QUASI DROP PROPERTIES, (α)-PROPERTIES AND THE STRICT MACKEY CONVERGENCE

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Abstract. The Mackey (α)-property and quasi-weak (α)-property are defined. For disks in locally convex spaces with the strict Mackey convergence condition, the quasi drop property (resp. quasi weak drop property) implies Mackey (α)-property (resp. quasi weak (α)-property). In Frechet spaces, the quasi drop property and Mackey (α)-property are equivalent. Other equivalences are given.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the drop $D(x, B_X)$ defined by an element $x \in X \setminus B_X$ we mean the set *conv* $(\{x\} \cup B_X)$. Danes [2] proved that, for any Banach space $(X, \|\cdot\|)$ and every non-empty closed set $A \subset X$ at positive distance from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. Motivated by Danes theorem, Rolewicz [21] introduced the notion of drop property for the norm of a Banach space: the norm $\|\cdot\|$ in X has the drop property if for every non-empty closed set A disjoint from B_X there exists $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. He proved that if the norm has the drop property then $(X, \|\cdot\|)$ is reflexive (see [21] Theorem 5). Later, Montesinos (see [14] Theorem 4) proved that a Banach space is reflexive if and only if it can be renormed to have the drop property.

Let B be a subset of a Banach space $(X, \|\cdot\|)$. The Kuratowski index of noncompactness of B, $\alpha(B)$, is the infimum of all positive numbers r such that B can be covered by a finite number of sets of diameter less than r. Given $f \in X^*$ such that $\|f\| = 1$ and $0 < \delta \leq 2$, consider the slice $S(f, B_X, \delta) = \{x \in B_X : f(x) \geq 1 - \delta\}$. The norm $\|\cdot\|$ in a Banach space X has property (α), if $\lim_{\delta \to 0} \alpha(S(f, B_X, \delta)) = 0$ for every $f \in X^*$, $\|f\| = 1$. Also, Rolewicz ([21] Theorem 4), proved that if the norm has the drop property then it has property (α), and Montesinos ([14] Theorem 3) established that these two properties are equivalent.

Giles, Sims and Yorke [3] said that the norm has the weak drop property if for every non-empty weakly sequentially closed set A disjoint from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$, and they proved that this property is equivalent to $(X, \|\cdot\|)$ being reflexive. Kutzarova [8] and Giles and Kutzarova [4] extended the discussion of these drop properties to closed bounded convex sets in Banach spaces. Cheng, Zhou and Zang [1], Zheng [23] and other authors studied those drop properties in locally convex spaces: a bounded, convex and closed subset

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B of a locally convex space (E,τ) is said to have the drop property if it is non-empty and for every non-empty sequentially closed subset $A \subset E$ disjoint from B there exists $a \in A$ such that $D(a,B) \cap A = \{a\}$.

Qiu, in [17] and Monterde and Montesinos in [13], introduced another drop properties in locally convex spaces: a non-empty closed bounded convex subset B of a locally convex space (E,τ) is said to have the quasi weak drop (resp. quasi drop) property if for every non-empty weakly closed (resp. closed) subset $A \subset E$ disjoint from B, there exists an $x_0 \in A$ such that $D(x_0,B) \cap A = \{x_0\}$. In [17] and [18], Qiu established a number of equivalences for the quasi-weak drop property in Frèchet spaces and in quasi-complete locally convex spaces. He characterized reflexivity of those spaces by the condition that every closed bounded convex subset of the space must satisfy the quasi weak drop property. Concerning drop properties and their applications, see for example [1], [2], [9], [10], [11],[12], [16], [19] and [20].

Also, in [17], Qiu proved that for every non-empty bounded, closed and convex subset of a Frechet space $B \subset (E,\tau)$ quasi-weak drop property and weak compactness are equivalent, and he asked if this and other properties can be extended to locally complete spaces. In [18] he answered these questions in negative. In order to extend some results on quasi drop properties to a bigger family of locally convex spaces, based on techniques of Qiu and Rolewickz, we consider locally convex spaces satisfying the strict Mackey convergence condition (sMc), and finally, based on a theorem of A. Martellotti we characterize quasi drop property for Frechet spaces.

2. PRELIMINARIES

Throughout this paper, (E,τ) is a Hausdorff locally convex space over $\mathbb{R}$. A closed, bounded and absolutely convex subset will be called a disk. If D is a disk in the space (E,τ) then we let E_D denote the linear span of D, equipped with the topology given by ρ_D the gauge (Minkowski's functional) of D. This topology has a base of zero neighborhoods of the form $\{aD : a > 0\}$, and makes E_D into a normed space such that $\tau|_{E_D} \leq \rho_D|_{E_D}$, for τ the original topology of E. And (E,τ) is said to be locally complete if every disk $D \subset E$, is a Banach disk, that is (E_D, ρ_D) is a Banach space. Note that for metrizable spaces, completeness and local completeness are equivalent. For local completeness, see [7] and [15].

According to Grothendieck (see [6]), we have that a space (E,τ) satisfies the strict Mackey convergence condition (sMc) if for every bounded subset $B \subset (E,\tau)$, there exists a disk $D \subset E$ containing B such that the topologies of E and E_D agree on B, i.e. $\tau|_B = \rho_D|_B$. Note that every metrizable space satisfies the sMc (see [15], 5.1.27(ii)). And following Gilsdorf [5], every space with a boundedly compatible web satisfies the sMc (see [5] and [7] for webs). In particular, every strictly barrelled space satisfies the sMc (see [22] for strictly barrelled spaces).

Let $B \subset (E,\tau)$ be a disk, for $f \in (E,\tau)' \setminus \{0\}$ let $M_f = \sup\{f(x) : x \in B\}$, and for $\delta > 0$ consider the slice $S(f,B,\delta) = \{x \in B : f(x) \geq M_f - \delta\}$. The disk B is said to have the (α)-property if for every $f \in (E,\tau)' \setminus \{0\}$ and for every neighborhood U of 0 in τ, there exists $\delta > 0$ such that $S(f,B,\delta)$ can be covered by a finite number of translates of U.

Suppose now that (E,τ) is a space that satisfies sMc and $B \subset E$ is a disk. So, there exists a disk $D \subset E$ containing B such that $\tau|_B = \rho_D|_B$. The Kuratowski index of noncompactness of $S(f,B,\delta)$ associated to the disk D is $\alpha_D(S(f,B,\delta))$ the infimum of all positive numbers r such that $S(f,B,\delta)$ is covered by a finite number of sets of ρ_D-diameter less than r. The disk $B\subset E$ is said to have the Mackey (α)-property if for D as above $\lim_{\delta\to 0}\alpha_D(S(f,B,\delta)=0$ for every $f\in(E,\tau)'\smallsetminus\{0\}$. In this case, due to the fact that ρ_D and τ induce the same topology on B, we get that B has the (α)-property with respect to τ. Obviously, if $(E,\|\cdot\|)$ is a normed space both (α)-properties coincide.

3. RESULTS

Let (E,τ) be a locally convex space and $B\subset E$ a disk.

a) Suppose there exists a disk $D\subset E$ such that $B\subset D$ and B has the quasi drop (resp. quasi weak drop) property in (E_D,ρ_D). Note that for every non-empty subset $A\subset E$, τ-closed (resp. $\sigma(E,E')$-closed), we have $A_D:=A\cap E_D$ is ρ_D-closed (resp. $\sigma(E_D,E'_D)$-closed, where $E'_D=(E_D,\rho_D)')$. Then B has the quasi drop (resp. quasi-weak drop) property in (E,τ).

b) Now, for $f\in(E,\tau)'\setminus\{0\}$, find $x_0\in E$ such that $f(x_0)>M_f$; where $M_f:=\sup\{f(x):x\in B\}$. Suppose that for the disk $C=\overline{abconv}\,\{B\cup\{x_0\}\}$ in E, there exists a disk $D\subset E$ containing C such that $\tau|_C=\rho_D|_C$; so, in particular, $\tau|_B=\rho_D|_B$. If $\inf_{\varepsilon>0}\alpha_D(S(f,B,\varepsilon))>2\delta_0$ for some $\delta_0>0$, then (see [21], Theorem 4) for every finite dimensional subspace $L\subset E_D$ we have:

$$\sup_{x\in S(f,B,\varepsilon)}(\inf_{y\in L}\rho_D(x-y))\geq\frac{1}{2}\inf_{\varepsilon>0}\alpha_D(S(f,B,\varepsilon))>\delta_0 \qquad \cdots(1)$$

Take $\varepsilon_1<f(x_0)-M_f$. And choose $\overline{x_1}\in S(f,B,\varepsilon_1)$ such that

$$\inf\{\rho_D(\overline{x_1}-z):z\in span\,\{x_0\}\}>\delta_0.$$

Let $x_1=\frac{x_0+\overline{x_1}}{2}$, then

$$f(x_1)=f(\frac{x_0+\overline{x_1}}{2})=\frac{f(x_0)}{2}+\frac{f(\overline{x_1})}{2}>\frac{M_f+\varepsilon_1}{2}+\frac{M_f-\varepsilon_1}{2}=M_f.$$

Moreover

$$\inf\{\rho_D(x_1-z):z\in span\,\{x_0\}\}=\frac{1}{2}\inf\{\rho_D(\overline{x_1}-z):z\in span\,\{x_0\}\}>\frac{\delta_0}{2}$$

Now, suppose we have $\{x_0,x_1,...,x_n\}$, such that $x_i\neq x_j$ if $i\neq j\leq n$, and
i) $f(x_i)>M_f$
ii) $\inf\{\rho_D(x_i-z):z\in span\,\{x_0,...,x_{i-1}\}\}>\frac{\delta_0}{2}$
iii) $x_i\in D(x_{i-1},B)$
for every $i\leq n$. Take $\varepsilon_{n+1}<f(x_n)-M_f$ and by (1) find $\overline{x_{n+1}}\in S(f,B,\varepsilon_{n+1})$ such that

$$\inf\{\rho_D(\overline{x_{n+1}}-z):z\in span\,\{x_0,x_1,...,x_n\}\}>\delta_0.$$

Let $x_{n+1}=\frac{x_n+\overline{x_{n+1}}}{2}$ then, in an analogous way to x_1, $f(x_{n+1})>M_f$ and

$$\begin{aligned}&\inf\{\rho_D(x_{n+1}-z):z\in span\,\{x_0,...,x_n\}\}\\ =\;&\frac{1}{2}\inf\{\rho_D(\overline{x_{n+1}}-z):z\in span\,\{x_0,...,x_n\}\}>\frac{\delta_0}{2}.\end{aligned}$$

Then the sequence $(x_n)_n$ satisfies (i,ii,iii) and the set $A=\{x_0,x_1,...,x_n,...\}\subset C$ is ρ_D-closed. Since the topologies τ and ρ_D agree on C, A is τ-closed and $A\cap B=\emptyset$. Hence B, does not have the quasi drop property. So, we conclude

Proposition 1. *Let (E,τ) be a locally convex space satisfying the sMc and $B \subset E$ a disk. Suppose that B has the quasi drop property. Then there exists a disk $D \subset E$ containing B such that $\tau|_B = \rho_D|_B$ and B has the Mackey (α)-property associated to the disk D. B has (α)-property respect to τ, too.*

For the next proposition, we will suppose that the disk $B \subset E$ has the Mackey (α)-property associated to a Banach disk D. So, if we take $x_n \in S(f, B, \frac{1}{n})$ for every $n \in \mathbb{N}$, there exists a subsequence $(x_{n_k})_k \subset (x_n)_n$ convergent to some $x_0 \in B$ respect to ρ_D, and hence $x_{n_k} \to x_0 \in B$ respect to τ.

Proposition 2. *Let (E,τ) be a locally complete space satisfying the sMc. Let $B, D \subset E$ be disks such that $B \subset D$, $\tau|_B = \rho_D|_B$ and for every $f \in (E,\tau)' \setminus \{0\}$, $\alpha_D(S(f,B,\delta)) \to 0$; as $\delta \to 0$. Then B has the quasi drop property on (E,τ).*

Proof. Suppose that B does not have the quasi drop property on (E,τ). It implies, there exists a τ-closed set $A \subset E$, $A \cap B = \emptyset$, such that for every $x \in A$, $D(x,B) \cap A \neq \{x\}$. Let $A_D := A \cap E_D$. So A_D is ρ_D-closed in E_D and $A_D \cap B = \emptyset$. Note that $D(x,B) \cap A_D \neq \{x\}$, for every $x \in A_D$. By [17] lemma 2.2, $\rho_D(D(z,B) \cap A_D, B) = 0$, for every $z \in A_D$. Take $x_1 \in A_D$ fixed and find $f \in (E,\tau)'$ such that $f(x_1) = M_f + 1$, for $M_f = \sup\{f(x) : x \in B\}$. We may suppose $M_f > 1$. Since $\rho_D(D(x_1,B) \cap A_D, B) = 0$, take $y_1 \in S(f,B,1)$ and let $R = \rho_D(x_1 - y_1)$. Take $x_2 \in D(x_1,B) \cap A_D \setminus \{x_1\}$, then $\rho_D(D(x_2,B) \cap A_D, B) = 0$. And find $y_2 \in S(f,B,\frac{1}{2}) \setminus \{y_1\}$ such that $\rho_D(x_2 - y_2) < \frac{R}{2}$. Continue this process inductively, to construct sequences $(x_n)_n \in A_D \subset (E \setminus B)$ and $(y_n)_n \in B$, $y_n \in S(f,B,\frac{1}{n})$ such that $x_n \neq x_m$ and $y_n \neq y_m$ if $n \neq m$, and $\rho_D(x_n - y_n) < \frac{R}{n}$, for every $n \in \mathbb{N}$. Since B has the Mackey (α)-property associated to the Banach disk D, there exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ convergent to some $y_0 \in B$, respect to ρ_D and so, respect to τ. Since $\rho_D(x_{n_k} - y_{n_k}) < \frac{R}{n_k}$ then $x_{n_k} \to y_0 \in B$, respect to ρ_D and respect to τ. Recall that A_D is ρ_D-closed and A is τ-closed which implies $y_0 \in A_D \subset A$. Hence $y_0 \in A \cap B$. It is a contradiction, since A and B are disjoint. Hence B has the quasi drop property in (E,τ) □

In [12], A. Martellotti characterizes drop property in Banach spaces in the following way:

Theorem 1. ([12], *Theorem 3.7*) *Let $(X, \|\cdot\|)$ be a Banach space and $B \subset X$ a non-empty, closed and convex subset. The following are equivalent:*
i) B has the drop property
ii) For every non-empty closed subset $C \subset X$, with $B \cap C = \emptyset$ there exists $x \in C$ such that $D(x,B) \cap C$ is compact
iii) For every non-empty closed subset $C \subset X$, with $B \cap C = \emptyset$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in C$ such that $\alpha(D(x_\varepsilon, B) \cap C) < \varepsilon$

By Martellotti's theorem and propositions 1 and 2, we obtain the following characterization of quasi drop property for disks in Frechet spaces.

Theorem 2. *Let (E,τ) be a Frechet space and $B \subset E$ a disk. The following are equivalent:*
i) B has the quasi drop property
ii) B has the Mackey (α)-property
iii) For every non-empty τ-closed subset $C \subset E$, with $B \cap C = \emptyset$ there exists $x \in C$ such that $D(x,B) \cap C$ is compact

iv) For every non-empty τ-closed subset $C \subset E$, with $B \cap C = \emptyset$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in C$ and a Banach disk $D \subset E$ such that $D(x_\varepsilon, B) \subset D$, $\tau \big|_{D(x_\varepsilon,B)} = \rho_D \big|_{D(x_\varepsilon,B)}$ and $\alpha_D(D(x_\varepsilon, B) \cap C) < \varepsilon$

Proof. (i$\Leftrightarrow$ii) Follows from propositions 1 and 2.

(i$\Longleftrightarrow$iii$\Longleftrightarrow$iv) Follows from Martellotti's theorem, since (E,τ) satisfies the strict Mackey convergence condition and every disk $D \subset E$ is a Banach disk. □

Corollary 1. *Let (E,τ) be a Frèchet space and $B \subset E$ a disk. Then B has the quasi drop property if and only if there exists a Banach disk $D \subset E$ containing B such that $\tau|_B = \rho_D|_B$ and $\alpha_D(S(f,B,\delta)) \to 0$; as $\delta \to 0$, for every $f \in (E,\tau)' \setminus \{0\}$.*

Definition 1. *A disk C in the locally convex space (E,τ) is said to have the quasi weak (α)-property if for every $f \in (E,\tau)' \setminus \{0\}$ and every $(\alpha_n)_n \in \mathbb{R}^+$ such that $\alpha_n \to 0$, every non eventually constant sequence $(x_n)_n \in C$ such that $x_n \in S(f,C,\alpha_n)$, for every $n \in \mathbb{N}$, has a weakly cluster point in C.*

Proposition 3. *Let (E,τ) be a locally convex space and $C \subset E$ a disk with the quasi-weak drop property. Then C has the quasi-weak (α)-property.*

Proof. By [17] in the proof of Theorem 2.1(i), for every disk C with the quasi-weak drop property (and with no one additional condition) every stream in $E \setminus C$ has a weakly cluster point. Suppose that C does not have the quasi-weak (α)-property. So, there exists $f \in (E,\tau)' \setminus \{0\}$ such that for every $n \in \mathbb{N}$ there exists $x_n \in S(f,C,\frac{1}{4^n})$, $x_n \neq x_m$ if $n \neq m$ and $\{x_n : n \in \mathbb{N}\}$ has no weakly cluster points. Let $M = \sup\{f(x) : x \in C\}$. Find $x_0 \in E$ such that $f(x_0) = M + 2$. Define $y_n = \frac{1}{2^n}x_0 + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}}x_i$. Then

$$\begin{aligned} f(y_n) &= \frac{1}{2^n}f(x_0) + \sum_{i=1}^{n}\frac{1}{2^{n-i+1}}f(x_i) \geq \frac{M+2}{2^n} + \sum_{i=1}^{n}\frac{1}{2^{n-i+1}}(M - \frac{1}{4^i}) \\ &= M + \frac{3}{2^{n+1}} + \frac{1}{2^{2n+1}} > M + \frac{1}{2^{n+1}} \end{aligned}$$

So, $(y_n)_n$ is a stream. If there exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ with no weakly cluster points, then $A = \{y_{n_k} : k \in \mathbb{N}\}$ is weakly closed set such that does not satisfy the quasi weak drop condition for C. It would be a contradiction, and we would have finished. So, suppose that every subsequence $(y_{n_k})_k \subset (y_n)_n$ has at least a weakly cluster point. Let $(y_{n_k})_k$, $(y_{n_k+1})_k$ subsequences of $(y_n)_n$. Then there exists z_1, z_2 weakly cluster points of $(y_{n_k})_k$ and $(y_{n_k+1})_k$ respectively. Note that $y_{n_k+1} = \frac{1}{2}(y_{n_k} + x_{n_k+1})$. So, for every $\varphi \in (E,\tau)' \setminus \{0\}$, $\varphi(z_1)$ is a cluster point of the set $\{\varphi(y_{n_k}) : k \in \mathbb{N}\}$ and $\varphi(z_2)$ is a cluster point of the set $\{\varphi(y_{n_k+1}) : k \in \mathbb{N}\}$. Note that $\varphi(y_{n_k+1}) = \frac{1}{2}\varphi(y_{n_k}) + \frac{1}{2}\varphi(x_{n_k+1})$, so $2(\varphi(z_2) - \frac{1}{2}\varphi(z_1)) = \varphi(2z_2 - z_1)$ is a cluster point of $\{\varphi(2y_{n_k+1} - y_{n_k}) = \varphi(x_{n_k+1}) : k \in \mathbb{N}\}$, for every $\varphi \in (E,\tau)'$. Then $2z_2 - z_1$ is a weakly cluster point of $\{x_{n_k+1} : k \in \mathbb{N}\}$ and of $\{x_{n_k} : k \in \mathbb{N}\}$. A contradiction. Hence, C has the quasi-weak (α)-property. □

If we would want to give a converse to this result for locally convex spaces (E,τ) with sMc and local completeness conditions, we should consider $B \subset E$ a disk with the quasi-weak (α)-property. So, suppose that B does not have the quasi-weak drop property, then there exists a subset $A \subset E$, $\sigma(E,E')$-closed and disjoint from B such that $D(x,B) \cap A \neq \{x\}$ for every $x \in A$. Take $x_0 \in A$ fixed. Let $D \subset E$ be a Banach disk such that $B\cup\{x_0\}\subset D$ and $\tau\left|_{D(x_0,B)}=\rho_D\right|_{D(x_0,B)}$. Note that $A_D:=A\cap E_D$ is $\sigma(E_D,E'_D)$-closed, where $E'_D=(E_D,\rho_D)'$ and A_D is disjoint from B. Moreover, $D(x_0,B)\cap A_D$ is $\sigma(E_D,E'_D)$-closed and different from $\{x_0\}$. And for every $x\in D(x_0,B)\cap A_D$ we have $\{x\}\neq D(x,B)\cap[D(x_0,B)\cap A_D]=D(x,B)\cap A_D$. By [16] lemma 2.2, $\rho_D(B,D(z,B)\cap A_D)=0$, for every $z\in A_D$. In particular if $z=x_0$. Find $f\in(E,\tau)'$ such that $f(x_0)>M_f>1$, where $M_f=\sup\{f(y):y\in B\}$. Since $\rho_D(B,D(x_0,B)\cap A_D)=0$ then there exists $x_1\in D(x_0,B)\cap A_D$ and $y_1\in S(f,B,1)$ such that $\rho_D(x_1,y_1)<1$. In an analogous way, $\rho_D(B,D(x_1,B)\cap A_D)=0$ and there exist $x_2\in D(x_1,B)\cap A_D$ and $y_2\in S(f,B,\frac{1}{2})$ such that $\rho_D(x_2,y_2)<\frac{1}{2}$. So, inductively, construct sequences $(x_n)_n$, $(y_n)_n$; $x_n\neq x_m$ and $y_n\neq y_m$ if $n\neq m$, such that $x_n\in D(x_{n-1},B)\cap A_D$ and $y_n\in S(f,B,\frac{1}{n})$ and $\rho_D(x_n,y_n)<\frac{1}{n}$. Since B has the quasi-weak (α)-property, there exists $y^*\in B$ weakly cluster point of $\{y_n:n\in\mathbb{N}\}$ and since $\rho_D(x_n,y_n)\to 0$, then $x_n-y_n\to 0$ respect to $\sigma(E_D,E'_D)$ and respect to $\sigma(E,E')$. Then $y^*\in B$ is a weakly cluster point of $\{x_n:n\in\mathbb{N}\}\subset A_D$. Since A_D is $\sigma(E_D,E'_D)$-closed, then $y^*\in A_D\subset A$, i.e. $y^*\in A\cap B$, a contradiction. And we have that B has the quasi weak drop property. But a theorem of Qiu in [16], ensures that a Mackey complete disk is weakly compact if, and only if, it has the quasi weak drop property. So, if the space (E,τ) is locally complete and satisfies the sMc, then the considered disk B is weakly compact and the equivalence should be almost obvious.

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