) = D_0(A|B)$. We remark that $D_v(A,B) = \mathcal{R}(v)D_0(A|B)$ holds for $v \in \mathbf{R}$ by (2) in Corollary 3.8 and (1-a) in Theorem 3.12.

4 α-divergence for operator distributions.

On operator entropies for operator distributions $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$, in [9], we obtained that the relations

$$S(\mathbb{A}|\mathbb{B}) \le I_\alpha(\mathbb{A}|\mathbb{B}) \le T_\alpha(\mathbb{A}|\mathbb{B}) \le 0,$$
$$0 \le -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) \le -I_{1-\alpha}(\mathbb{B}|\mathbb{A}) \le S_1(\mathbb{A}|\mathbb{B})$$

and

$$T_\alpha(\mathbb{A}|\mathbb{B}) \le S_\alpha(\mathbb{A}|\mathbb{B}) \le -T_{1-\alpha}(\mathbb{B}|\mathbb{A})$$

hold for $0 < \alpha < 1$, where $I_\alpha(\mathbb{A}|\mathbb{B}) \equiv \frac{1}{\alpha}\log\sum_{i=1}^n A_i \sharp_\alpha B_i$ is Rényi relative operator entropy for operator distributions. By these inequalities and Corollary 3.5, we have

$$S(\mathbb{A}|\mathbb{B}) \le T_\alpha(\mathbb{A}|\mathbb{B}) \le S_\alpha(\mathbb{A}|\mathbb{B}) \le -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) = T^\alpha_{1-\alpha}(\mathbb{A}|\mathbb{B}) \le S_1(\mathbb{A}|\mathbb{B})$$

for $0 < \alpha < 1$, where $T^v_\alpha(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n \mathcal{R}_i(v)T_\alpha(A_i|B_i)$ for $v \in \mathbf{R}$ and $\mathcal{R}_i(v) = \mathcal{R}(v; A_i, B_i)$, as used in section 3. In this section, we investigate fundamental properties and relations between α-divergences and relative operator entropies for operator distributions.

Here, we define α-divergence for operator distributions.

Definition 4.1. For operator distributions $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$, and for $\alpha \in (0,1)$, α-divergence for operator distributions is defined as follows:

$$D_\alpha(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n D_\alpha(A_i|B_i) = \sum_{i=1}^n \frac{A_i\ \nabla_\alpha\ B_i - A_i\ \sharp_\alpha\ B_i}{\alpha(1-\alpha)}.$$

As in section 2, we show fundamental properties of α-divergences for operator distributions.

Proposition 4.2. *Let $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$ be operator distributions. Then,*

$$D_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{\alpha\to 0} D_\alpha(\mathbb{A}|\mathbb{B}) = -S(\mathbb{A}|\mathbb{B}), \tag{1}$$

$$D_1(\mathbb{A}|\mathbb{B}) \equiv \lim_{\alpha\to 1} D_\alpha(\mathbb{A}|\mathbb{B}) = S_1(\mathbb{A}|\mathbb{B}) \tag{2}$$

hold.

Proof. We only show the proof of equality (1) since the equality (2) can be shown similarly. By Proposition 2.2, we have

$$D_0(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n D_0(A_i|B_i) = \sum_{i=1}^n \{B_i - A_i - S(A_i|B_i)\} = -S(\mathbb{A}|\mathbb{B}).$$

□

By Proposition 2.3, Theorem 2.4, Theorem 2.5 and Corollary 2.6, we get the following Proposition 4.3, Theorem 4.4, Theorem 4.5 and Corollary 4.6, respectively.

Proposition 4.3. *Let $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$ be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) = -\frac{1}{1-\alpha} T_\alpha(\mathbb{A}|\mathbb{B}) = -\frac{1}{\alpha} T_{1-\alpha}(\mathbb{B}|\mathbb{A}),\ \textit{for } \alpha \in (0,1), \tag{1}$$

$$D_{1-\alpha}(\mathbb{B}|\mathbb{A}) = D_\alpha(\mathbb{A}|\mathbb{B}),\ \textit{for } \alpha \in [0,1] \tag{2}$$

hold.

Theorem 4.4. *Let $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$ be operator distributions. Then,*

$$0 \le D_\alpha(\mathbb{A}|\mathbb{B}) \le \frac{1}{1-\alpha} D_0(\mathbb{A}|\mathbb{B}), \tag{1}$$

$$0 \le D_\alpha(\mathbb{A}|\mathbb{B}) \le \frac{1}{\alpha} D_1(\mathbb{A}|\mathbb{B}) \tag{2}$$

hold for $\alpha \in (0,1)$.

Theorem 4.5. *Let $\mathbb{A} = (A_1,\ A_2,\ \cdots,\ A_n)$ and $\mathbb{B} = (B_1,\ B_2,\ \cdots,\ B_n)$ be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) = -\{T_\alpha(\mathbb{A}|\mathbb{B}) + T_{1-\alpha}(\mathbb{B}|\mathbb{A})\}$$

holds for $\alpha \in (0,1)$.

Corollary 4.6. *Let* $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ *be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) \leq S_1(\mathbb{A}|\mathbb{B}) - S(\mathbb{A}|\mathbb{B})$$

holds for $\alpha \in (0,1)$.

From above discussion, we remark that the relations

$$\alpha T_\alpha(\mathbb{A}|\mathbb{B}) = (1-\alpha)T_{1-\alpha}(\mathbb{B}|\mathbb{A})$$

and

$$T_\alpha(\mathbb{A}|\mathbb{B}) \geq -(1-\alpha)\{S_1(\mathbb{A}|\mathbb{B}) - S(\mathbb{A}|\mathbb{B})\}$$

hold for $\alpha \in (0,1)$.

Finally, we apply noncommutative ratio translation to α-divergence for operator distributions by the following notation:

Definition 4.7. Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. For $v \in \mathbf{R}$ and $\alpha \in (0,1)$, we define $D_\alpha^v(\mathbb{A}|\mathbb{B})$ as follows:

$$D_\alpha^v(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n \mathcal{R}_i(v) D_\alpha(A_i|B_i).$$

Then, we get the following from Theorem 3.12.

Corollary 4.8. *Let* $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ *be operator distributions. Then,*

(1-a) $$D_0^v(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n \mathcal{R}_i(v) D_0(A_i|B_i),$$

(1-b) $$D_1^v(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n \mathcal{R}_i(v) D_1(A_i|B_i),$$

(2) $$D_\alpha^v(\mathbb{A}|\mathbb{B}) = -\sum_{i=1}^n \mathcal{R}_i(v)\{T_\alpha(A_i|B_i) + T_{1-\alpha}(B_i|A_i)\},$$

(3-a) $$0 \leq D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \frac{1}{1-\alpha} D_0^v(\mathbb{A}|\mathbb{B}),$$

(3-b) $$0 \leq D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \frac{1}{\alpha} D_1^v(\mathbb{A}|\mathbb{B}),$$

(4) $$D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \sum_{i=1}^n \mathcal{R}_i(v)\{S_1(A_i|B_i) - S(A_i|B_i)\}$$

hold for $\alpha \in (0,1)$ *and* $v \in \mathbf{R}$.

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(1) Maebashi Institute of Technology, 460-1, Kamisadori-machi, Maebashi, Gunma, Japan, 371-0816. isa@maebashi-it.ac.jp

(2) Maebashi Institute of Technology, 460-1, Kamisadori-machi, Maebashi, Gunma, Japan, 371-0816. m-ito@maebashi-it.ac.jp

(3) 1-1-3, Sakuragaoka, Kanmakicho, Kitakaturagi-gun, Nara, Japan, 639-0202. ekamei1947@yahoo.co.jp

(4) Maebashi Institute of Technology, 460-1, Kamisadori-machi, Maebashi, Gunma, Japan, 371-0816. tohyama@maebashi-it.ac.jp

(5) Maebashi Institute of Technology, 460-1, Kamisadori-machi, Maebashi, Gunma, Japan, 371-0816. masayukiwatanabe@maebashi-it.ac.jp

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